# Vector Calculus (H.1)
# Surface Integrals

20160118

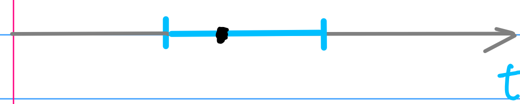
References

[1] Paul's online math note

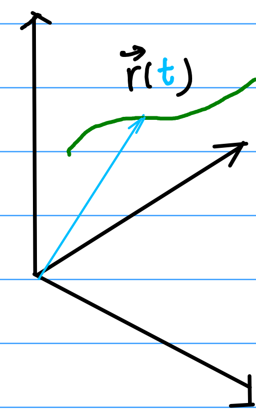
# Parametric Surfaces

$t$ parametric **Curve**

$$\boxed{\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}}$$

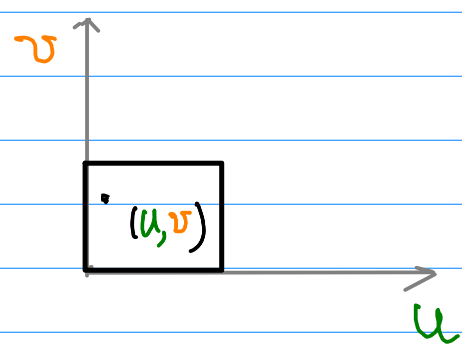



| |
|---|
| $x = x(t)$ |
| $y = y(t)$ |
| $z = z(t)$ |

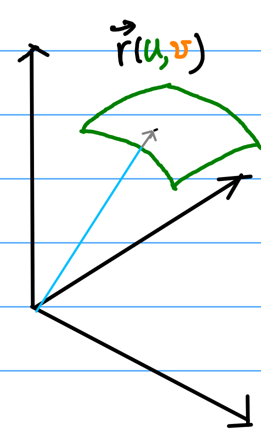



$u, v$ parametric **surface**

$$\boxed{\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}}$$




| |
|---|
| $x = x(u,v)$ |
| $y = y(u,v)$ |
| $z = z(u,v)$ |

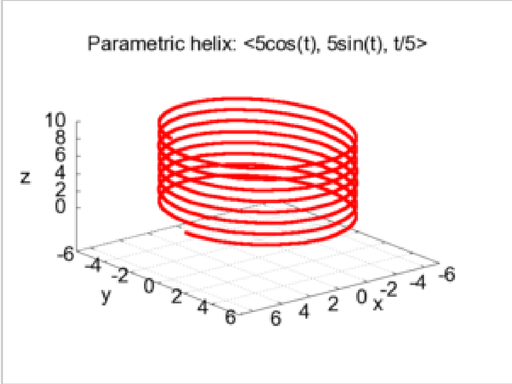


## Helix [ edit ]

Parametric equations are convenient for describing curves in higher-dimensional spaces. For example:

$$x = a\cos(t)$$
$$y = a\sin(t)$$
$$z = bt$$

describes a three-dimensional curve, the helix, with a radius of *a* and rising by $2\pi b$ units per turn. Note that the equations are identical in the plane to those for a circle. Such expressions as the one above are commonly written as



Parametric helix

$$\mathbf{r}(t) = (x(t), y(t), z(t)) = (a\cos(t), a\sin(t), bt),$$

where **r** is a three-dimensional vector.

https://en.wikipedia.org/wiki/Parametric_equation

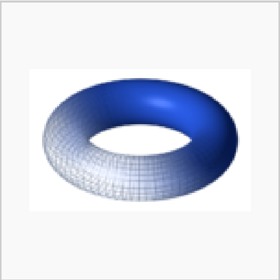
## Parametric surfaces [ edit ]

*Main article: Parametric surface*

A torus with major radius *R* and minor radius *r* may be defined parametrically as

$$x = \cos[t]\,[R + r\cos(u)]\,,$$
$$y = \sin[t]\,[R + r\cos(u)]\,,$$
$$z = r\sin[u].$$

where the two parameters t and u both vary between 0 and 2π.



R=2, r=1/2

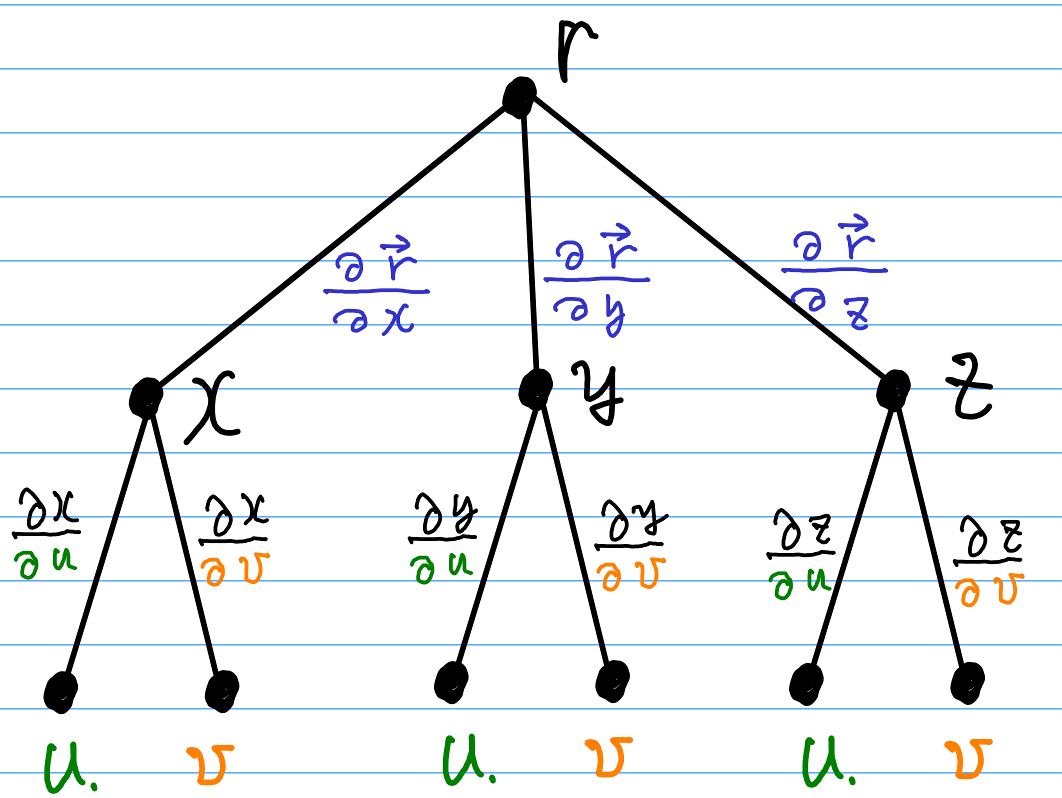
As u varies from 0 to 2π the point on the surface moves about a short circle passing through the hole in the torus. As t varies from 0 to 2π the point on the surface moves about a long circle around the hole in the torus.

https://en.wikipedia.org/wiki/Parametric_equation

parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$\boxed{\vec{r}_u = \frac{\partial \vec{r}}{\partial u}} \qquad \boxed{\vec{r}_v = \frac{\partial \vec{r}}{\partial v}}$$




$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial \vec{r}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \vec{r}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \vec{r}}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial \vec{r}}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \vec{r}}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \vec{r}}{\partial z}\frac{\partial z}{\partial v}$$

parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial \vec{r}}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial \vec{r}}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial \vec{r}}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial \vec{r}}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial \vec{r}}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial \vec{r}}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial \vec{r}}{\partial x} = \frac{\partial}{\partial x}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \vec{i}$$

$$\frac{\partial \vec{r}}{\partial y} = \frac{\partial}{\partial y}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \vec{j}$$

$$\frac{\partial \vec{r}}{\partial z} = \frac{\partial}{\partial z}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \vec{k}$$

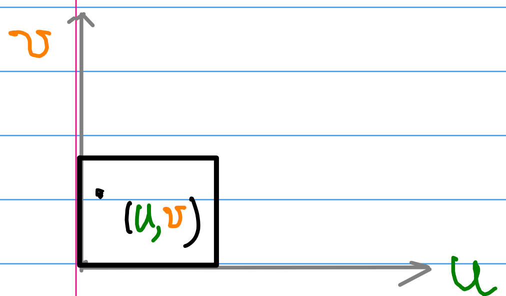
$$\boxed{\vec{r}_u = \frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k}}$$

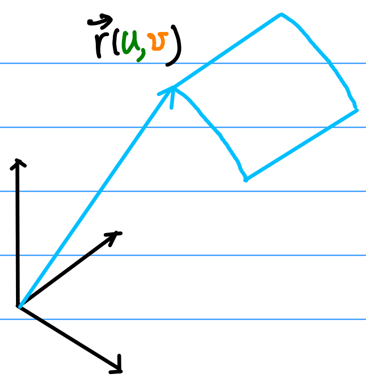
$$\boxed{\vec{r}_v = \frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}}$$

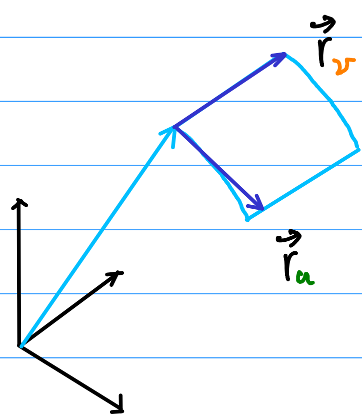
parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

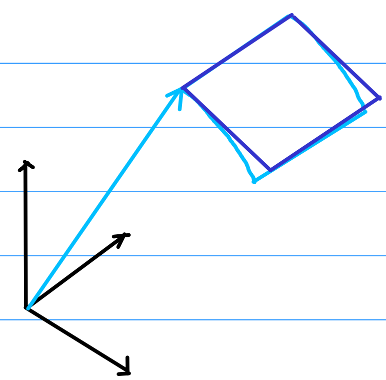
$$\boxed{\vec{r}_u = \frac{\partial \vec{r}}{\partial u}} \qquad \boxed{\vec{r}_v = \frac{\partial \vec{r}}{\partial v}}$$




tangent vectors



tangent plane

area: $\|\vec{r}_u \times \vec{r}_v\|$

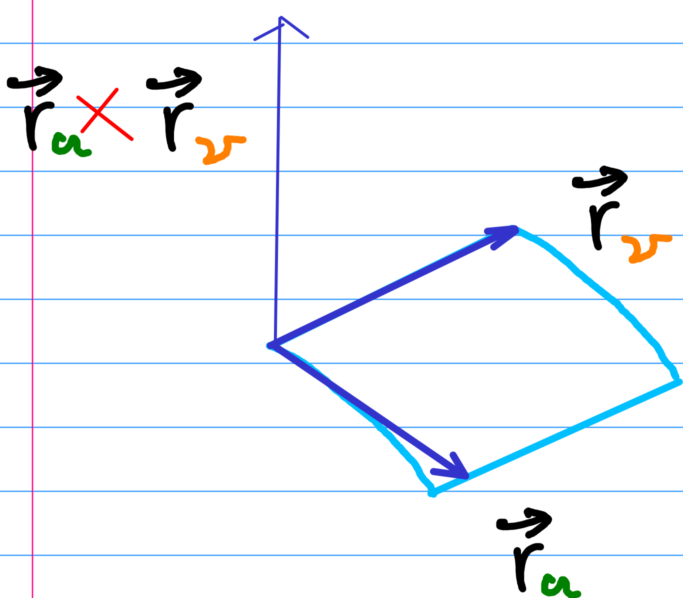
# Regular Parametrization

$\{\vec{r}_u, \vec{r}_v\}$ linearly independent

any tangent vector can be uniquely decomposed into a linear combination of $\vec{r}_u$ & $\vec{r}_v$

a normal vector $\vec{r}_u \times \vec{r}_v$

a unit normal vector $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$



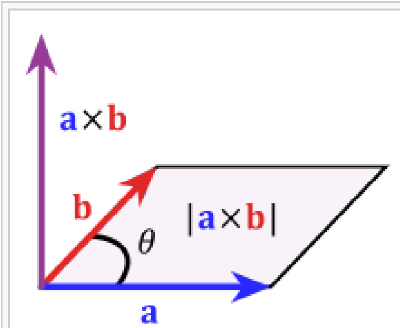



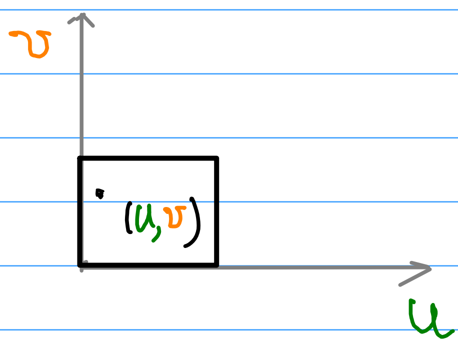

Figure 1. The area of a parallelogram as the magnitude of a cross product

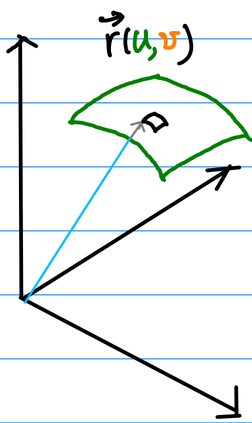
parametric surface

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

Surface Area

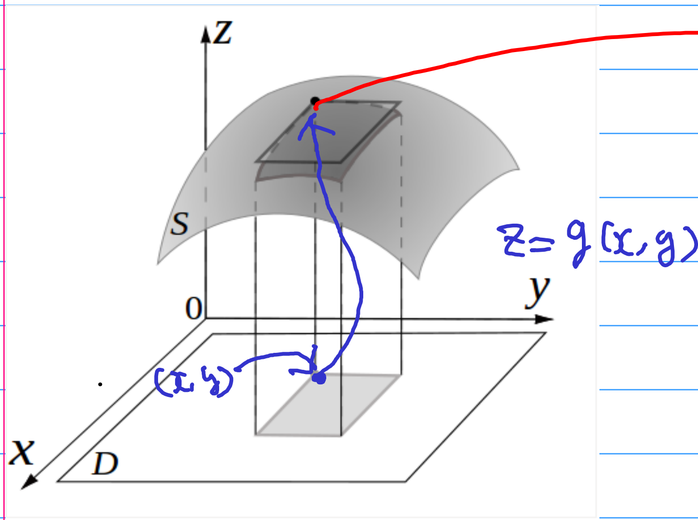
$$A(D) = \iint_D \left\| \vec{r}_u \times \vec{r}_v \right\| \, du\, dv$$



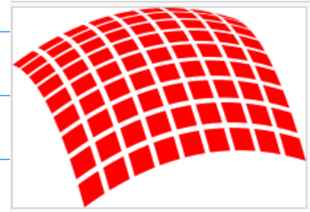




$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k}$$

# Surface Integrals




$f(x, y, g(x, y))$



The definition of surface integral relies on splitting the surface into small surface elements.

$dS$

https://en.wikipedia.org/wiki/Surface_integral

$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\; dA$$

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(u,v)) \left\| \vec{r}_u \times \vec{r}_v \right\| du\, dv$$

$$= \iint_D f(\vec{r}(u,v)) \left\| \vec{r}_u \times \vec{r}_v \right\| dA$$

$$\left\| \vec{r}_u \times \vec{r}_v \right\| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$