Divergence and Curl (3B)

- Divergence
- Curl
- Green's Theorem

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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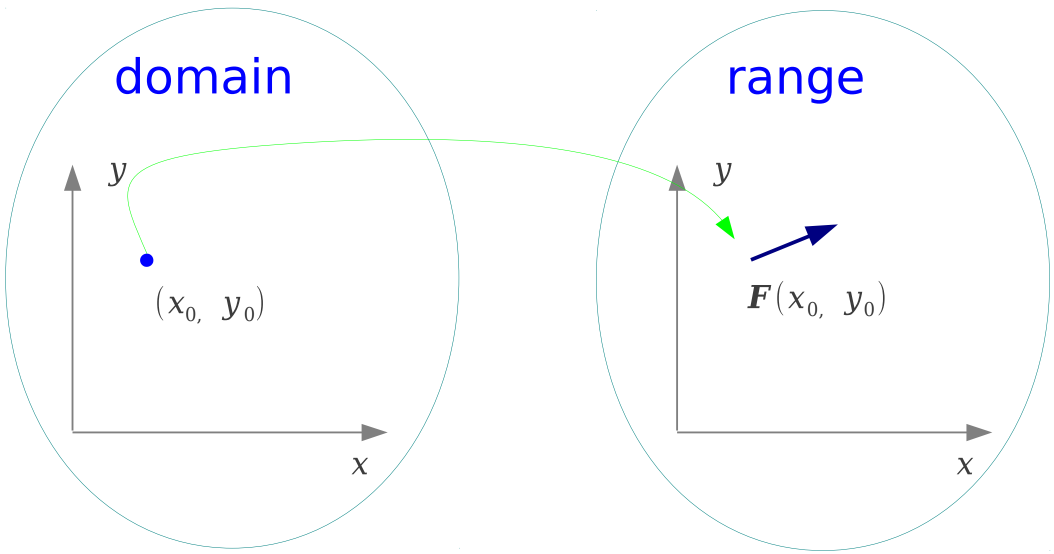
2-D Vector Field

a given point in a 2-d space ➡ A vector

$(x_{0,}\ \ y_0)$ $\quad\quad$ $\langle\, M(x_{0,}y_0),\ \ N(x_{0,}y_0)\,\rangle$



2 functions

$(x_{0,}\ \ y_0) \longrightarrow M(x_{0,}y_0)$

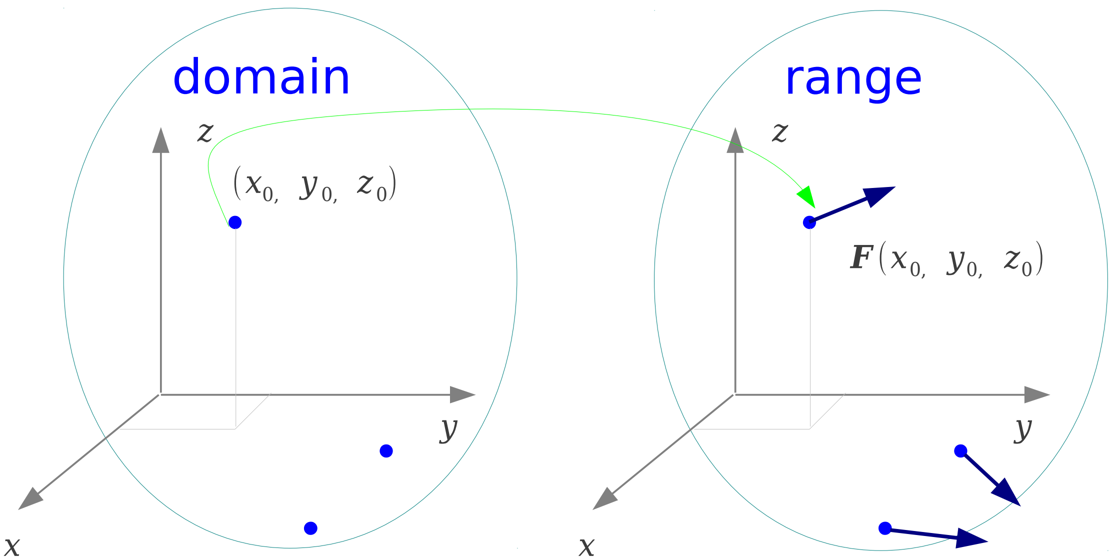
$(x_{0,}\ \ y_0) \longrightarrow N(x_{0,}y_0)$

$$(x_{0,}\ \ y_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_0) = M(x_{0,}y_0)\boldsymbol{i} + N(x_{0,}y_0)\boldsymbol{j}$$

3-D Vector Field

A given point in a 3-d space ➡ A vector

$$(x_0,\ y_0,\ z_0) \qquad \langle\, M(x_0, y_0, z_0),\ N(x_0, y_0, z_0),\ P(x_0, y_0, z_0) \,\rangle$$



3 functions

$$(x_0,\ y_0,\ z_0) \longrightarrow M(x_0, y_0, Z_0)$$

$$(x_0,\ y_0,\ z_0) \longrightarrow N(x_0, y_0, z_0)$$

$$(x_0,\ y_0,\ z_0) \longrightarrow P(x_0, y_0, z_0)$$

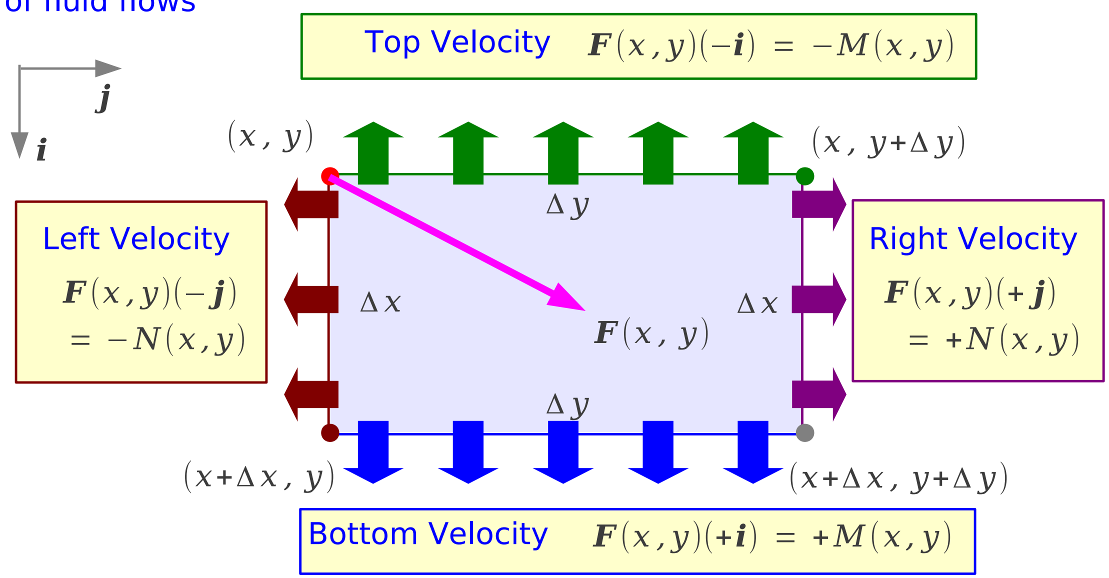
$$(x_0,\ y_0,\ z_0) \longrightarrow \boldsymbol{F}(x_0,\ y_0,\ z_0) = M(x_0, y_0, z_0)\boldsymbol{i} + N(x_0, y_0, z_0)\boldsymbol{j} + P(x_0, y_0, z_0)\boldsymbol{k}$$

Inward & Outward Bound

2-D Divergence (5)

Velocity Fields of fluid flows

$$\boldsymbol{F}(x,\,y) \;=\; M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$$



$$\{M(x+\Delta x, y) - M(x,\,y)\}\Delta y$$
$$= \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\{N(x,\,y+\Delta y) - N(x,\,y)\}\Delta x$$
$$= \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y \;+\; \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x\,\Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

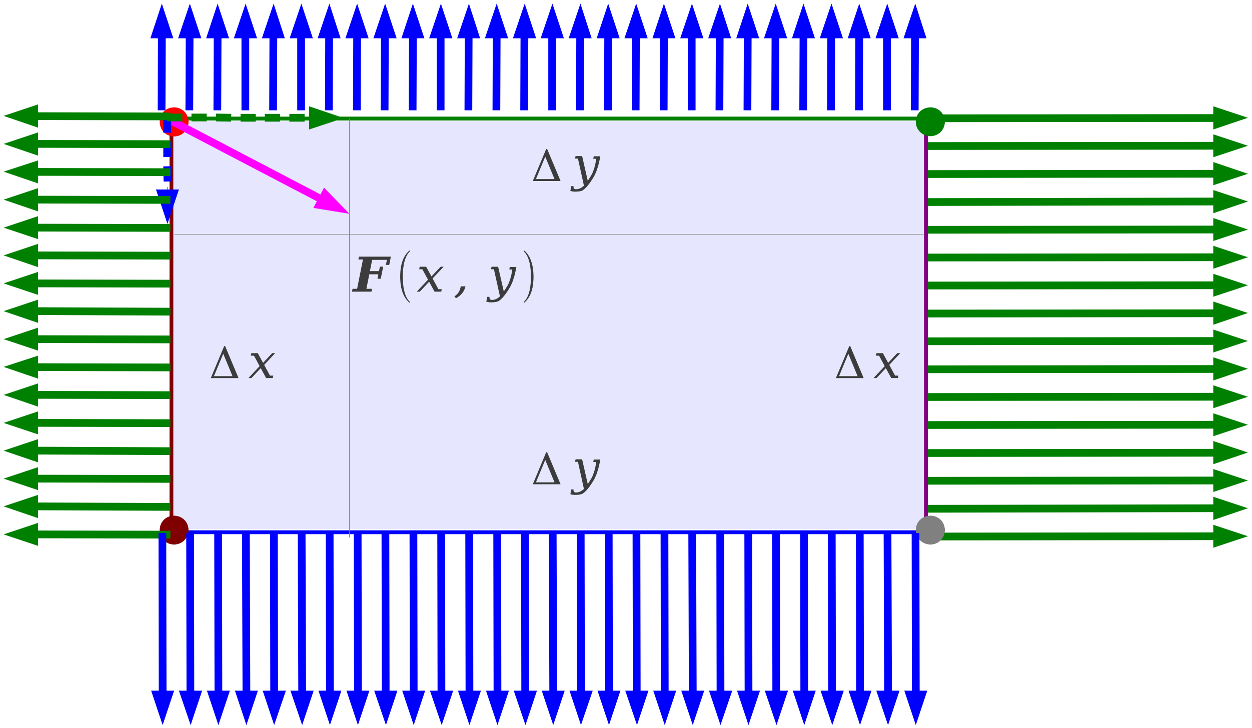
Divergence of **F**

Flux Density

2-D Divergence (d)

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$



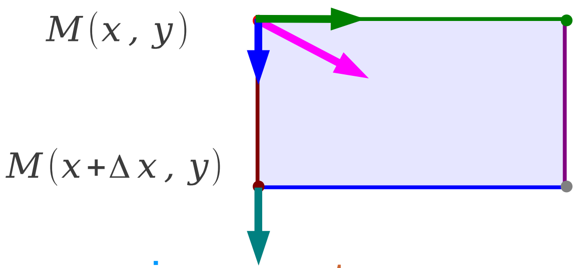
Flux density $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$ Divergence of **F** Flux Density

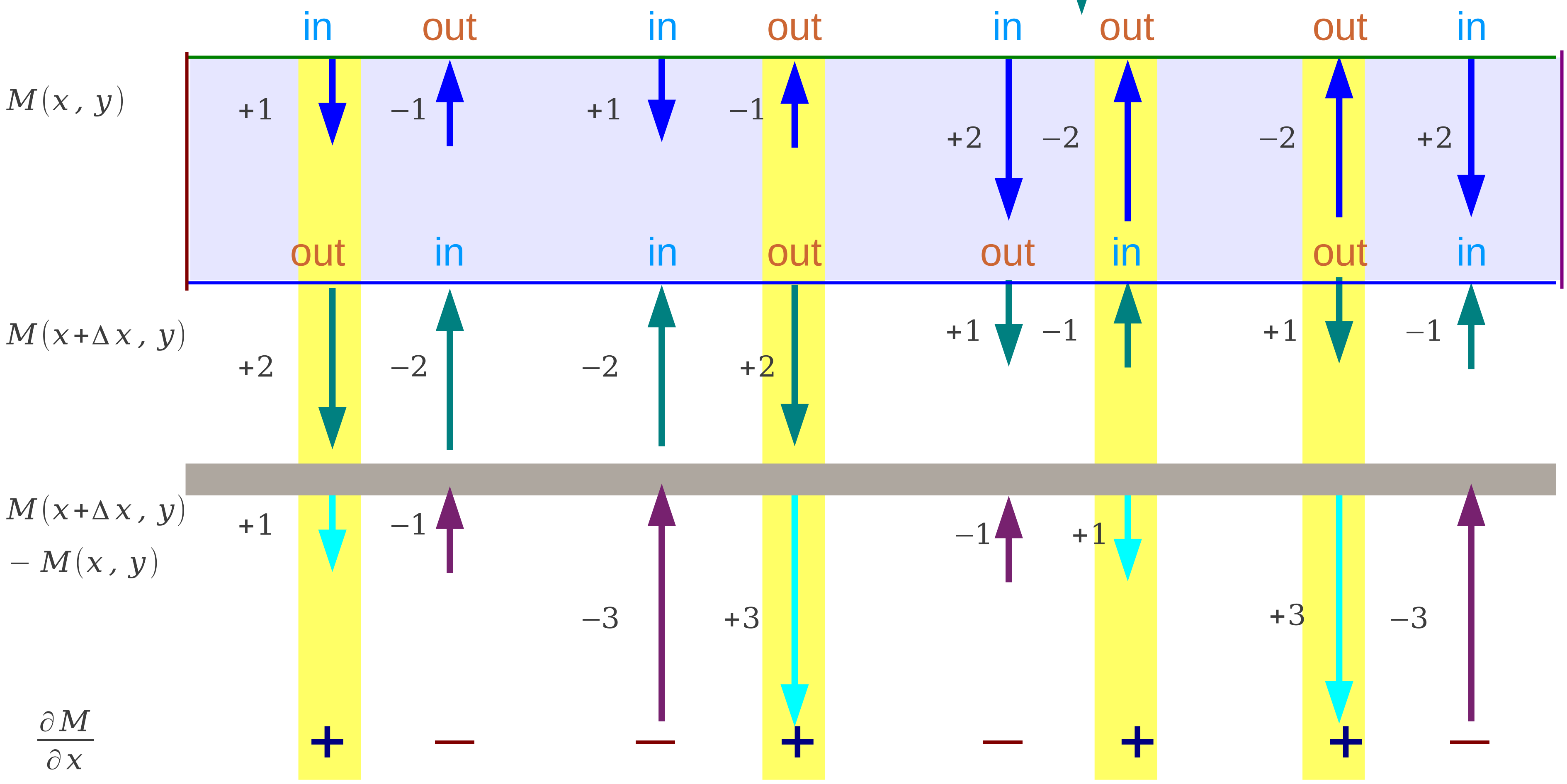
Inward & Outward Bound

Inward & Outward Bound – X (i)

$$\mathbf{F}(x, y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$$

$$\{M(x+\Delta x, y) - M(x, y)\}$$

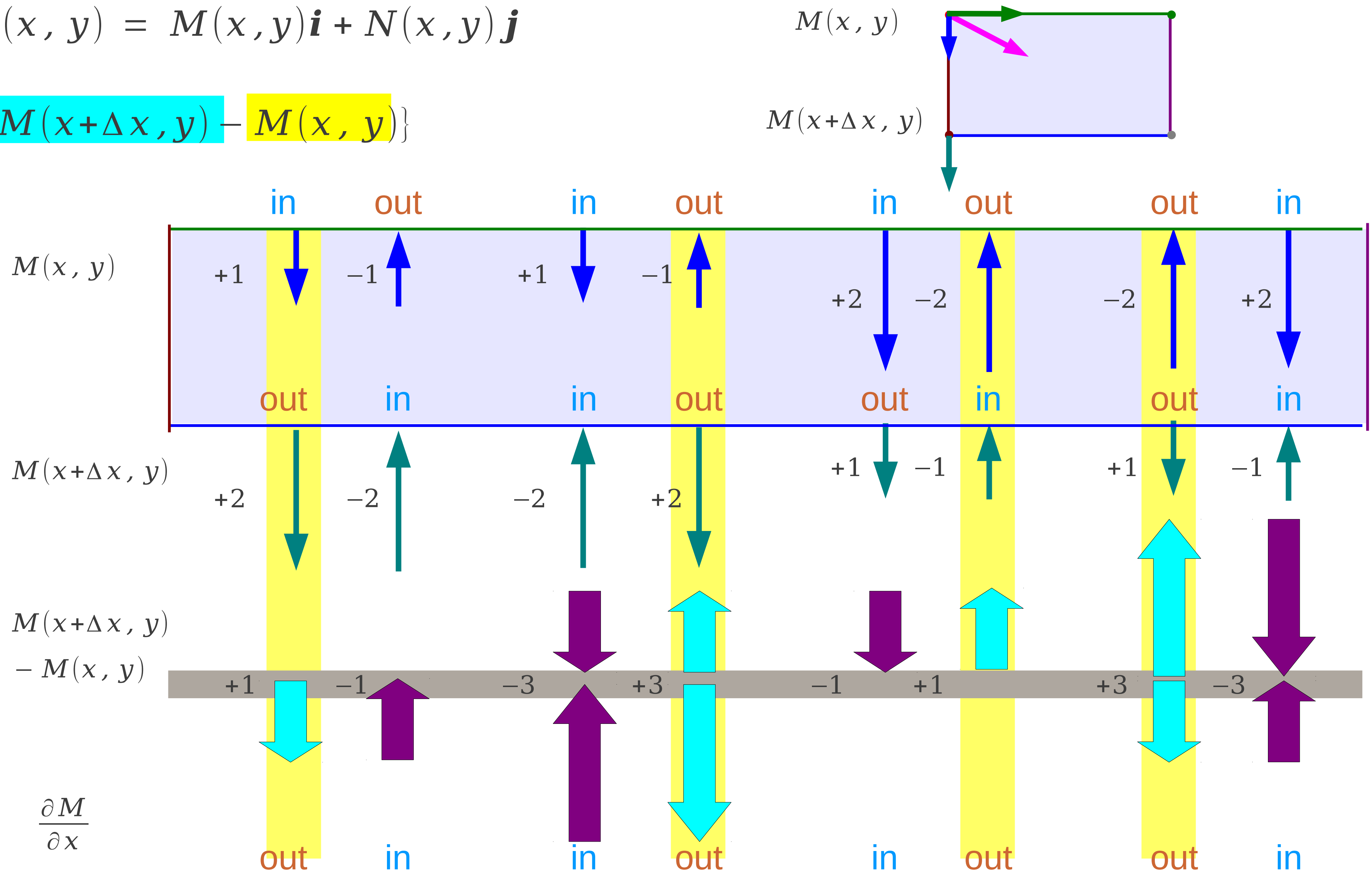




Inward & Outward Bound – X (ii)

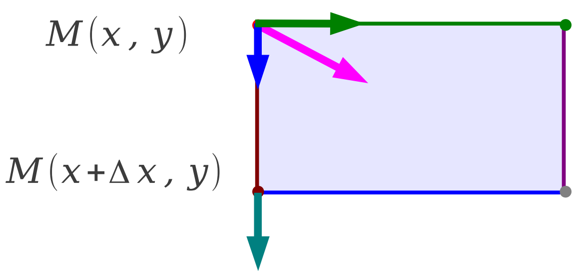
$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\{M(x+\Delta x, y) - M(x, y)\}$$



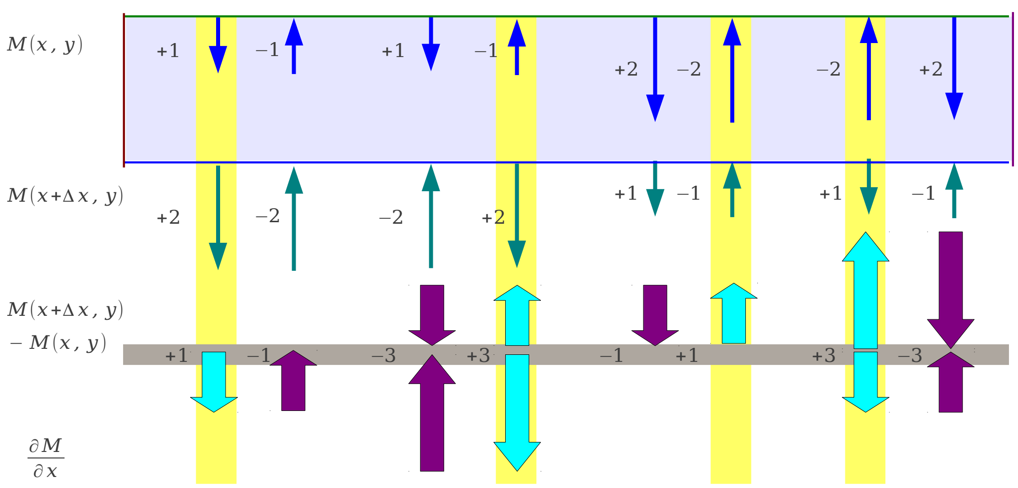
Inward & Outward Bound – X (iii)

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$



$\{M(x+\Delta x, y) - M(x, y)\} > 0 \quad \Delta x > 0$ **Outward** bound net flow

$\{M(x+\Delta x, y) - M(x, y)\} < 0 \quad \Delta x > 0$ **Inward** bound net flow

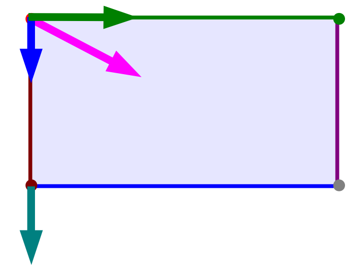


$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0$$ **Outward**

$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0$$ **Inward**

Inward & Outward Bound – X (iv)

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$



$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0$$

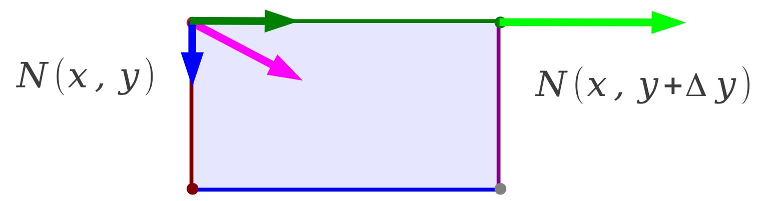
- **Positive Slope** of a tangent line parallel to the x axis
- **Outward** bound net flow along the x axis

$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0$$

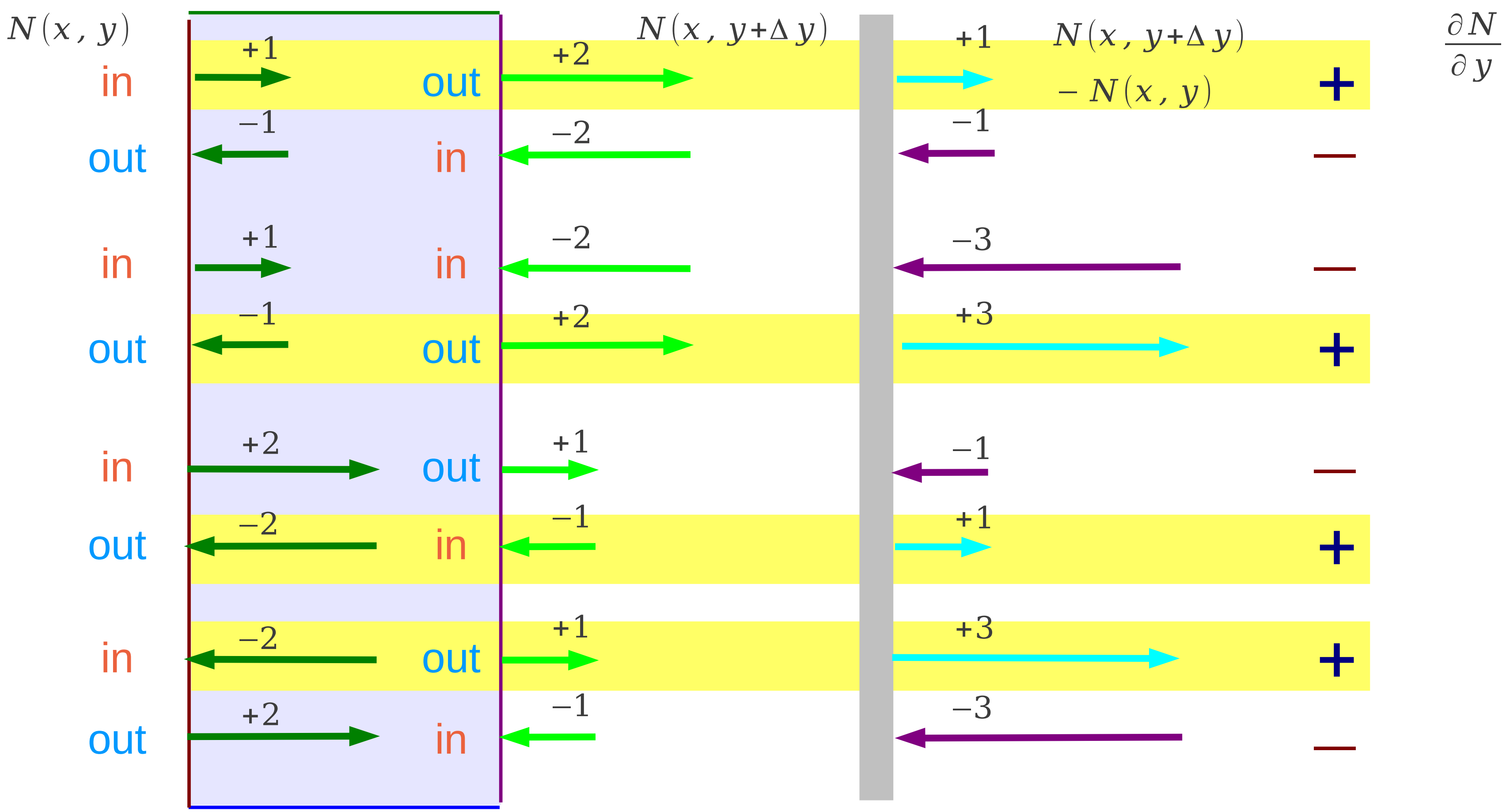
- **Negative Slope** of a tangent line parallel to the x axis
- **Inward** bound net flow along the x axis

Inward & Outward Bound – Y (i)

$$\boldsymbol{F}(x,y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$

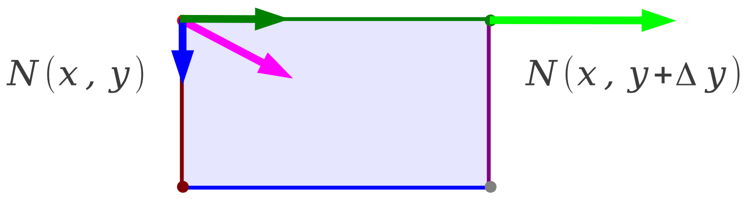


$$\{M(x+\Delta x, y) - M(x,y)\}$$

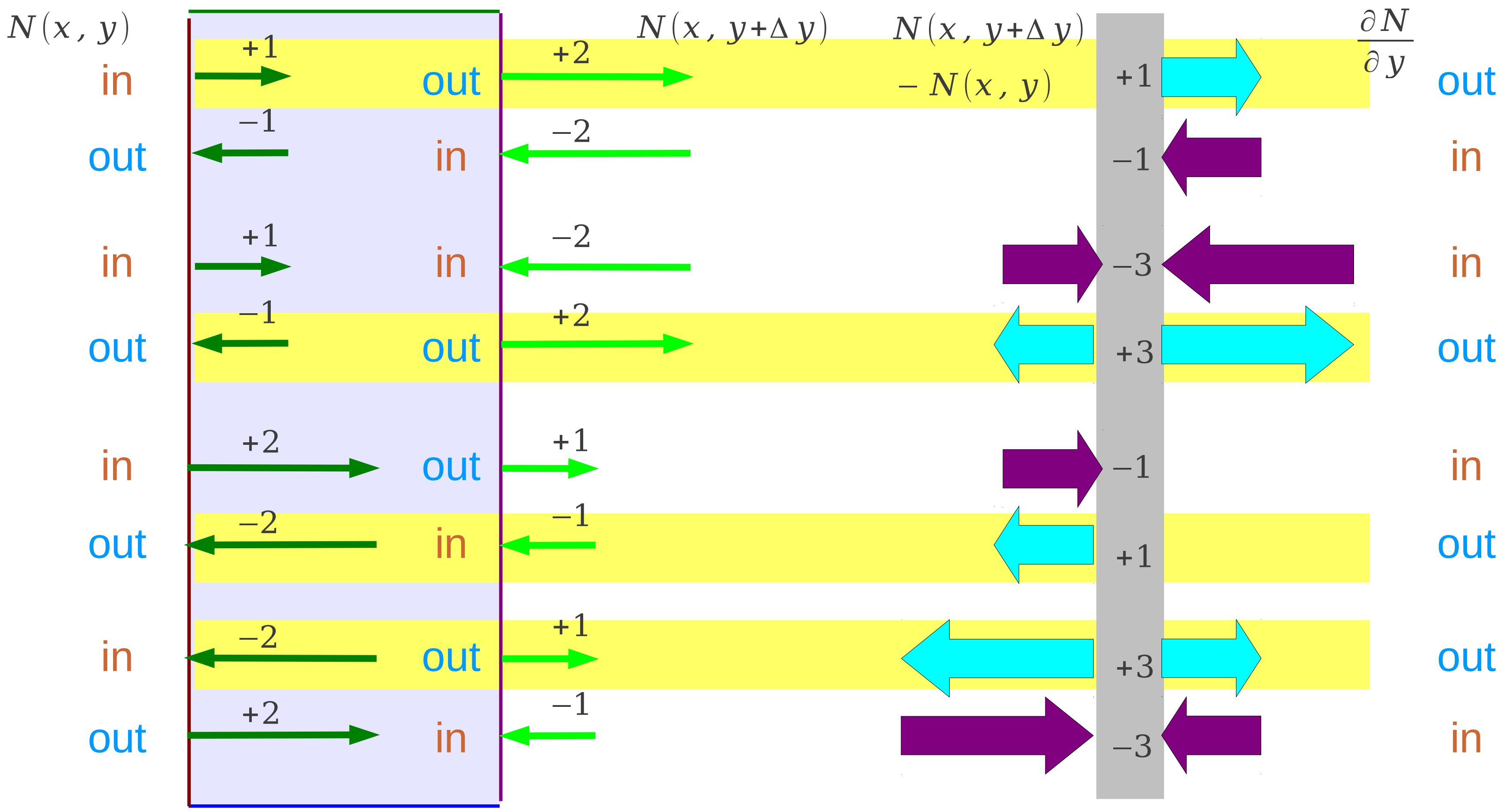


Inward & Outward Bound – Y (ii)

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$

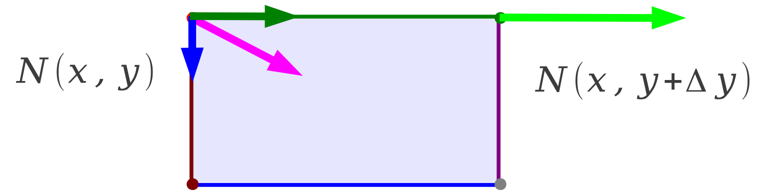


$$\{M(x+\Delta x, y) - M(x, y)\}$$



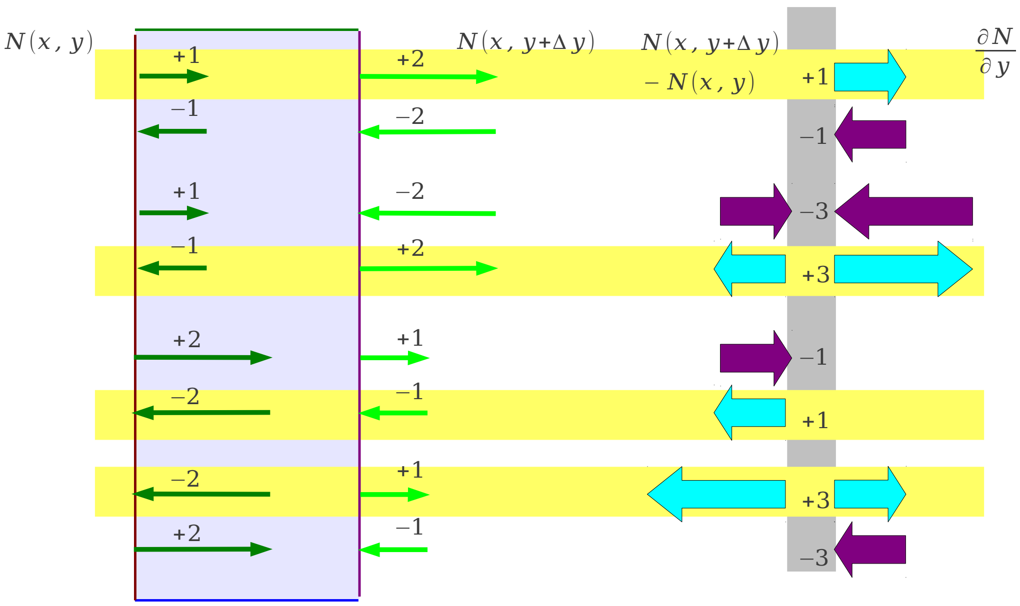
Inward & Outward Bound – Y (iii)

$$\boldsymbol{F}(x,y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



$\{N(x,y+\Delta y) - N(x,y)\} > 0 \quad \Delta y > 0$ **Outward** bound net flow

$\{N(x,y+\Delta y) - N(x,y)\} < 0 \quad \Delta y > 0$ **Inward** bound net flow

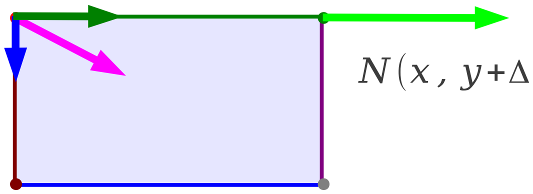


$$\frac{\{N(x,y+\Delta y) - N(x,y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0$$ **Outward**

$$\frac{\{N(x,y+\Delta y) - N(x,y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0$$ **Inward**

Inward & Outward Bound – Y (iv)

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



$$\frac{\{N(x,y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0$$

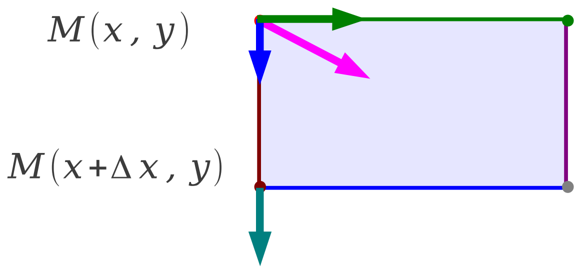
- **Positive Slope** of a tangent line parallel to the y axis
- **Outward** bound net flow along the y axis

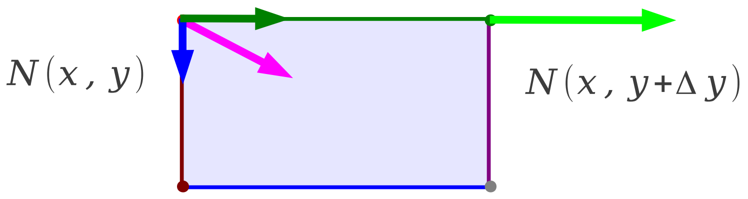
$$\frac{\{N(x,y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0$$

- **Negative Slope** of a tangent line parallel to the y axis
- **Inward** bound net flow along the y axis

Inward & Outward Bound

Inward & Outward Bound





$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} > 0$$ **Outward**

$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x} < 0$$ **Inward**

$$\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i})$$

$$= \nabla \cdot (M\boldsymbol{i})$$

$$\frac{\{N(x, y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} > 0$$ **Outward**

$$\frac{\{N(x, y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} < 0$$ **Inward**

$$\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (N\boldsymbol{j})$$

$$= \nabla \cdot (N\boldsymbol{j})$$

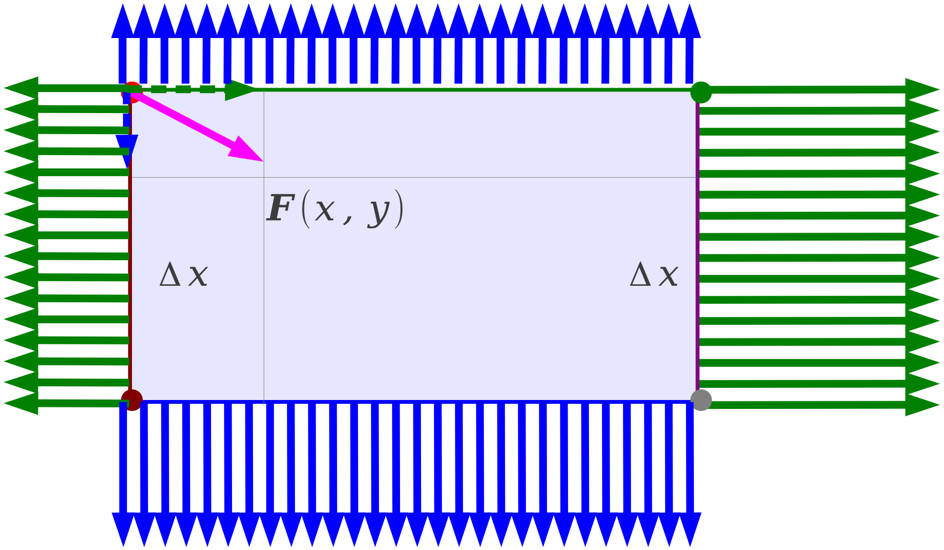
2-D Divergence and Del Operator

2-D Vector Field

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$



Flux density $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j})$$

$$= \nabla \cdot \boldsymbol{F}$$

Divergence of **F**

3-D Divergence and Del Operator

2-D Vector Field $\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$

Divergence of **F** $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$

$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j}) = \nabla \cdot \boldsymbol{F}$

3-D Vector Field $\boldsymbol{F}(x, y, z) = M(x, y, z)\boldsymbol{i} + N(x, y, z)\boldsymbol{j} + P(x, y, z)\boldsymbol{k}$

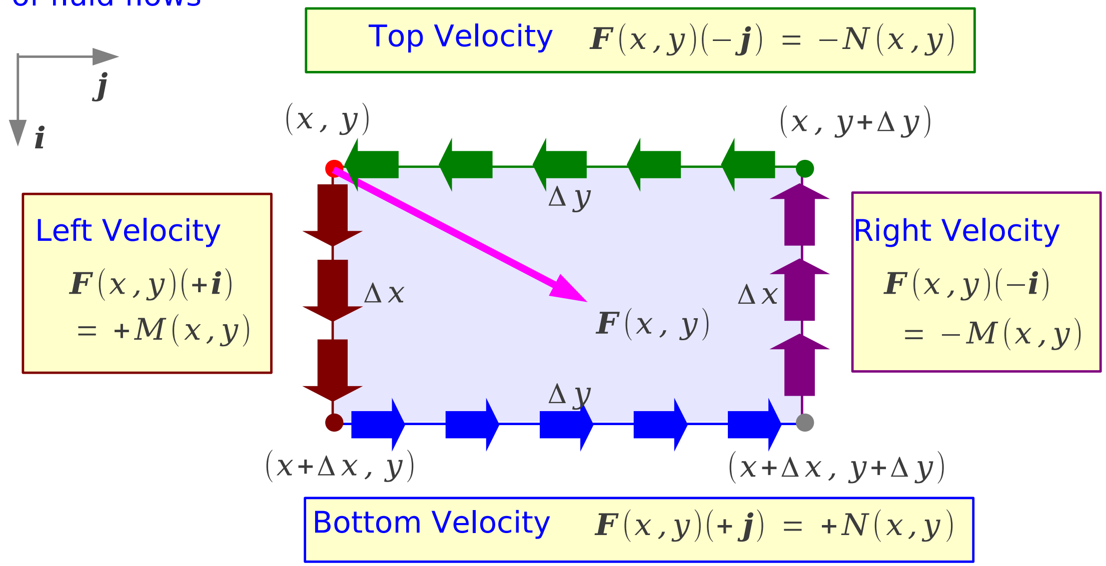
Divergence of **F** $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)$

$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) = \nabla \cdot \boldsymbol{F}$

2-D Curl (4)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\{N(x+\Delta x, y) - N(x, y)\}\Delta y = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$-\{M(x, y+\Delta y) - M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

Flow rate of counter clock wise circulating fluid

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

Circulation density

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

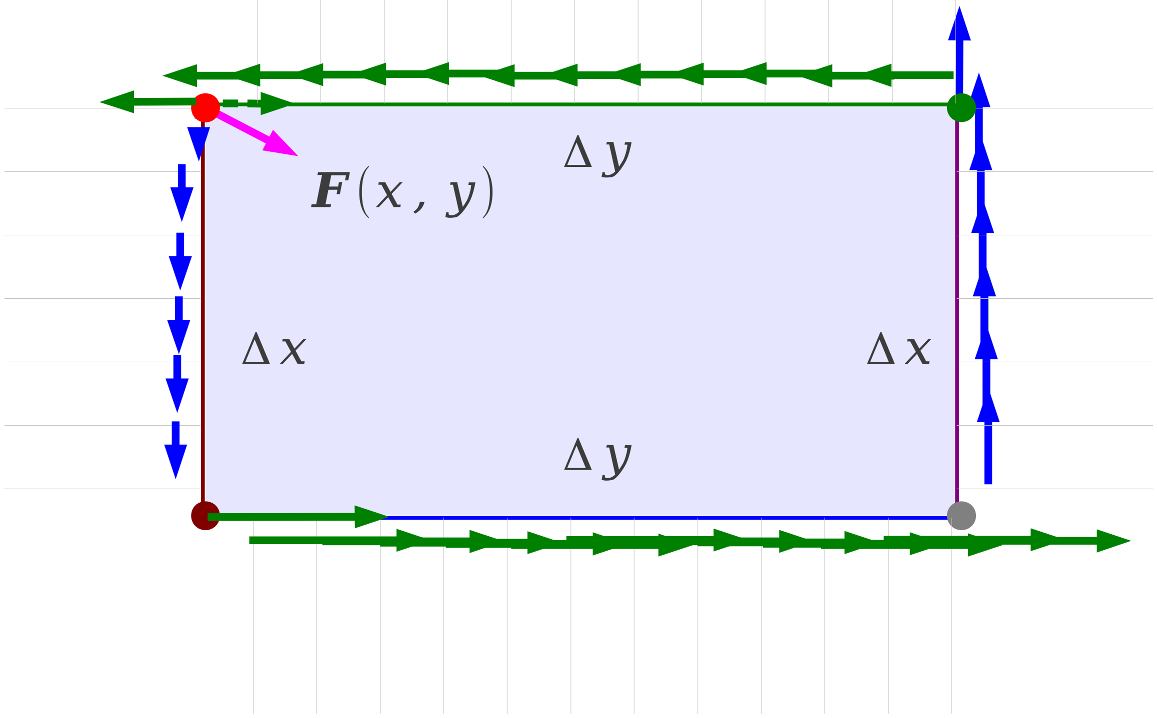
k-component
Curl of **F**

Circulation Density

2-D Curl (d)

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y \;-\; \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$



Circulation density $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

k-component
Curl of **F**

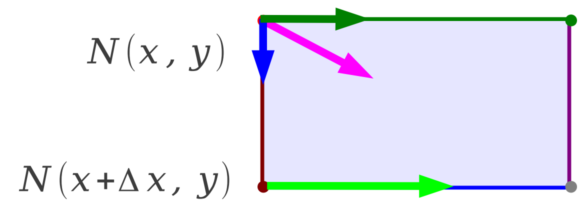
Circulation Density

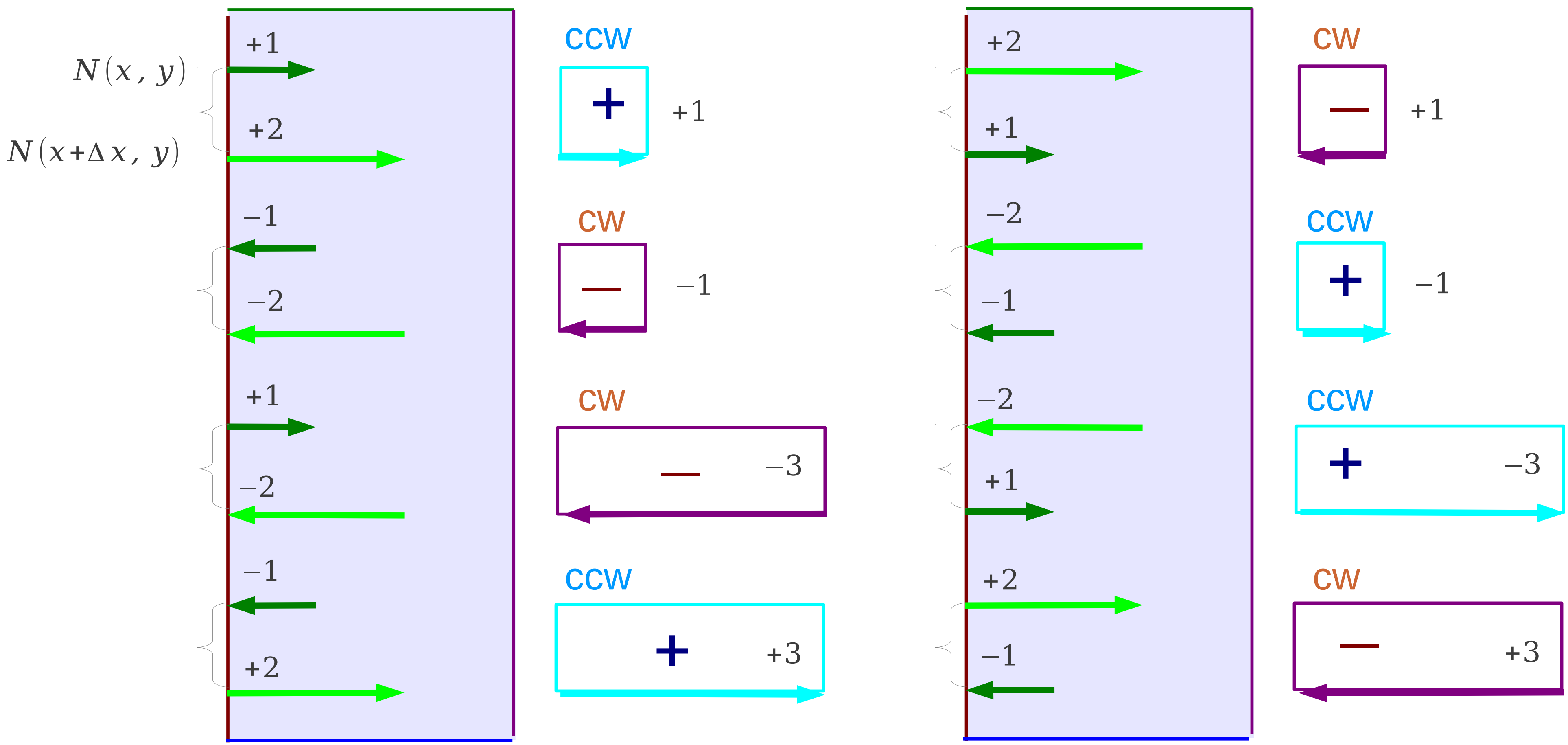
Clock-Wise & Counter-Clock-Wise

Clock-Wise & Counter-Clock-Wise – X (i)

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$

$$\{N(x+\Delta x, y) - N(x, y)\}$$

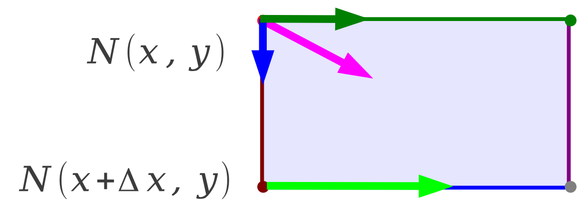


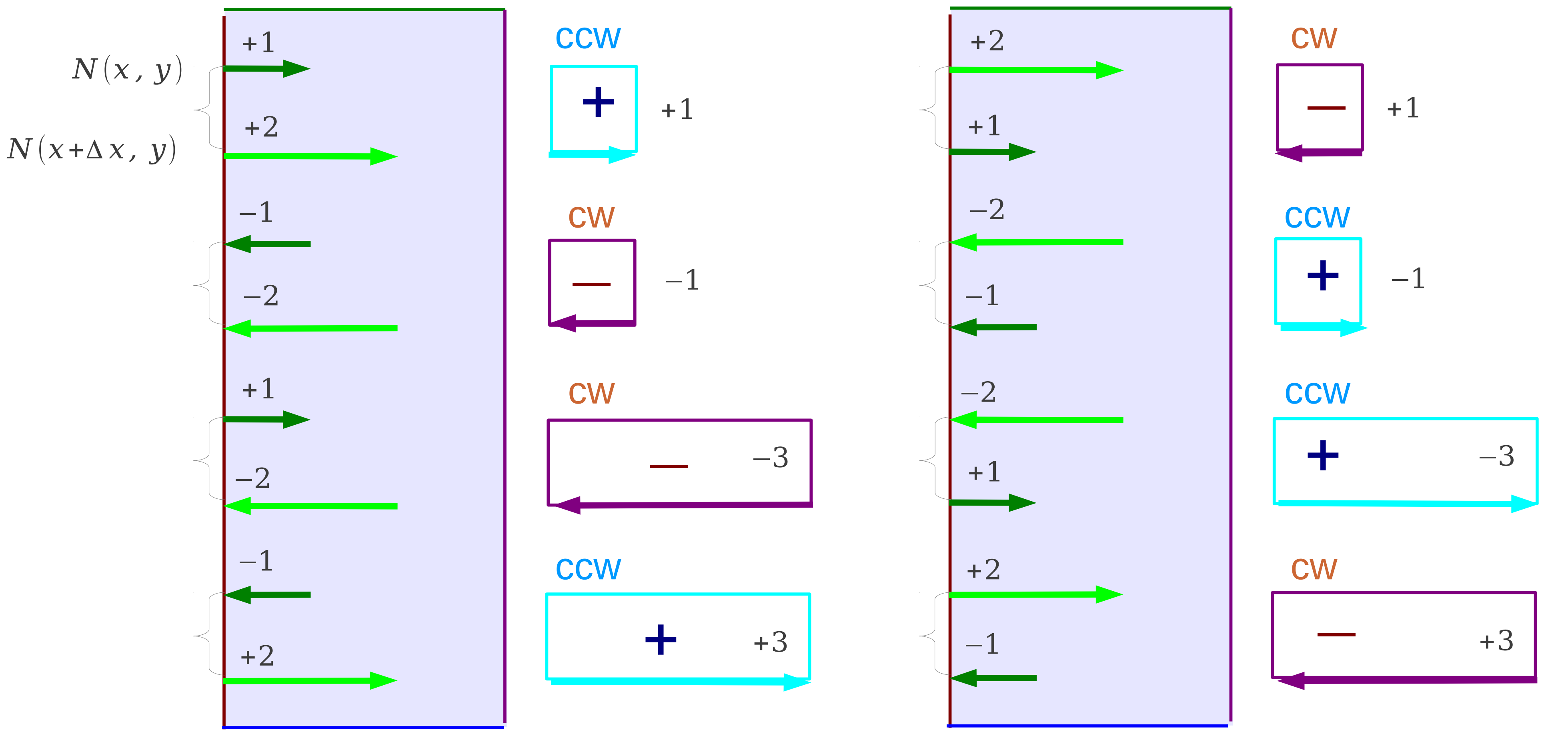


Clock-Wise & Counter-Clock-Wise – X (i)

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$

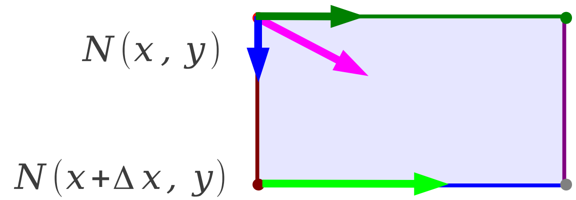
$$\{N(x+\Delta x, y) - N(x, y)\}$$





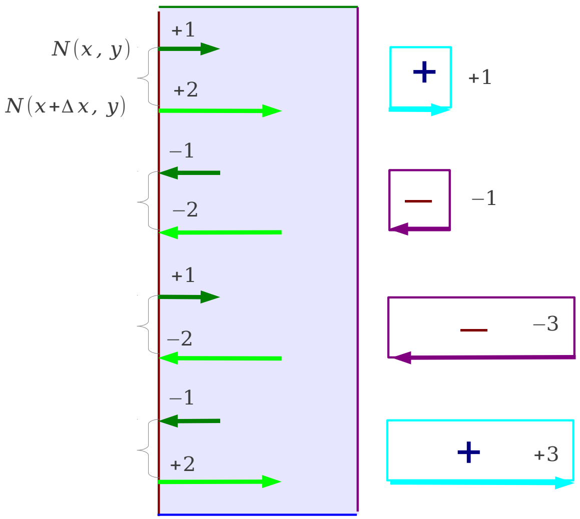
Clock-Wise & Counter-Clock-Wise – X (ii)

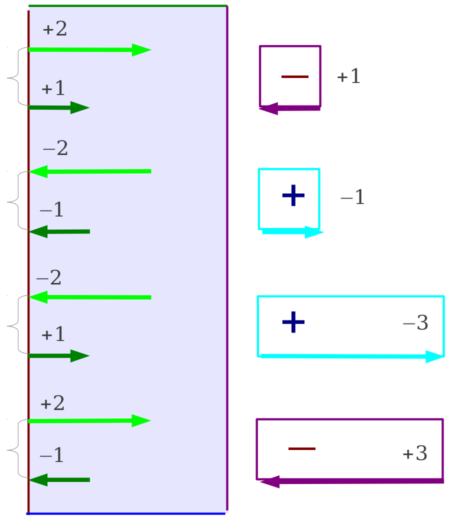
$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$



$\{N(x+\Delta x, y) - N(x, y)\} > 0 \quad \Delta x > 0$ **CCW** bound net flow

$\{N(x+\Delta x, y) - N(x, y)\} < 0 \quad \Delta x > 0$ **CW** bound net flow



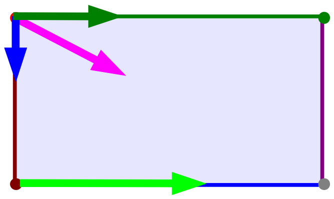


$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0 \qquad \textbf{CCW}$$

$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0 \qquad \textbf{CW}$$

Clock-Wise & Counter-Clock-Wise – X (iii)

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$



$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0$$

- **Positive Slope** of a tangent line parallel to the x axis
- **CCW** bound net flow along the z axis

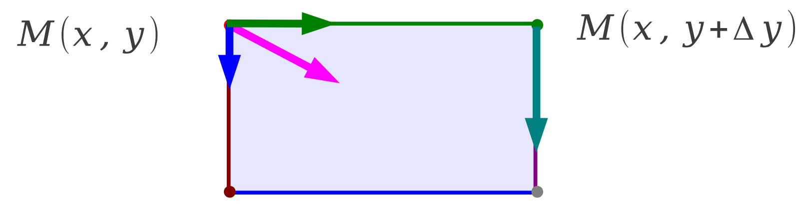
$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0$$

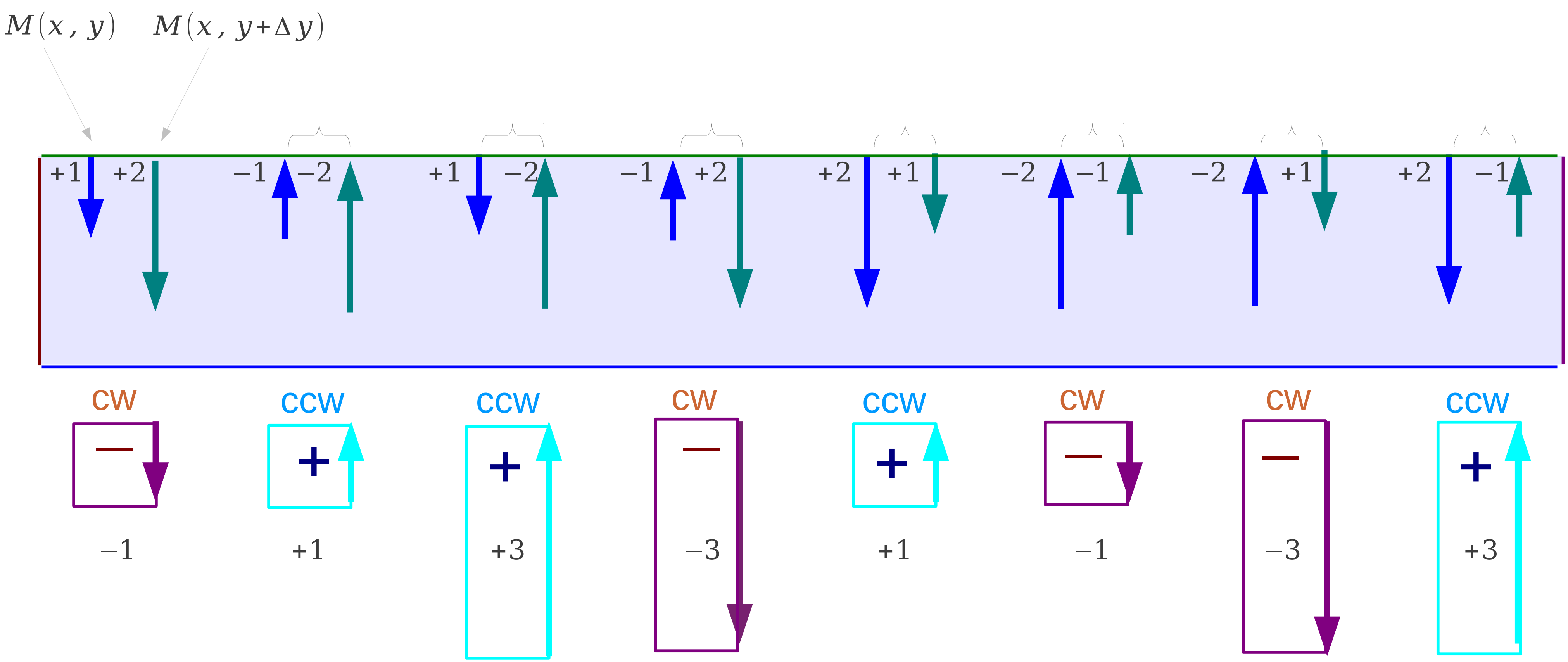
- **Negative Slope** of a tangent line parallel to the x axis
- **CW** bound net flow along the z axis

Clock-Wise & Counter-Clock-Wise – Y (i)

$$\boldsymbol{F}(x, y) = M(x, y)\,\boldsymbol{i} + N(x, y)\,\boldsymbol{j}$$

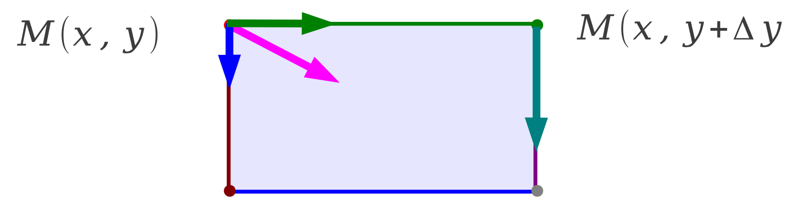
$$-\{M(x, y+\Delta y) - M(x, y)\}$$





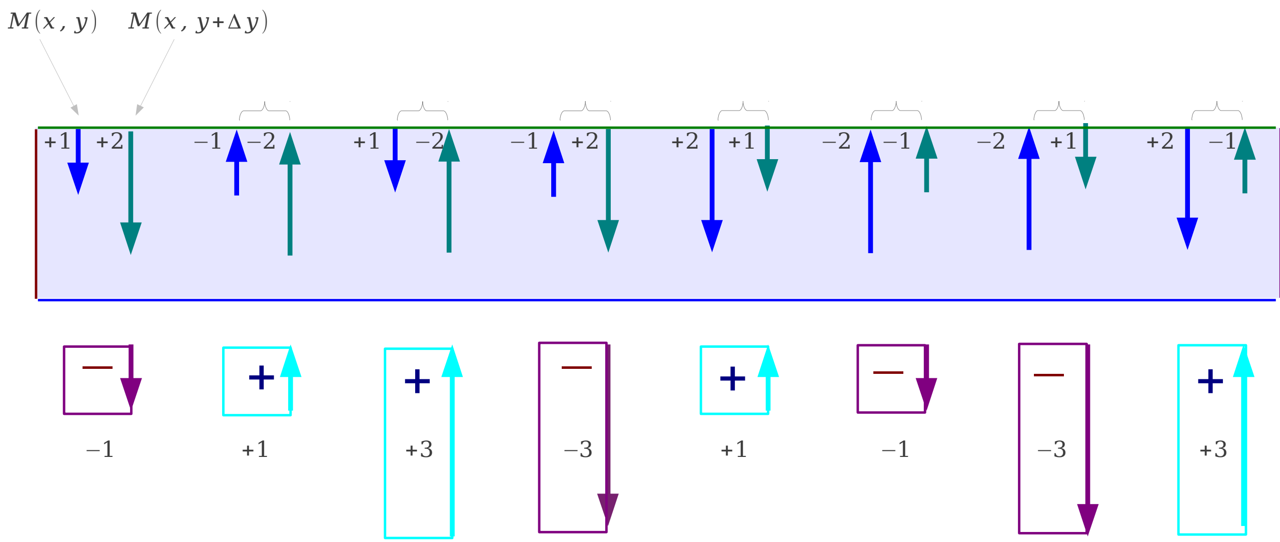
Clock-Wise & Counter-Clock-Wise – Y (ii)

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$



$-\{M(x, y+\Delta y) - M(x, y)\} > 0 \quad \Delta x > 0$ **CCW** bound net flow

$-\{M(x, y+\Delta y) - M(x, y)\} < 0 \quad \Delta x > 0$ **CW** bound net flow

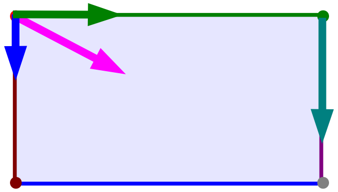


$$\frac{-\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} > 0 \quad \textbf{CCW}$$

$$\frac{-\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} < 0 \quad \textbf{CW}$$

Clock-Wise & Counter-Clock-Wise – Y (iii)

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$



$$-\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} > 0$$

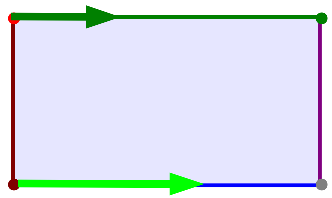
- **Positive Slope** of a tangent line parallel to the y axis
- **CCW** bound net flow along the z axis

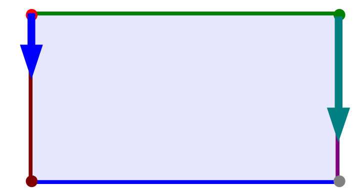
$$-\frac{\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} < 0$$

- **Negative Slope** of a tangent line parallel to the y axis
- **CW** bound net flow along the z axis

Clock-Wise & Counter-Clock-Wise

Clock-Wise & Counter-Clock-Wise





$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} > 0 \quad \textbf{CCW}$$

$$\frac{\{N(x+\Delta x, y) - N(x, y)\}}{\Delta x} \approx \frac{\partial N}{\partial x} < 0 \quad \textbf{CW}$$

$$\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (N\boldsymbol{j})$$

$$= \nabla \times (N\boldsymbol{j})$$

$$\frac{-\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} > 0$$

$$\frac{-\{M(x, y+\Delta y) - M(x, y)\}}{\Delta y} \approx -\frac{\partial M}{\partial y} < 0$$

$$\left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i})$$

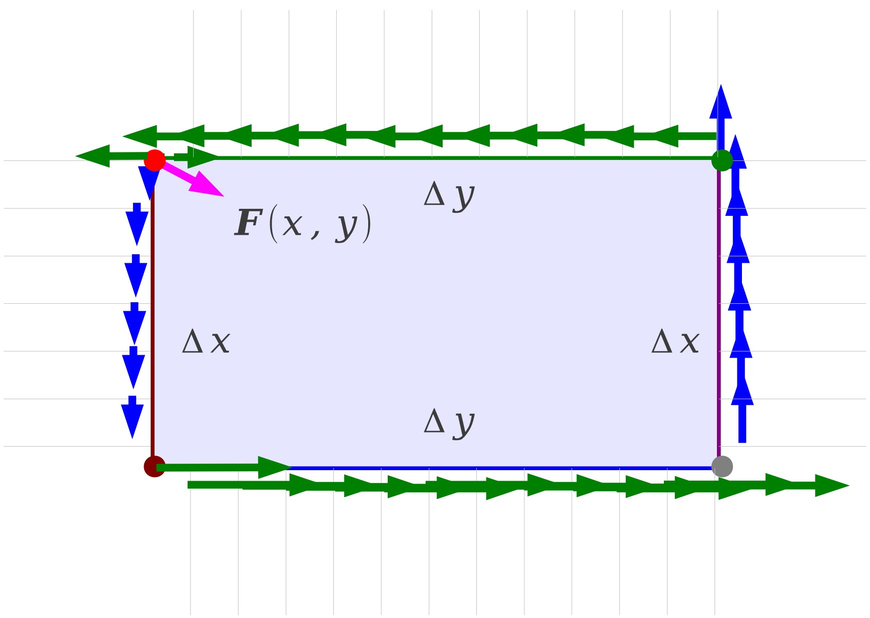
$$= \nabla \times (M\boldsymbol{i})$$

2-D Curl and Del Operator

2-D Vector Field $\quad \boldsymbol{F}(x, y) = M(x,y)\,\boldsymbol{i} + N(x,y)\,\boldsymbol{j}$

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$



Circulation density $\quad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k})$$

Curl of **F** $\quad = \nabla \times \boldsymbol{F}$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

3-D Curl and Del Operator

2-D Vector Field $\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Curl of **F** $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}$ $\qquad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k}) \quad = \nabla \times \boldsymbol{F}$$

3-D Vector Field $\boldsymbol{F}(x, y, z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$

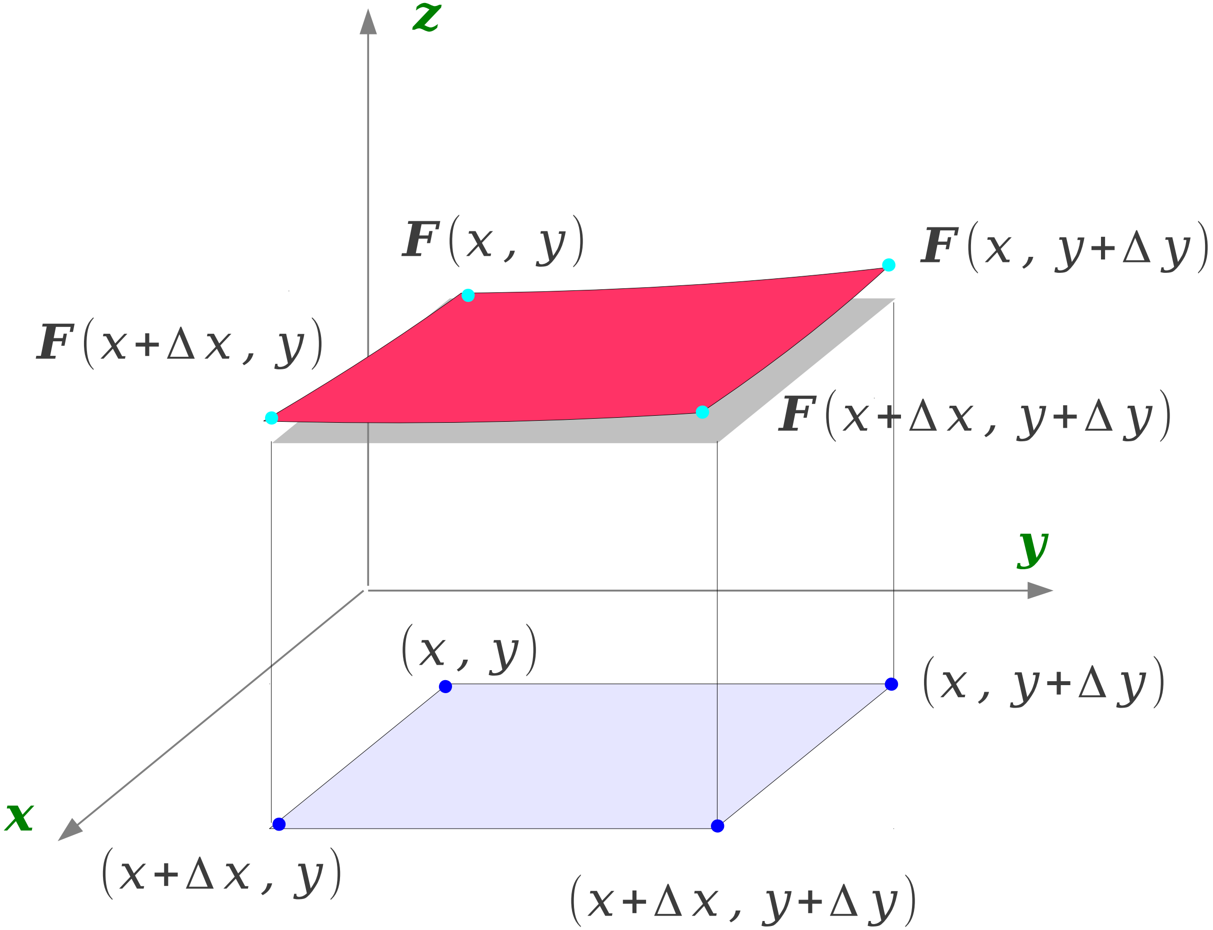
Curl of **F** $= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k}$ $\qquad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) = \nabla \times \boldsymbol{F}$$

2

i

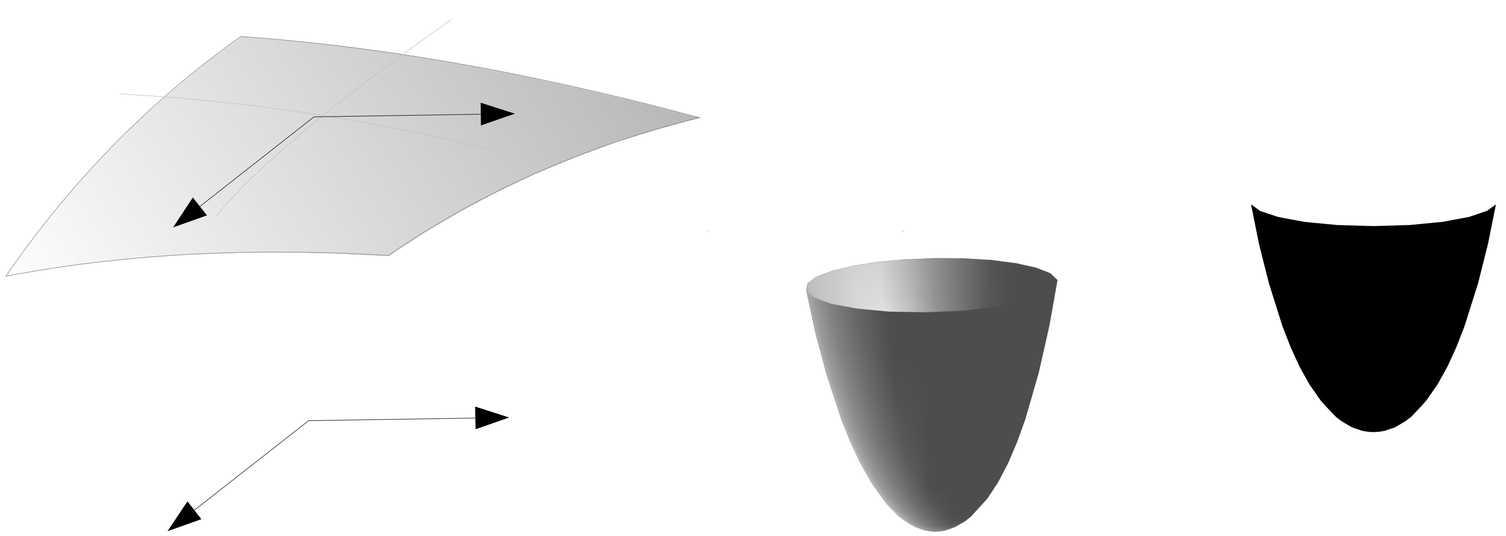
2-D Divergence



Chain Rule

Function of two variable $y = f(u, v)$

$u = g(x, y)$ $\qquad$ $v = h(x, y)$



References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[4] D.G. Zill, “Advanced Engineering Mathematics”