Propagating Wave (1B)

- 3-D Propagating Wave

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Maxwell Equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \mu \boldsymbol{H}}{\partial t} \qquad \nabla \cdot (\epsilon \boldsymbol{E}) = 0$$

$$\nabla \times \boldsymbol{H} = +\frac{\partial \epsilon \boldsymbol{E}}{\partial t} \qquad \nabla \cdot (\mu \boldsymbol{H}) = 0$$

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i}_x + \frac{\partial}{\partial y}\boldsymbol{i}_y + \frac{\partial}{\partial z}\boldsymbol{i}_z \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \xrightarrow{s(\boldsymbol{x}, t)} \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}$$

Wave Equation in Cartesian Coordinates

Assume a separable solution

$$
\begin{aligned}
s(x, y, z, t) &= A e^{j(\omega t - k_x x - k_y y - k_z z)} \\
&= A e^{-jk_x x} e^{-jk_y y} e^{-jk_z z} e^{j\omega t} \\
&= f(x) g(x) h(x) p(t)
\end{aligned}
$$

$$
\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 s}{\partial t^2}
$$

$$
k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}
$$

$$
k_x^2 s(x, y, z, t) + k_y^2 s(x, y, z, t) + k_z^2 s(x, y, z, t) = \frac{\omega^2 s(x, y, z, t)}{c^2}
$$

Monochrome Plane Wave (1)

$$s(x, y, z, t) = A e^{j(\omega t - k_x x - k_y y - k_z z)}$$

Fixed point $(x, y, z) = (0,0,0)$

$$s(0,0,0,t) = A e^{j\omega t} = A\cos\omega t + A\sin\omega t$$

Monochrome Wave

$$s(x, y, z, t) = A e^{j(\omega t - k_x x - k_y y - k_z z)}$$

Fixed time $t = t_0$

$$s(x, y, z, t_0) = A e^{j(\omega t_0 - [k_x x + k_y y + k_z z])}$$

points (x, y, z) such that $k_x x + k_y y + k_z z = C$ Plane Wave

planes of constant phase

$$s(x, y, z, t_0) = A e^{j(\omega t_0 - k_x x - k_y y - k_z z)}$$ has the same value $A e^{j(\omega t_0 - C)}$

Monochrome Plane Wave (2)

$$s(x, y, z, t) = A e^{j(\omega t - k_x x - k_y y - k_z z)} \Rightarrow s(\boldsymbol{x}, t) = A e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

planes of constant phase $\rightarrow$ $\boldsymbol{k}\cdot\boldsymbol{x} = C$

If truly a propagating wave

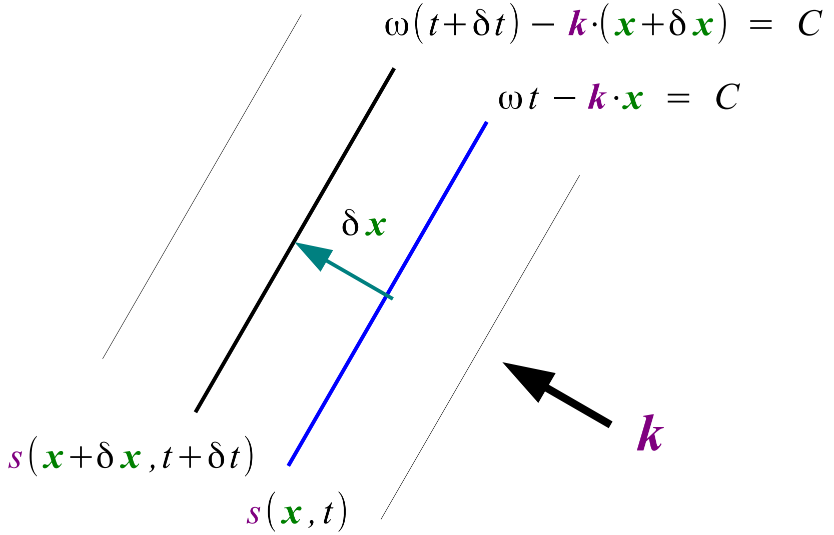
planes of constant phase move by $\delta \boldsymbol{x}$

as time advances by δt

$$\Rightarrow s(\boldsymbol{x}+\delta\boldsymbol{x}, t+\delta t) = s(\boldsymbol{x}, t)$$

$$\Rightarrow \omega(t+\delta t) - \boldsymbol{k}\cdot(\boldsymbol{x}+\delta\boldsymbol{x}) = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$$

$$\omega\,\delta t - \boldsymbol{k}\cdot\delta\boldsymbol{x} = 0$$



Monochrome Plane Wave (3)

constant phase → $\boldsymbol{k}\cdot\boldsymbol{x} = C$

planes of constant phase → perpendicular to $\boldsymbol{k}$

Plane Equation

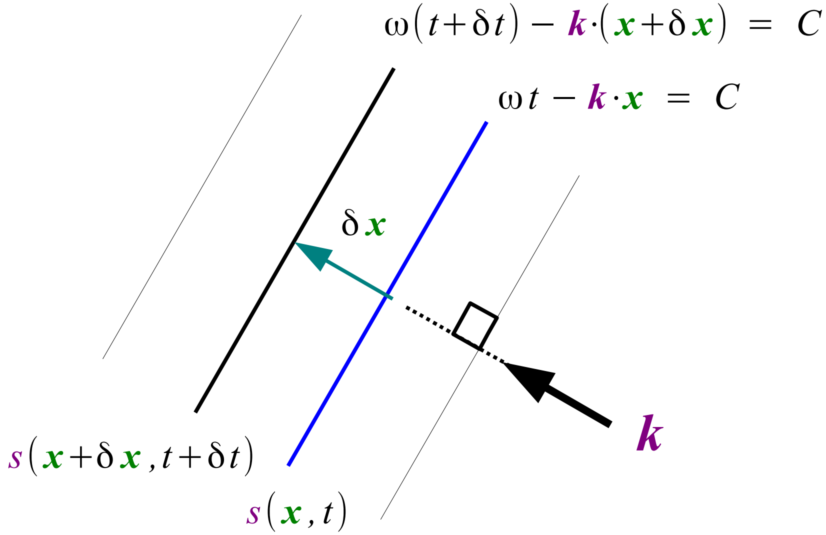
$$\boldsymbol{n}\cdot(\boldsymbol{r} - \boldsymbol{r}_0) = 0$$

Normal Vector

Position vector of a point in the plane

$\boldsymbol{n}\cdot\boldsymbol{r}_0$

$$\boldsymbol{k}\cdot\boldsymbol{x} - C = 0$$



Monochrome Plane Wave (4)

constant phase → $\boldsymbol{k} \cdot \boldsymbol{x} = C$

planes of constant phase → perpendicular to $\boldsymbol{k}$

If truly a propagating wave planes of constant phase move by $\delta \boldsymbol{x}$
as time advances by δt

$s(\boldsymbol{x} + \delta \boldsymbol{x}, t + \delta t) = s(\boldsymbol{x}, t)$ → $\omega \delta t - \boldsymbol{k} \cdot \delta \boldsymbol{x} = 0$

$\delta \boldsymbol{x}$ in the same direction $\boldsymbol{k}$: minimum $|\delta \boldsymbol{x}|$

The direction of propagation $\zeta_0 = \dfrac{\boldsymbol{k}}{|\boldsymbol{k}|}$

If in the same direction $\boldsymbol{k} \cdot \delta \boldsymbol{x} = |\boldsymbol{k}||\delta \boldsymbol{x}|$

Monochrome Plane Wave (5)

constant phase → $\boldsymbol{k} \cdot \boldsymbol{x} = C$

planes of constant phase → perpendicular to $\boldsymbol{k}$

$s(\boldsymbol{x}+\delta\boldsymbol{x}, t+\delta t) = s(\boldsymbol{x}, t)$ → $\omega\,\delta t - \boldsymbol{k}\cdot\delta\boldsymbol{x} = 0$

$\delta\boldsymbol{x}$ in the same direction $\boldsymbol{k}$: minimum $|\delta\boldsymbol{x}|$

The direction of propagation $\boldsymbol{\zeta}_0 = \dfrac{\boldsymbol{k}}{|\boldsymbol{k}|}$

If in the same direction $\boldsymbol{k}\cdot\delta\boldsymbol{x} = |\boldsymbol{k}||\delta\boldsymbol{x}|$

$$\omega\,\delta t = |\boldsymbol{k}||\delta\boldsymbol{x}|$$

$$\frac{\omega}{|\boldsymbol{k}|} = \frac{|\delta\boldsymbol{x}|}{\delta t}$$

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} \;\Rightarrow\; \boldsymbol{k}^2 = \frac{\omega^2}{c^2} \;\Rightarrow$$

$$c^2 = \frac{\omega^2}{\boldsymbol{k}^2} \;\Rightarrow\; c = \frac{\omega}{|\boldsymbol{k}|} \;\Rightarrow\; c = \frac{|\delta\boldsymbol{x}|}{\delta t}$$

The speed of propagation of the plane wave

Wave Number, Angular Frequency

wave number $\boxed{k = \frac{2\pi}{\lambda}}$

How many λ in 2π *(rad / m)*

angular frequency $\boxed{\omega = \frac{2\pi}{T}}$

How many T in 2π *(rad / sec)*

3-dimensional space

$$\omega\,\delta t - \boldsymbol{k}\cdot\delta\boldsymbol{x} = 0$$

period

wavelength

$$\delta t \Rightarrow \boxed{T = \frac{2\pi}{\omega}}$$

$$\delta x \Rightarrow \boxed{\lambda = \frac{2\pi}{\boldsymbol{k}}}$$

wave number vector

spatial frequency variable

Its magnitude represents the number of cycles (in rad) per meter of length that the monochromatic plane wave exhibits *in the direction of propagation*.

Wavelength, Frequency

$$s(\boldsymbol{x}, t) = A e^{j(\omega t - \boldsymbol{k}\cdot\boldsymbol{x})}$$

position vector

$$(\omega t - \boldsymbol{k}\cdot\boldsymbol{x}) = \omega\left(t - \left(\frac{\boldsymbol{k}}{\omega}\right)\cdot\boldsymbol{x}\right)$$

temporal angular frequency

3-d spatial frequency

slowness vector $\boldsymbol{\alpha}$

$$= [\omega(t - \boldsymbol{\alpha}\cdot\boldsymbol{x})]$$

$$s(\boldsymbol{x}, t) = A e^{j(\omega(t - \boldsymbol{\alpha}\cdot\boldsymbol{x}))}$$

Function of a single variable

$$s(u) = A e^{j(\omega u)}$$

$$s(t - \boldsymbol{\alpha}\cdot\boldsymbol{x}) = A e^{j(\omega(t - \boldsymbol{\alpha}\cdot\boldsymbol{x}))}$$

$$= s(\boldsymbol{x}, t)$$

Slowness Vector

$$\boldsymbol{\alpha} = \frac{\boldsymbol{k}}{\omega}$$

Speed Vector (Phase Velocity)

$$\boldsymbol{v}_p = \frac{\omega}{\boldsymbol{k}}$$

Summary

Propagating Plane Wave

$$s(t-\boldsymbol{\alpha}\cdot\boldsymbol{x}) = A e^{j(\omega(t-\boldsymbol{\alpha}\cdot\boldsymbol{x}))} = s(\boldsymbol{x}, t)$$

Propagating Sinusoidal Plane Wave

$$\sin(\omega(t-\boldsymbol{\alpha}\cdot\boldsymbol{x})) = \sin(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})$$

Slowness Vector

$$\boldsymbol{\alpha} = \frac{\boldsymbol{k}}{\omega} \qquad |\boldsymbol{\alpha}| = \frac{1}{c}$$

Wavenumber Vector

$$\boldsymbol{k} = \omega\boldsymbol{\alpha} \qquad |\boldsymbol{k}| = \frac{2\pi}{\lambda}$$

Frequency and wavelength

$$|\boldsymbol{\alpha}| = \frac{|\boldsymbol{k}|}{\omega} \qquad c = \frac{1}{|\boldsymbol{\alpha}|} = \frac{\omega}{|\boldsymbol{k}|}$$

$$c = \frac{\omega\cdot\lambda}{2\pi}$$

Maxwell Equations

$$A(t,\ t) = A_0 \cos(k x - \omega t)$$

Maxwell Equations

$$A(t, t) = A_0 \cos(kx - \omega t)$$

References

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