Contents

Articles

References

Article Licenses

Trigonometry

"Trig" redirects here. For other uses, see Trig (disambiguation).

The Canadarm2 robotic manipulator on the International Space Station is operated by controlling the angles of its joints. Calculating the final position of the astronaut at the end of the arm requires repeated use of trigonometric functions of those angles.

Trigonometry
• Outline • History • Usage • Functions (inverse) • Generalized trigonometry
Reference
• Identities • Exact constants • Tables
Laws and theorems
• Sines • Cosines • Tangents • Cotangents • Pythagorean theorem
Calculus
• Trigonometric substitution • Integrals (inverse functions) • Derivatives
• v • t • e [1]

Trigonometry (from Greek *trigōnon*, "triangle" + *metron*, "measure") is a branch of mathematics that studies relationships involving lengths and angles of triangles. The field emerged during the 3rd century BC from applications of geometry to astronomical studies.[2]

The 3rd-century astronomers first noted that the lengths of the sides of a right-angle triangle and the angles between those sides have fixed relationships: that is, if at least the length of one side and the value of one angle is known, then all other angles and lengths can be determined algorithmically. These calculations soon came to be defined as the trigonometric functions and today are pervasive in both pure and applied mathematics: fundamental methods of analysis such as the Fourier transform, for example, or the wave equation, use trigonometric functions to understand cyclical phenomena across many applications in fields as diverse as physics, mechanical and electrical engineering, music and acoustics, astronomy, ecology, and biology. Trigonometry is also the foundation of the practical art of surveying.

Trigonometry is most simply associated with planar right-angle triangles (each of which is a two-dimensional triangle with one angle equal to 90 degrees). The applicability to non-right-angle triangles exists, but, since any non-right-angle triangle (on a flat plane) can be bisected to create two right-angle triangles, most problems can be reduced to calculations on right-angle triangles. Thus the majority of applications relate to right-angle triangles. One exception to this is spherical trigonometry, the study of triangles on spheres, surfaces of constant positive curvature, in elliptic geometry (a fundamental part of astronomy and navigation). Trigonometry on surfaces of negative curvature is part of hyperbolic geometry.

Trigonometry basics are often taught in schools, either as a separate course or as a part of a precalculus course.

History

Main article: History of trigonometry

Hipparchus, credited with compiling the first trigonometric table, is known as "the father of trigonometry".

Sumerian astronomers studied angle measure, using a division of circles into 360 degrees.[3] They, and later the Babylonians, studied the ratios of the sides of similar triangles and discovered some properties of these ratios, but did not turn that into a systematic method for finding sides and angles of triangles. The ancient Nubians used a similar method. The ancient Greeks transformed trigonometry into an ordered science.[4]

In the 3rd century BCE, classical Greek mathematicians (such as Euclid and Archimedes) studied the properties of chords and inscribed angles in circles, and proved theorems that are equivalent to modern trigonometric formulae, although they presented them geometrically rather than algebraically. Claudius Ptolemy expanded upon Hipparchus' *Chords in a Circle* in his *Almagest*.[5]

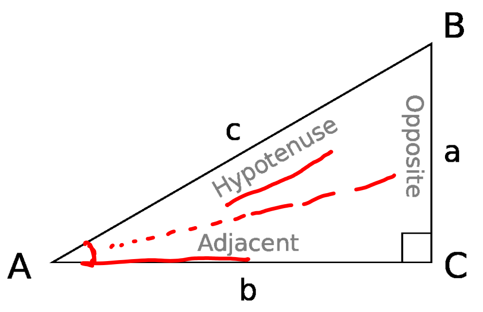
The modern sine function was first defined in the *Surya Siddhanta*, and its properties were further documented by the 5th century (CE) Indian mathematician and astronomer Aryabhata.[6] These Greek and Indian works were translated and expanded by medieval Islamic mathematicians. By the 10th century, Islamic mathematicians were using all six trigonometric functions, had tabulated their values, and were applying them to problems in spherical geometry.Wikipedia:Citation needed At about the same time, Chinese mathematicians developed trigonometry independently, although it was not a major field of study for them. Knowledge of trigonometric functions and methods reached Europe via Latin translations of the works of Persian and Arabic astronomers such as Al Battani and Nasir al-Din al-Tusi.[7] One of the earliest works on trigonometry by a European mathematician is *De Triangulis* by the 15th century German mathematician Regiomontanus. Trigonometry was still so little known in 16th-century Europe that Nicolaus Copernicus devoted two chapters of *De revolutionibus orbium coelestium* to explain its basic concepts.

Driven by the demands of navigation and the growing need for accurate maps of large geographic areas, trigonometry grew into a major branch of mathematics. Bartholomaeus Pitiscus was the first to use the word, publishing his *Trigonometria* in 1595. Gemma Frisius described for the first time the method of triangulation still used today in surveying. It was Leonhard Euler who fully incorporated complex numbers into trigonometry. The works of James Gregory in the 17th century and Colin Maclaurin in the 18th century were influential in the development of trigonometric series.[8] Also in the 18th century, Brook Taylor defined the general Taylor series.[9]

Overview

Main article: Trigonometric function

If one angle of a triangle is 90 degrees and one of the other angles is known, the third is thereby fixed, because the three angles of any triangle add up to 180 degrees. The two acute angles therefore add up to 90 degrees: they are complementary angles. The shape of a triangle is completely determined, except for similarity, by the angles. Once the angles are known, the ratios of the sides are determined, regardless of the overall size of the triangle. If the length of one of the sides is known, the other two are determined. These ratios are given by the following trigonometric functions of the known angle A, where a, b and c refer to the lengths of the sides in the accompanying figure:



In this right triangle: sin A = a/c; cos A = b/c; tan A = a/b.

- **Sine** function (sin), defined as the ratio of the side opposite the angle to the hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{c}.$$

- **Cosine** function (cos), defined as the ratio of the adjacent leg to the hypotenuse.

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{c}.$$

- **Tangent** function (tan), defined as the ratio of the opposite leg to the adjacent leg.

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b} = \frac{\sin A}{\cos A}.$$

The **hypotenuse** is the side opposite to the 90 degree angle in a right triangle; it is the longest side of the triangle, and one of the two sides adjacent to angle A. The **adjacent leg** is the other side that is adjacent to angle A. The **opposite side** is the side that is opposite to angle A. The terms **perpendicular** and **base** are sometimes used for the opposite and adjacent sides respectively. Many people find it easy to remember what sides of the right triangle are equal to sine, cosine, or tangent, by memorizing the word SOH-CAH-TOA (see below under Mnemonics).

The reciprocals of these functions are named the **cosecant** (csc or cosec), **secant** (sec), and **cotangent** (cot), respectively:

$$\csc A = \frac{1}{\sin A} = \frac{c}{a},$$

$$\sec A = \frac{1}{\cos A} = \frac{c}{b},$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A} = \frac{b}{a}.$$

The inverse functions are called the **arcsine**, **arccosine**, and **arctangent**, respectively. There are arithmetic relations between these functions, which are known as trigonometric identities. The cosine, cotangent, and cosecant are so named because they are respectively the sine, tangent, and secant of the complementary angle abbreviated to "co-".

With these functions one can answer virtually all questions about arbitrary triangles by using the law of sines and the law of cosines. These laws can be used to compute the remaining angles and sides of any triangle as soon as two sides and their included angle or two angles and a side or three sides are known. These laws are useful in all branches of geometry, since every polygon may be described as a finite combination of triangles.

Extending the definitions

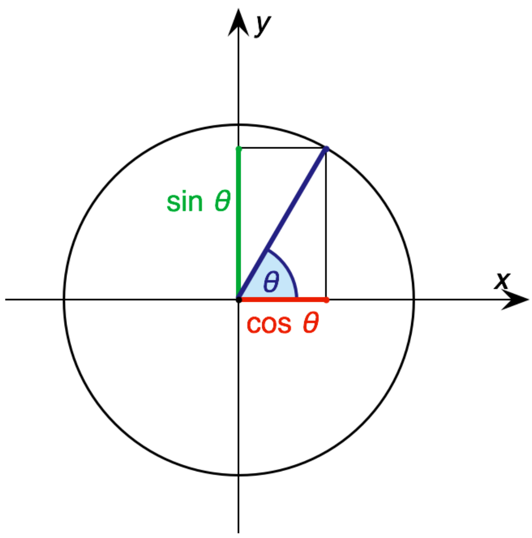


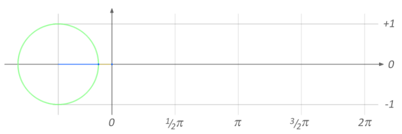
Fig. 1a – Sine and cosine of an angle θ defined using the unit circle.

The above definitions apply to angles between 0 and 90 degrees (0 and π/2 radians) only. Using the unit circle, one can extend them to all positive and negative arguments (see trigonometric function). The trigonometric functions are periodic, with a period of 360 degrees or 2π radians. That means their values repeat at those intervals. The tangent and cotangent functions also have a shorter period, of 180 degrees or π radians.

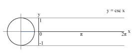
The trigonometric functions can be defined in other ways besides the geometrical definitions above, using tools from calculus and infinite series. With these definitions the trigonometric functions can be defined for complex numbers. The complex exponential function is particularly useful.

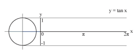
$$e^{x+iy} = e^x(\cos y + i \sin y).$$

See Euler's and De Moivre's formulas.



Graphing process of $y = \sin(x)$ using a unit circle.


Graphing process of $y = \csc(x)$, the reciprocal of sine, using a unit circle.


Graphing process of $y = \tan(x)$ using a unit circle.

Mnemonics

Main article: Mnemonics in trigonometry

A common use of mnemonics is to remember facts and relationships in trigonometry. For example, the *sine*, *cosine*, and *tangent* ratios in a right triangle can be remembered by representing them and their corresponding sides as strings of letters. For instance, a mnemonic is SOH-CAH-TOA:

Sine = **O**pposite ÷ **H**ypotenuse

Cosine = **A**djacent ÷ **H**ypotenuse

Tangent = **O**pposite ÷ **A**djacent

One way to remember the letters is to sound them out phonetically (i.e., *SOH-CAH-TOA*, which is pronounced 'so-kə-**toe**-uh' /soʊkəˈtoʊə/). Another method is to expand the letters into a sentence, such as "**S**ome **O**ld **H**ippy **C**aught **A**nother **H**ippy **T**rippin' **O**n **A**cid".[10]

Calculating trigonometric functions

Main article: Generating trigonometric tables

Trigonometric functions were among the earliest uses for mathematical tables. Such tables were incorporated into mathematics textbooks and students were taught to look up values and how to interpolate between the values listed to get higher accuracy. Slide rules had special scales for trigonometric functions.

Today scientific calculators have buttons for calculating the main trigonometric functions (sin, cos, tan, and sometimes cis and their inverses). Most allow a choice of angle measurement methods: degrees, radians, and sometimes grad. Most computer programming languages provide function libraries that include the trigonometric functions. The floating point unit hardware incorporated into the microprocessor chips used in most personal computers has built-in instructions for calculating trigonometric functions.

Applications of trigonometry

Main article: Uses of trigonometry

There is an enormous number of uses of trigonometry and trigonometric functions. For instance, the technique of triangulation is used in astronomy to measure the distance to nearby stars, in geography to measure distances between landmarks, and in satellite navigation systems. The sine and cosine functions are fundamental to the theory of periodic functions such as those that describe sound and light waves.

Sextants are used to measure the angle of the sun or stars with respect to the horizon. Using trigonometry and a marine chronometer, the position of the ship can be determined from such measurements.

Fields that use trigonometry or trigonometric functions include astronomy (especially for locating apparent positions of celestial objects, in which spherical trigonometry is essential) and hence navigation (on the oceans, in aircraft, and in space), music theory, audio synthesis, acoustics, optics, analysis of financial markets, electronics, probability theory, statistics, biology, medical imaging (CAT scans and ultrasound), pharmacy, chemistry, number theory (and hence cryptology), seismology, meteorology, oceanography, many physical sciences, land surveying and geodesy, architecture, image compression, phonetics, economics, electrical engineering, mechanical engineering, civil engineering, computer graphics, cartography, crystallography and game development.

Pythagorean identities

Identities are those equations that hold true for any value.

$$\sin^2 A + \cos^2 A = 1$$

(Note that the following two can be derived from the first)

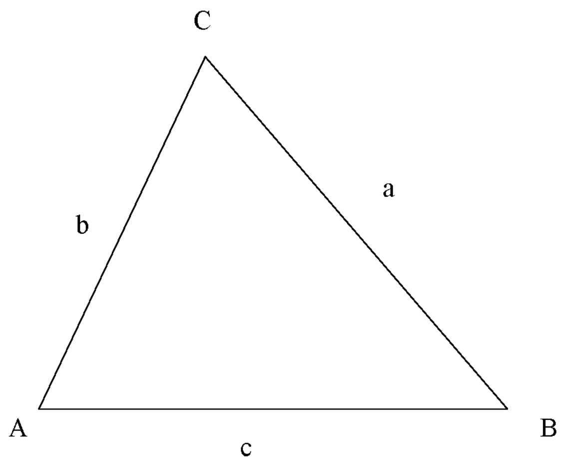
$$\sec^2 A - \tan^2 A = 1$$
$$\csc^2 A - \cot^2 A = 1$$

Angle transformation formulas

$$\sin(A \pm B) = \sin A \ \cos B \pm \cos A \ \sin B$$
$$\cos(A \pm B) = \cos A \ \cos B \mp \sin A \ \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \ \tan B}$$
$$\cot(A \pm B) = \frac{\cot A \ \cot B \mp 1}{\cot B \pm \cot A}$$

Common formulas

Certain equations involving trigonometric functions are true for all angles and are known as *trigonometric identities*. Some identities equate an expression to a different expression involving the same angles. These are listed in List of trigonometric identities. Triangle identities that relate the sides and angles of a given triangle are listed below.



Triangle with sides *a*,*b*,*c* and respectively opposite angles *A*,*B*,*C*

In the following identities, *A*, *B* and *C* are the angles of a triangle and *a*, *b* and *c* are the lengths of sides of the triangle opposite the respective angles (as shown in the diagram).

Law of sines

The **law of sines** (also known as the "sine rule") for an arbitrary triangle states:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

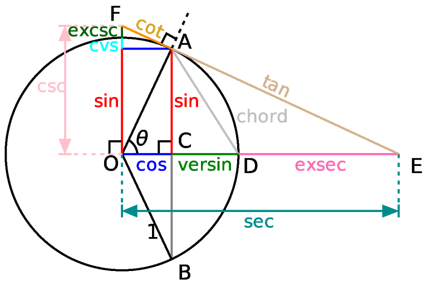
where *R* is the radius of the circumscribed circle of the triangle:

$$R = \frac{abc}{\sqrt{(a+b+c)(a-b+c)(a+b-c)(b+c-a)}}.$$

Another law involving sines can be used to calculate the area of a triangle. Given two sides *a* and *b* and the angle between the sides *C*, the area of the triangle is given by half the product of the lengths of two sides and the sine of the angle between the two sides:

$$\text{Area} = \frac{1}{2}ab\sin C.$$

Law of cosines



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at O.

The **law of cosines** (known as the cosine formula, or the "cos rule") is an extension of the Pythagorean theorem to arbitrary triangles:

$$c^2 = a^2 + b^2 - 2ab\cos C,$$

or equivalently:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The law of cosines may be used to prove Heron's Area Formula, which is another method that may be used to calculate the area of a triangle. This formula states that if a triangle has sides of lengths a, b, and c, and if the semiperimeter is

$$s = \frac{1}{2}(a+b+c),$$

then the area of the triangle is:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Law of tangents

The **law of tangents**:

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

Euler's formula

Euler's formula, which states that $e^{ix} = \cos x + i\sin x$, produces the following analytical identities for sine, cosine, and tangent in terms of e and the imaginary unit i:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \tan x = \frac{i(e^{-ix} - e^{ix})}{e^{ix} + e^{-ix}}.$$

References

[1] http://en.wikipedia.org/w/index.php?title=Template:Trigonometry&action=edit
[2] R. Nagel (ed.), *Encyclopedia of Science*, 2nd Ed., The Gale Group (2002)
[3] Aaboe, Asger. Episodes from the Early History of Astronomy. New York: Springer, 2001. ISBN 0-387-95136-9
[4] " The Beginnings of Trigonometry (http://www.math.rutgers.edu/~cherlin/History/Papers2000/hunt.html)". Rutgers, The State University of New Jersey.
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[6] Boyer p. 215
[7] Boyer pp. 237, 274
[8] William Bragg Ewald (2008). *From Kant to Hilbert: a source book in the foundations of mathematics* (http://books.google.com/books?id=AcuF0w-Qg08C&pg=PA93). Oxford University Press US. p. 93. ISBN 0-19-850535-3
[9] Kelly Dempski (2002). *Focus on Curves and Surfaces* (http://books.google.com/books?id=zxdigX-KSZYC&pg=PA29). p. 29. ISBN 1-59200-007-X
[10] A sentence more appropriate for high schools is "Some old horse came a'hopping through our alley".

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External links

Library resources about **Trigonometry**
• Resources in your library (http://tools.wmflabs.org/ftl/cgi-bin/ftl?st=wp&su=Trigonometry)

- Khan Academy: Trigonometry, free online micro lectures (http://www.khanacademy.org/math/trigonometry)
- Trigonometry (http://baqaqi.chi.il.us/buecher/mathematics/trigonometry/index.html) by Alfred Monroe Kenyon and Louis Ingold, The Macmillan Company, 1914. In images, full text presented.
- Benjamin Banneker's Trigonometry Puzzle (http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=212&bodyId=81) at Convergence (http://mathdl.maa.org/convergence/1/)
- Dave's Short Course in Trigonometry (http://www.clarku.edu/~djoyce/trig/) by David Joyce of Clark University
- Trigonometry, by Michael Corral, Covers elementary trigonometry, Distributed under GNU Free Documentation License (http://www.mecmath.net/trig/trigbook.pdf)

Proofs of trigonometric identities

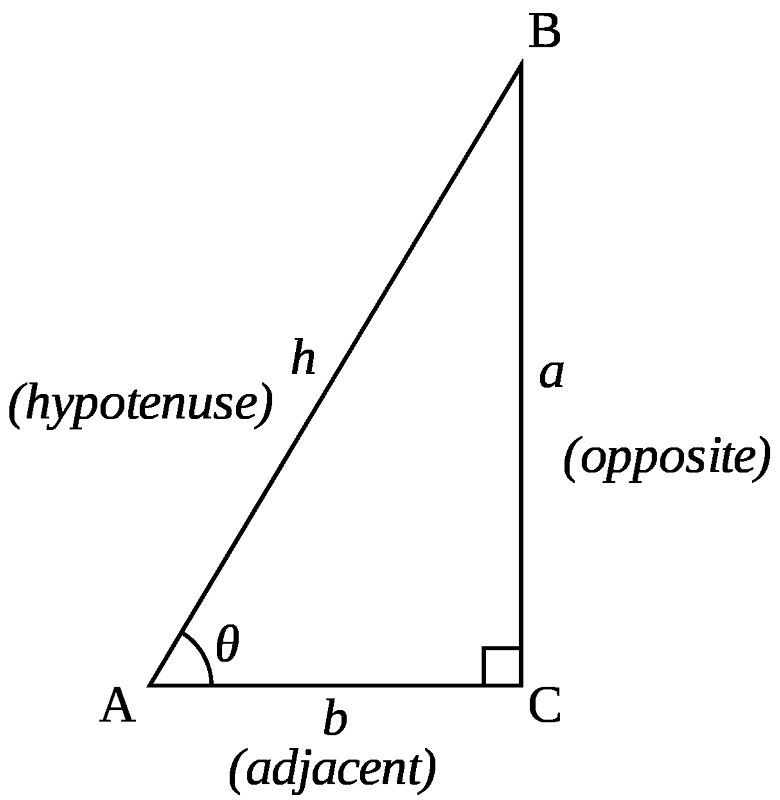
Proofs of trigonometric identities are used to show relations between trigonometric functions. This article will list trigonometric identities and prove them.

Elementary trigonometric identities

Definitions

Referring to the diagram at the right, the six trigonometric functions of θ are:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{h}$$
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{h}$$
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{a}{b}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{h}{b}$$
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{h}{a}$$



Trigonometric functions specify the relationships between side lengths and interior angles of a right triangle. For example, the sine of angle θ is defined as being the length of the opposite side divided by the length of the hypotenuse.

Ratio identities

The following identities are trivial algebraic consequences of these definitions and the division identity. They rely on multiplying or dividing the numerator and denominator of fractions by a variable. Ie,

$$\frac{a}{b} = \frac{\left(\frac{a}{c}\right)}{\left(\frac{b}{c}\right)}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)}{\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)} = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{\left(\frac{\text{adjacent}}{\text{adjacent}}\right)}{\left(\frac{\text{opposite}}{\text{adjacent}}\right)} = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\left(\frac{\text{opposite}\times\text{hypotenuse}}{\text{opposite}\times\text{adjacent}}\right)}{\left(\frac{\text{adjacent}\times\text{hypotenuse}}{\text{opposite}\times\text{adjacent}}\right)} = \frac{\left(\frac{\text{hypotenuse}}{\text{adjacent}}\right)}{\left(\frac{\text{hypotenuse}}{\text{opposite}}\right)} = \frac{\sec\theta}{\csc\theta}$$

Or

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{1}{\csc\theta}\right)}{\left(\frac{1}{\sec\theta}\right)} = \frac{\left(\frac{\csc\theta\sec\theta}{\csc\theta}\right)}{\left(\frac{\csc\theta\sec\theta}{\sec\theta}\right)} = \frac{\sec\theta}{\csc\theta}$$

$$\cot\theta = \frac{\csc\theta}{\sec\theta}$$

Complementary angle identities

Two angles whose sum is π/2 radians (90 degrees) are *complementary*. In the diagram, the angles at vertices A and B are complementary, so we can exchange a and b, and change θ to π/2 – θ, obtaining:

$$\sin(\pi/2-\theta) = \cos\theta$$
$$\cos(\pi/2-\theta) = \sin\theta$$
$$\tan(\pi/2-\theta) = \cot\theta$$
$$\cot(\pi/2-\theta) = \tan\theta$$
$$\sec(\pi/2-\theta) = \csc\theta$$
$$\csc(\pi/2-\theta) = \sec\theta$$

Pythagorean identities

Identity 1:

$$\sin^2(x) + \cos^2(x) = 1$$

Proof 1:

Refer to the triangle diagram above. Note that $a^2 + b^2 = h^2$ by Pythagorean theorem.

$$\sin^2(x) + \cos^2(x) = \frac{a^2}{h^2} + \frac{b^2}{h^2} = \frac{a^2+b^2}{h^2} = \frac{h^2}{h^2} = 1.$$

The following two results follow from this and the ratio identities. To obtain the first, divide both sides of $\sin^2(x) + \cos^2(x) = 1$ by $\cos^2(x)$; for the second, divide by $\sin^2(x)$.

$$\tan^2(x) + 1 \ = \sec^2(x)$$
$$\sec^2(x) - \tan^2(x) = 1$$

Similarly

$$1 \ + \cot^2(x) = \csc^2(x)$$
$$\csc^2(x) - \cot^2(x) = 1$$

Proof 2:

Differentiating the left-hand side of the identity yields:

$$2\sin x \cdot \cos x - 2\sin x \cdot \cos x = 0$$

Integrating this shows that the original identity is equal to a constant, and this constant can be found by plugging in any arbitrary value of x.

Identity 2:

The following accounts for all three reciprocal functions.

$$\csc^2(x) + \sec^2(x) - \cot^2(x) = 2\ + \tan^2(x)$$

Proof 1:

Refer to the triangle diagram above. Note that $a^2 + b^2 = h^2$ by Pythagorean theorem.

$$\csc^2(x) + \sec^2(x) = \frac{h^2}{a^2} + \frac{h^2}{b^2} = \frac{a^2+b^2}{a^2} + \frac{a^2+b^2}{b^2} = 2\ + \frac{b^2}{a^2} + \frac{a^2}{b^2}$$

Substituting with appropriate functions -

$$2\ + \frac{b^2}{a^2} + \frac{a^2}{b^2} = 2\ + \tan^2(x) + \cot^2(x)$$

Rearranging gives:

$$\csc^2(x) + \sec^2(x) - \cot^2(x) = 2\ + \tan^2(x)$$

Angle sum identities

Sine

Draw a horizontal line (the *x*-axis); mark an origin O. Draw a line from O at an angle α above the horizontal line and a second line at an angle β above that; the angle between the second line and the *x*-axis is $\alpha + \beta$.

Place P on the line defined by $\alpha + \beta$ at a unit distance from the origin.

Let PQ be a line perpendicular to line defined by angle α, drawn from point Q on this line to point P. $\therefore$OQP is a right angle.

Let QA be a perpendicular from point A on the *x*-axis to Q and PB be a perpendicular from point B on the *x*-axis to P. $\therefore$OAQ and OBP are right angles.

Draw QR parallel to the *x*-axis.

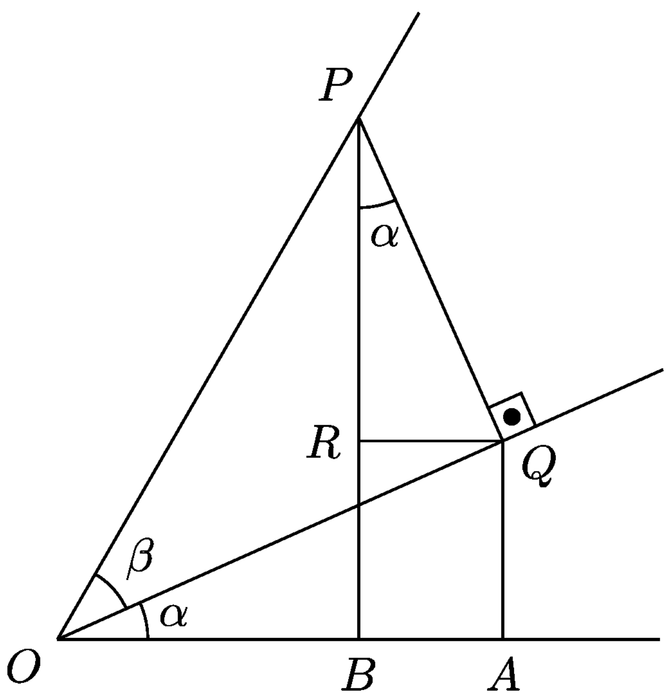


Illustration of the sum formula.

Now angle $RPQ = \alpha$ (because $OQA = 90 - \alpha$, making $RQO = \alpha, RQP = 90 - \alpha$, and finally $RPQ = \alpha$)

$$RPQ = \tfrac{\pi}{2} - RQP = \tfrac{\pi}{2} - (\tfrac{\pi}{2} - RQO) = RQO = \alpha$$

$$OP = 1$$

$$PQ = \sin\beta$$

$$OQ = \cos\beta$$

$$\frac{AQ}{OQ} = \sin\alpha \text{, so } AQ = \sin\alpha\cos\beta$$

$$\frac{PR}{PQ} = \cos\alpha \text{, so } PR = \cos\alpha \sin\beta$$

$$\sin(\alpha + \beta) = PB = RB + PR = AQ + PR = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

By substituting $-\beta$ for β and using Symmetry, we also get:

$$\sin(\alpha - \beta) = \sin\alpha\cos -\beta + \cos\alpha\sin -\beta$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Another simple "proof" can be given using Euler's formula known from complex analysis: Euler's formula is:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Although it is more precise to say that Euler's formula entails the trigonometric identities, it follows that for angles α and β we have:

$$e^{i(\alpha+\beta)} = \cos(\alpha + \beta) + i\sin(\alpha + \beta)$$

Also using the following properties of exponential functions:

$$e^{i(\alpha+\beta)} = e^{i\alpha}e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$

Evaluating the product:

$$e^{i(\alpha+\beta)} = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha)$$

Equating real and imaginary parts:

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

Cosine

Using the figure above,

$$OP = 1$$

$$PQ = \sin\beta$$

$$OQ = \cos\beta$$

$$\frac{OA}{OQ} = \cos\alpha \text{, so } OA = \cos\alpha\cos\beta$$

$$\frac{RQ}{PQ} = \sin\alpha \text{, so } RQ = \sin\alpha\sin\beta$$

$$\cos(\alpha + \beta) = OB = OA - BA = OA - RQ = \cos\alpha\cos\beta\ - \sin\alpha\sin\beta$$

By substituting $-\beta$ for β and using Symmetry, we also get:

$$\cos(\alpha - \beta) = \cos\alpha\cos -\beta\ - \sin\alpha\sin -\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Also, using the complementary angle formulae,

$$\begin{aligned}\cos(\alpha + \beta) &= \sin\left(\pi/2 - (\alpha + \beta)\right)\\ &= \sin\left((\pi/2 - \alpha) - \beta\right)\\ &= \sin\left(\pi/2 - \alpha\right)\cos\beta - \cos\left(\pi/2 - \alpha\right)\sin\beta\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

Tangent and cotangent

From the sine and cosine formulae, we get

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

Dividing both numerator and denominator by $\cos\alpha\cos\beta$, we get

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

Subtracting β from α , using $\tan(-\beta) = -\tan\beta$,

$$\tan(\alpha-\beta) = \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)} = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Similarly from the sine and cosine formulae, we get

$$\cot(\alpha+\beta) = \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta}$$

Then by dividing both numerator and denominator by $\sin\alpha\sin\beta$, we get

$$\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

Or, using $\cot\theta = \frac{1}{\tan\theta}$,

$$\cot(\alpha+\beta) = \frac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta} = \frac{\frac{1}{\tan\alpha\tan\beta} - 1}{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}} = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$$

Using $\cot(-\beta) = -\cot\beta$,

$$\cot(\alpha-\beta) = \frac{\cot\alpha\cot(-\beta) - 1}{\cot\alpha + \cot(-\beta)} = \frac{\cot\alpha\cot\beta + 1}{\cot\beta - \cot\alpha}$$

Double-angle identities

From the angle sum identities, we get

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

and

θ+θ

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

The Pythagorean identities give the two alternative forms for the latter of these:

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

The angle sum identities also give

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2}{\cot\theta - \tan\theta}$$

$$\cot(2\theta) = \frac{\cot^2\theta - 1}{2\cot\theta} = \frac{\cot\theta - \tan\theta}{2}$$

It can also be proved using Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

Squaring both sides yields

$$e^{i2\varphi} = (\cos\varphi + i\sin\varphi)^2$$

But replacing the angle with its doubled version, which achieves the same result in the left side of the equation, yields

$$e^{i2\varphi} = \cos 2\varphi + i \sin 2\varphi$$

It follows that

$$(\cos\varphi + i\sin\varphi)^2 = \cos 2\varphi + i\sin 2\varphi\,.$$

Expanding the square and simplifying on the left hand side of the equation gives

$$i(2\sin\varphi\cos\varphi) + \cos^2\varphi - \sin^2\varphi\ = \cos 2\varphi + i\sin 2\varphi\,.$$

Because the imaginary and real parts have to be the same, we are left with the original identities

$$\cos^2\varphi - \sin^2\varphi\ = \cos 2\varphi\,,$$

and also

$$2\sin\varphi\cos\varphi = \sin 2\varphi\,.$$

Half-angle identities

The two identities giving the alternative forms for cos 2θ lead to the following equations:

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}},$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}.$$

The sign of the square root needs to be chosen properly—note that if π is added to θ, the quantities inside the square roots are unchanged, but the left-hand-sides of the equations change sign. Therefore the correct sign to use depends on the value of θ.

For the tan function, the equation is:

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}.$$

Then multiplying the numerator and denominator inside the square root by (1 + cos θ) and using Pythagorean identities leads to:

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}.$$

Also, if the numerator and denominator are both multiplied by (1 - cos θ), the result is:

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}.$$

This also gives:

$$\tan\frac{\theta}{2} = \csc\theta - \cot\theta.$$

Similar manipulations for the cot function give:

$$\cot\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta} = \csc\theta + \cot\theta.$$

Miscellaneous -- the triple tangent identity

If ψ , θ and ϕ are the angles of a triangle, i.e. $\psi+\theta+\phi=\pi=$half circle,

$$\tan(\psi)+\tan(\theta)+\tan(\phi)=\tan(\psi)\tan(\theta)\tan(\phi).$$

Proof:[1]

$$\begin{aligned}
\psi &= \pi-\theta-\phi \\
\tan(\psi) &= \tan(\pi-\theta-\phi) \\
&= -\tan(\theta+\phi) \\
&= \frac{-\tan\theta-\tan\phi}{1-\tan\theta\tan\phi} \\
&= \frac{\tan\theta+\tan\phi}{\tan\theta\tan\phi-1} \\
(\tan\theta\tan\phi-1)\tan\psi &= \tan\theta+\tan\phi \\
\tan\psi\tan\theta\tan\phi-\tan\psi &= \tan\theta+\tan\phi \\
\tan\psi\tan\theta\tan\phi &= \tan\psi+\tan\theta+\tan\phi
\end{aligned}$$

Miscellaneous -- the triple cotangent identity

If $\psi+\theta+\phi=\tfrac{\pi}{2}=$quarter circle,

$$\cot(\psi)+\cot(\theta)+\cot(\phi)=\cot(\psi)\cot(\theta)\cot(\phi).$$

Proof:

Replace each of ψ , θ , and ϕ with their complementary angles, so cotangents turn into tangents and vice-versa.

Given

$$\psi+\theta+\phi=\tfrac{\pi}{2}$$
$$\therefore (\tfrac{\pi}{2}-\psi)+(\tfrac{\pi}{2}-\theta)+(\tfrac{\pi}{2}-\phi)=\tfrac{3\pi}{2}-(\psi+\theta+\phi)=\tfrac{3\pi}{2}-\tfrac{\pi}{2}=\pi$$

so the result follows from the triple tangent identity.

Prosthaphaeresis identities

- $\sin\theta\pm\sin\phi=2\sin\left(\frac{\theta\pm\phi}{2}\right)\cos\left(\frac{\theta\mp\phi}{2}\right)$
- $\cos\theta+\cos\phi=2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$
- $\cos\theta-\cos\phi=-2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$

Proof of sine identities

First, start with the sum-angle identities:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$
$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

By adding these together,

$$\sin(\alpha+\beta)+\sin(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta=2\sin\alpha\cos\beta$$

Similarly, by subtracting the two sum-angle identities,

$$\sin(\alpha+\beta)-\sin(\alpha-\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta=2\cos\alpha\sin\beta$$

Let $\alpha+\beta=\theta$ and $\alpha-\beta=\phi$,

$$\therefore \alpha = \frac{\theta+\phi}{2} \text{ and } \beta = \frac{\theta-\phi}{2}$$

Substitute θ and ϕ

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right) = 2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$$

Therefore,

$$\sin\theta \pm \sin\phi = 2\sin\left(\frac{\theta\pm\phi}{2}\right)\cos\left(\frac{\theta\mp\phi}{2}\right)$$

Proof of cosine identities

Similarly for cosine, start with the sum-angle identities:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta\ - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Again, by adding and substracting

$$\cos(\alpha+\beta)+\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 2\cos\alpha\cos\beta$$
$$\cos(\alpha+\beta)-\cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta = -2\sin\alpha\sin\beta$$

Substitute θ and ϕ as before,

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right)$$
$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right)$$

Inequalities

The figure at the right shows a sector of a circle with radius 1. The sector is θ/(2π) of the whole circle, so its area is θ/2.

$$OA = OD = 1$$
$$AB = \sin\theta$$
$$CD = \tan\theta$$

The area of triangle OAD is AB/2, or sinθ/2. The area of triangle OCD is CD/2, or tanθ/2.

Since triangle OAD lies completely inside the sector, which in turn lies completely inside triangle OCD, we have

$$\sin\theta < \theta < \tan\theta$$

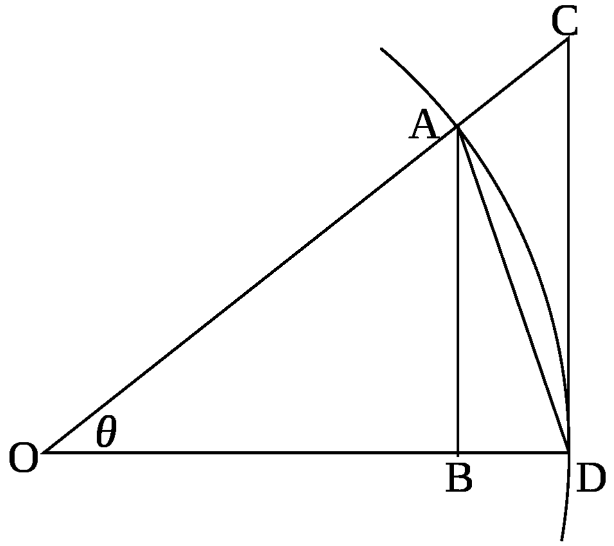


Illustration of the sine and tangent inequalities.

This geometric argument applies if 0<θ<π/2. It relies on definitions of arc length and area, which act as assumptions, so it is rather a condition imposed in construction of trigonometric functions than a provable property. For the sine function, we can handle other values. If θ>π/2, then θ>1. But sinθ≤1 (because of the Pythagorean identity), so sinθ<θ. So we have

$$\frac{\sin\theta}{\theta} < 1 \quad \text{if} \quad 0 < \theta$$

For negative values of θ we have, by symmetry of the sine function

$$\frac{\sin\theta}{\theta} = \frac{\sin(-\theta)}{-\theta} < 1$$

Hence

$$\frac{\sin\theta}{\theta} < 1 \quad \text{if} \quad \theta \neq 0$$

$$\frac{\tan\theta}{\theta} > 1 \quad \text{if} \quad 0 < \theta < \frac{\pi}{2}$$

Identities involving calculus

Preliminaries

$$\lim_{\theta\to 0} \sin\theta = 0$$

$$\lim_{\theta\to 0} \cos\theta = 1$$

Sine and angle ratio identity

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Proof: From the previous inequalities, we have, for small angles

$$\sin\theta < \theta < \tan\theta,$$

Therefore,

$$\frac{\sin\theta}{\theta} < 1 < \frac{\tan\theta}{\theta},$$

Consider the right-hand inequality. Since

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\therefore 1 < \frac{\sin\theta}{\theta\cos\theta}$$

Multply through by $\cos\theta$

$$\cos\theta < \frac{\sin\theta}{\theta}$$

Combining with the left-hand inequality:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Taking $\cos\theta$ to the limit as $\theta \to 0$

$$\lim_{\theta\to 0} \cos\theta = 1$$

Therefore,

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Cosine and angle ratio identity

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = 0$$

Proof:

$$\begin{aligned} \frac{1 - \cos\theta}{\theta} &= \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)} \\ &= \frac{\sin^2\theta}{\theta(1 + \cos\theta)} \\ &= \left(\frac{\sin\theta}{\theta}\right) \times \sin\theta \times \left(\frac{1}{1 + \cos\theta}\right) \end{aligned}$$

The limits of those three quantities are 1, 0, and 1/2, so the resultant limit is zero.

Cosine and square of angle ratio identity

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta^2} = \frac{1}{2}$$

Proof:

As in the preceding proof,

$$\frac{1 - \cos\theta}{\theta^2} = \frac{\sin\theta}{\theta} \times \frac{\sin\theta}{\theta} \times \frac{1}{1 + \cos\theta}.$$

The limits of those three quantities are 1, 1, and 1/2, so the resultant limit is 1/2.

Proof of Compositions of trig and inverse trig functions

All these functions follow from the Pythagorean trigonometric identity. We can prove for instance the function

$$\sin[\arctan(x)] = \frac{x}{\sqrt{1 + x^2}}$$

Proof:

We start from

$$\sin^2\theta + \cos^2\theta = 1$$

Then we divide this equation by $\cos^2\theta$

$$\cos^2\theta = \frac{1}{\tan^2\theta + 1}$$

Then use the substitution $\theta = \arctan(x)$, also use the Pythagorean trigonometric identity:

$$1 - \sin^2[\arctan(x)] = \frac{1}{\tan^2[\arctan(x)] + 1}$$

Then we use the identity $\tan[\arctan(x)] \equiv x$

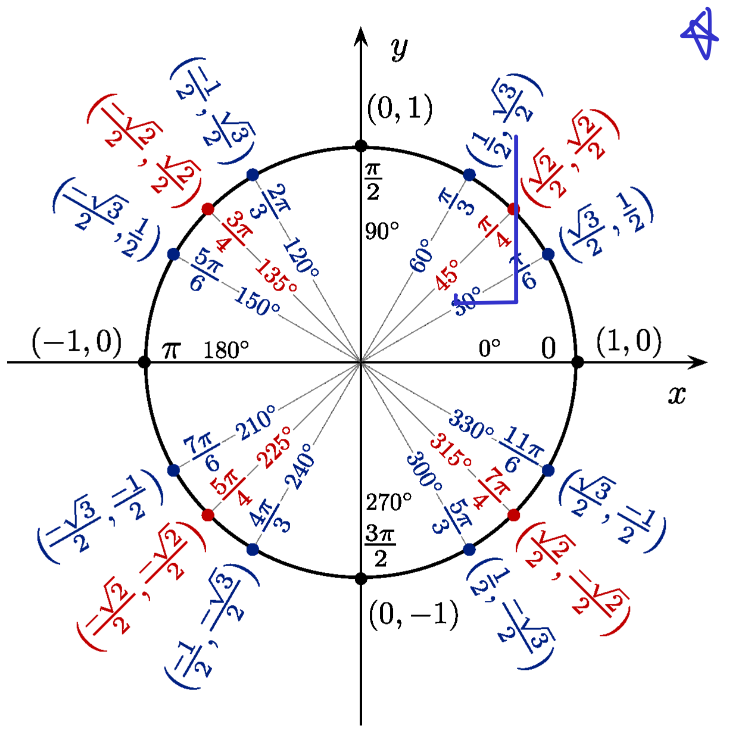
$$\sin[\arctan(x)] = \frac{x}{\sqrt{x^2 + 1}}$$

References

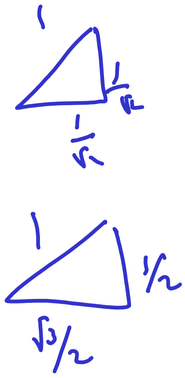
- E. T. Whittaker and G. N. Watson. *A course of modern analysis*, Cambridge University Press, 1952

[1] http://mathlaoshi.com/tags/tangent-identity/

List of trigonometric identities



Cosines and sines around the unit circle



Trigonometry
• Outline • History • Usage • Functions (inverse) • Generalized trigonometry
Reference
• Identities • Exact constants • Tables
Laws and theorems
• Sines • Cosines • Tangents • Cotangents • Pythagorean theorem
Calculus

- Trigonometric substitution
- Integrals (inverse functions)
- Derivatives

- v
- t
- e [1]

In mathematics, **trigonometric identities** are equalities that involve trigonometric functions and are true for every single value of the occurring variables. Geometrically, these are identities involving certain functions of one or more angles. They are distinct from triangle identities, which are identities involving both angles and side lengths of a triangle. Only the former are covered in this article.

These identities are useful whenever expressions involving trigonometric functions need to be simplified. An important application is the integration of non-trigonometric functions: a common technique involves first using the substitution rule with a trigonometric function, and then simplifying the resulting integral with a trigonometric identity.

Notation

Angles

This article uses Greek letters such as alpha (α), beta (β), gamma (γ), and theta (θ) to represent angles. Several different units of angle measure are widely used, including degrees, radians, and grads:

1 full circle = 360 degrees = 2 π radians = 400 grads.

The following table shows the conversions for some common angles:

Degrees	30°	60°	120°	150°	210°	240°	300°	330°
Radians	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$
Grads	~~33⅓ grad~~	~~66⅔ grad~~	133⅓ grad	166⅔ grad	233⅓ grad	266⅔ grad	333⅓ grad	366⅔ grad
Degrees	45°	90°	135°	180°	225°	270°	315°	360°
Radians	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
Grads	50 grad	~~100 grad~~	~~150 grad~~	~~200 grad~~	250 grad	300 grad	350 grad	400 grad

Unless otherwise specified, all angles in this article are assumed to be in radians, but angles ending in a degree symbol (°) are in degrees. Per Niven's theorem multiples of 30° are the only angles that are a rational multiple of one degree and also have a rational sin/cos, which may account for their popularity in examples.[1]

Trigonometric functions

The primary trigonometric functions are the sine and cosine of an angle. These are sometimes abbreviated $\sin(\theta)$ and $\cos(\theta)$, respectively, where θ is the angle, but the parentheses around the angle are often omitted, e.g., $\sin\theta$ and $\cos\theta$.

The sine of an angle is defined in the context of a right triangle, as the ratio of the length of the side that is opposite to the angle divided by the length of the longest side of the triangle (the hypotenuse).

The cosine of an angle is also defined in the context of a right triangle, as the ratio of the length of the side the angle is in divided by the length of the longest side of the triangle (the hypotenuse).

The tangent (tan) of an angle is the ratio of the sine to the cosine:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Finally, the reciprocal functions secant (sec), cosecant (csc), and cotangent (cot) are the reciprocals of the cosine, sine, and tangent:

$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}.$$

These definitions are sometimes referred to as ratio identities.

Inverse functions

Main article: Inverse trigonometric functions

The inverse trigonometric functions are partial inverse functions for the trigonometric functions. For example, the inverse function for the sine, known as the **inverse sine** ($\sin^{-1}$) or **arcsine** (arcsin or asin), satisfies

$$\sin(\arcsin x) = x \quad \text{for} \quad |x| \le 1$$

and

$$\arcsin(\sin x) = x \quad \text{for} \quad |x| \le \pi/2.$$

This article uses the notation below for inverse trigonometric functions:

Function	sin	cos	tan	sec	csc	cot
Inverse	arcsin	arccos	arctan	arcsec	arccsc	arccot

Pythagorean identity

The basic relationship between the sine and the cosine is the Pythagorean trigonometric identity:

$$\cos^2\theta + \sin^2\theta = 1$$

where $\cos^2\theta$ means $(\cos(\theta))^2$ and $\sin^2\theta$ means $(\sin(\theta))^2$.

This can be viewed as a version of the Pythagorean theorem, and follows from the equation $x^2 + y^2 = 1$ for the unit circle. This equation can be solved for either the sine or the cosine:

$$\sin\theta = \pm\sqrt{1-\cos^2\theta} \quad \text{and} \quad \cos\theta = \pm\sqrt{1-\sin^2\theta}.$$

Related identities

Dividing the Pythagorean identity by either $\cos^2\theta$ or $\sin^2\theta$ yields two other identities:

$$1+\tan^2\theta=\sec^2\theta \quad \text{and} \quad 1+\cot^2\theta=\csc^2\theta.$$

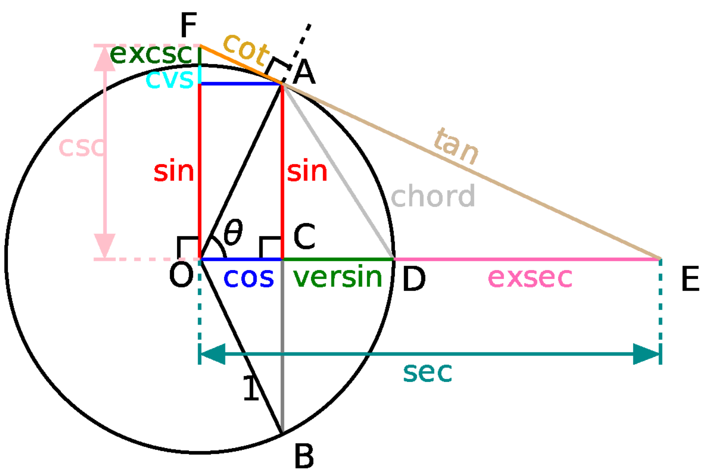
Using these identities together with the ratio identities, it is possible to express any trigonometric function in terms of any other (up to a plus or minus sign):

Each trigonometric function in terms of the other five.[2]

in terms of	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\sin\theta=$	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\csc\theta}$	$\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm\frac{1}{\sqrt{1+\cot^2\theta}}$
$\cos\theta=$	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\frac{1}{\sqrt{1+\tan^2\theta}}$	$\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\sec\theta}$	$\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$
$\tan\theta=$	$\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\frac{1}{\sqrt{\csc^2\theta-1}}$	$\pm\sqrt{\sec^2\theta-1}$	$\frac{1}{\cot\theta}$
$\csc\theta=$	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\csc\theta$	$\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm\sqrt{1+\cot^2\theta}$
$\sec\theta=$	$\pm\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\sec\theta$	$\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$
$\cot\theta=$	$\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\pm\sqrt{\csc^2\theta-1}$	$\pm\frac{1}{\sqrt{\sec^2\theta-1}}$	$\cot\theta$

Historic shorthands

The versine, coversine, haversine, and exsecant were used in navigation. For example the haversine formula was used to calculate the distance between two points on a sphere. They are rarely used today.



All of the trigonometric functions of an angle θ can be constructed geometrically in terms of a unit circle centered at O. Many of these terms are no longer in common use.

Name(s)	Abbreviation(s)	Value[3]
versed sine, versine	$\operatorname{versin}(\theta)$ $\operatorname{vers}(\theta)$ $\operatorname{ver}(\theta)$	$1-\cos(\theta)$
versed cosine, vercosine	$\operatorname{vercosin}(\theta)$	$1+\cos(\theta)$
coversed sine, coversine	$\operatorname{coversin}(\theta)$ $\operatorname{cvs}(\theta)$	$1-\sin(\theta)$
coversed cosine, covercosine	$\operatorname{covercosin}(\theta)$	$1+\sin(\theta)$
half versed sine, haversine	$\operatorname{haversin}(\theta)$	$\frac{1-\cos(\theta)}{2}$
half versed cosine, havercosine	$\operatorname{havercosin}(\theta)$	$\frac{1+\cos(\theta)}{2}$
half coversed sine, hacoversine cohaversine	$\operatorname{hacoversin}(\theta)$	$\frac{1-\sin(\theta)}{2}$
half coversed cosine, hacovercosine cohavercosine	$\operatorname{hacovercosin}(\theta)$	$\frac{1+\sin(\theta)}{2}$
exterior secant, exsecant	$\operatorname{exsec}(\theta)$	$\sec(\theta)-1$
exterior cosecant, excosecant	$\operatorname{excsc}(\theta)$	$\csc(\theta)-1$
chord	$\operatorname{crd}(\theta)$	$2\sin\frac{\theta}{2}$

Ancient Indian mathematicians used Sanskrit terms Jyā, koti-jyā and utkrama-jyā, based on the resemblance of the chord, arc, and radius to the shape of a bow and bowstring drawn back.

Symmetry, shifts, and periodicity

By examining the unit circle, the following properties of the trigonometric functions can be established.

Symmetry

When the trigonometric functions are reflected from certain angles, the result is often one of the other trigonometric functions. This leads to the following identities:

Reflected in $\theta = 0$[4]	Reflected in $\theta = \pi/2$ (co-function identities)[5]	Reflected in $\theta = \pi$
$\sin(-\theta) = -\sin\theta$	$\sin(\tfrac{\pi}{2}-\theta) = +\cos\theta$	$\sin(\pi-\theta) = +\sin\theta$
$\cos(-\theta) = +\cos\theta$	$\cos(\tfrac{\pi}{2}-\theta) = +\sin\theta$	$\cos(\pi-\theta) = -\cos\theta$
$\tan(-\theta) = -\tan\theta$	$\tan(\tfrac{\pi}{2}-\theta) = +\cot\theta$	$\tan(\pi-\theta) = -\tan\theta$
$\csc(-\theta) = -\csc\theta$	$\csc(\tfrac{\pi}{2}-\theta) = +\sec\theta$	$\csc(\pi-\theta) = +\csc\theta$
$\sec(-\theta) = +\sec\theta$	$\sec(\tfrac{\pi}{2}-\theta) = +\csc\theta$	$\sec(\pi-\theta) = -\sec\theta$
$\cot(-\theta) = -\cot\theta$	$\cot(\tfrac{\pi}{2}-\theta) = +\tan\theta$	$\cot(\pi-\theta) = -\cot\theta$

Shifts and periodicity

By shifting the function round by certain angles, it is often possible to find different trigonometric functions that express particular results more simply. Some examples of this are shown by shifting functions round by π/2, π and 2π radians. Because the periods of these functions are either π or 2π, there are cases where the new function is exactly the same as the old function without the shift.

Shift by π/2	**Shift by π** **Period for tan and cot**[6]	**Shift by 2π** **Period for sin, cos, csc and sec**[7]
$\sin(\theta + \frac{\pi}{2}) = +\cos\theta$	$\sin(\theta + \pi) = -\sin\theta$	$\sin(\theta + 2\pi) = +\sin\theta$
$\cos(\theta + \frac{\pi}{2}) = -\sin\theta$	$\cos(\theta + \pi) = -\cos\theta$	$\cos(\theta + 2\pi) = +\cos\theta$
$\tan(\theta + \frac{\pi}{2}) = -\cot\theta$	$\tan(\theta + \pi) = +\tan\theta$	$\tan(\theta + 2\pi) = +\tan\theta$
$\csc(\theta + \frac{\pi}{2}) = +\sec\theta$	$\csc(\theta + \pi) = -\csc\theta$	$\csc(\theta + 2\pi) = +\csc\theta$
$\sec(\theta + \frac{\pi}{2}) = -\csc\theta$	$\sec(\theta + \pi) = -\sec\theta$	$\sec(\theta + 2\pi) = +\sec\theta$
$\cot(\theta + \frac{\pi}{2}) = -\tan\theta$	$\cot(\theta + \pi) = +\cot\theta$	$\cot(\theta + 2\pi) = +\cot\theta$

Angle sum and difference identities

See also: § Product-to-sum and sum-to-product identities

These are also known as the *addition and subtraction theorems* or *formulae*. They were originally established by the 10th century Persian mathematician Abū al-Wafā' Būzjānī. One method of proving these identities is to apply Euler's formula. The use of the symbols $\pm$ and $\mp$ is described in the article plus-minus sign.

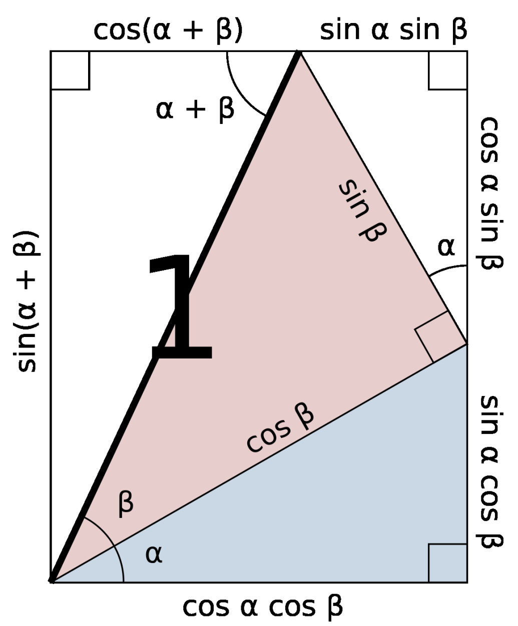


Illustration of angle addition formulae for the sine and cosine. Emphasized segment is of unit length.

For the angle addition diagram for the sine and cosine, the line in bold with the 1 on it is of length 1. It is the hypotenuse of a right angle triangle with angle β which gives the sin β and cos β. The cos β line is the hypotenuse of a right angle triangle with angle α so it has sides sin α and cos α both multiplied by cos β. This is the same for the sin β line. The original line is also the hypotenuse of a right angle triangle with angle α+β, the opposite side is the sin(α+β) line up from the origin and the adjacent side is the cos(α+β) segment going horizontally from the top left.

Overall the diagram can be used to show the sine and cosine of sum identities

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

because the opposite sides of the rectangle are equal.

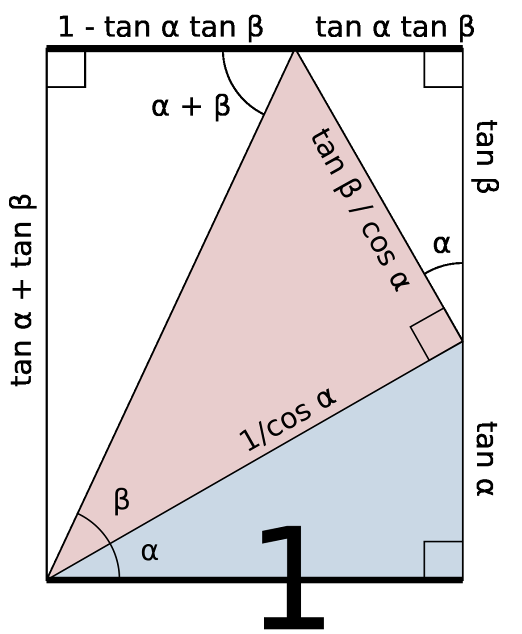


Illustration of the angle addition formula for the tangent. Emphasized segments are of unit length.

Sine	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ [8]
Cosine	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ [9]
Tangent	$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ [10]
Arcsine	$\arcsin\alpha \pm \arcsin\beta = \arcsin\left(\alpha\sqrt{1-\beta^2} \pm \beta\sqrt{1-\alpha^2}\right)$ [11]
Arccosine	$\arccos\alpha \pm \arccos\beta = \arccos\left(\alpha\beta \mp \sqrt{(1-\alpha^2)(1-\beta^2)}\right)$ [12]
Arctangent	$\arctan\alpha \pm \arctan\beta = \arctan\left(\frac{\alpha \pm \beta}{1 \mp \alpha\beta}\right)$ [13]

Matrix form

See also: matrix multiplication

The sum and difference formulae for sine and cosine can be written in matrix form as:

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}.$$

This shows that these matrices form a representation of the rotation group in the plane (technically, the special orthogonal group $SO(2)$), since the composition law is fulfilled: subsequent multiplications of a vector with these two matrices yields the same result as the rotation by the sum of the angles.

Sines and cosines of sums of infinitely many terms

$$\sin\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{odd } k\ge 1} (-1)^{(k-1)/2} \sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}} \left(\prod_{i\in A}\sin\theta_i \prod_{i\notin A}\cos\theta_i\right)$$

$$\cos\left(\sum_{i=1}^{\infty}\theta_i\right) = \sum_{\text{even } k\ge 0} (-1)^{k/2} \sum_{\substack{A\subseteq\{1,2,3,\dots\}\\ |A|=k}} \left(\prod_{i\in A}\sin\theta_i \prod_{i\notin A}\cos\theta_i\right)$$

In these two identities an asymmetry appears that is not seen in the case of sums of finitely many terms: in each product, there are only finitely many sine factors and cofinitely many cosine factors.

If only finitely many of the terms θ_i are nonzero, then only finitely many of the terms on the right side will be nonzero because sine factors will vanish, and in each term, all but finitely many of the cosine factors will be unity.

Tangents of sums

Let e_k (for k = 0, 1, 2, 3, ...) be the kth-degree elementary symmetric polynomial in the variables

$$x_i = \tan\theta_i$$

for i = 0, 1, 2, 3, ..., i.e.,

$$\begin{aligned}
e_0 &= 1 \\
e_1 &= \sum_i x_i &&= \sum_i \tan\theta_i \\
e_2 &= \sum_{i<j} x_i x_j &&= \sum_{i<j} \tan\theta_i\tan\theta_j \\
e_3 &= \sum_{i<j<k} x_i x_j x_k &&= \sum_{i<j<k} \tan\theta_i\tan\theta_j\tan\theta_k \\
&\ \ \vdots && \ \ \vdots
\end{aligned}$$

Then

$$\tan\left(\sum_i \theta_i\right) = \frac{e_1 - e_3 + e_5 - \cdots}{e_0 - e_2 + e_4 - \cdots}.$$

The number of terms on the right side depends on the number of terms on the left side.

For example:

$$\begin{aligned}
\tan(\theta_1+\theta_2) &= \frac{e_1}{e_0-e_2} = \frac{x_1+x_2}{1\ -\ x_1x_2} = \frac{\tan\theta_1+\tan\theta_2}{1\ -\ \tan\theta_1\tan\theta_2}, \\
\tan(\theta_1+\theta_2+\theta_3) &= \frac{e_1-e_3}{e_0-e_2} = \frac{(x_1+x_2+x_3)\ -\ (x_1x_2x_3)}{1\ -\ (x_1x_2+x_1x_3+x_2x_3)}, \\
\tan(\theta_1+\theta_2+\theta_3+\theta_4) &= \frac{e_1-e_3}{e_0-e_2+e_4} \\
&= \frac{(x_1+x_2+x_3+x_4)\ -\ (x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)}{1\ -\ (x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)\ +\ (x_1x_2x_3x_4)},
\end{aligned}$$

and so on. The case of only finitely many terms can be proved by mathematical induction.

Secants and cosecants of sums

$$\sec\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_0 - e_2 + e_4 - \cdots}$$

$$\csc\left(\sum_i \theta_i\right) = \frac{\prod_i \sec\theta_i}{e_1 - e_3 + e_5 - \cdots}$$

where e_k is the kth-degree elementary symmetric polynomial in the n variables $x_i = \tan\theta_i$, $i = 1, \ldots, n$, and the number of terms in the denominator and the number of factors in the product in the numerator depend on the number of terms in the sum on the left. The case of only finitely many terms can be proved by mathematical induction on the number of such terms. The convergence of the series in the denominators can be shown by writing the secant identity in the form

$$e_0 - e_2 + e_4 - \cdots = \frac{\prod_i \sec\theta_i}{\sec\left(\sum_i \theta_i\right)}$$

and then observing that the left side converges if the right side converges, and similarly for the cosecant identity.

For example,

$$\sec(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{1 - \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$$

$$\csc(\alpha+\beta+\gamma) = \frac{\sec\alpha\sec\beta\sec\gamma}{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}.$$

Multiple-angle formulae

T_n is the nth Chebyshev polynomial	$\cos n\theta = T_n(\cos\theta)$
S_n is the nth spread polynomial	$\sin^2 n\theta = S_n(\sin^2\theta)$
de Moivre's formula, i is the imaginary unit	$\cos n\theta + i\sin n\theta = (\cos(\theta) + i\sin(\theta))^n$ [14]

Double-angle, triple-angle, and half-angle formulae

See also: Tangent half-angle formula

These can be shown by using either the sum and difference identities or the multiple-angle formulae.

Double-angle formulae[15]			
$\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$	$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$	$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$	$\cot 2\theta = \frac{\cot^2\theta - 1}{2\cot\theta}$
Triple-angle formulae[16]			
$\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta = -4\sin^3\theta + 3\sin\theta$	$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta = 4\cos^3\theta - 3\cos\theta$	$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$	$\cot 3\theta = \frac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta}$
Half-angle formulae[17]			

$\sin\frac{\theta}{2} = \operatorname{sgn}\left(2\pi-\theta+4\pi\left\lfloor\frac{\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1-\cos\theta}{2}}$ $\left(\text{or } \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\right)$	$\cos\frac{\theta}{2} = \operatorname{sgn}\left(\pi+\theta+4\pi\left\lfloor\frac{\pi-\theta}{4\pi}\right\rfloor\right)\sqrt{\frac{1+\cos\theta}{2}}$ $\left(\text{or } \cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\right)$	$\tan\frac{\theta}{2} = \csc\theta - \cot\theta = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$ $\tan\frac{\eta+\theta}{2} = \frac{\sin\eta+\sin\theta}{\cos\eta+\cos\theta}$ $\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right) = \sec\theta + \tan\theta$ $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan(\theta/2)}{1+\tan(\theta/2)}$ $\tan\tfrac{1}{2}\theta = \frac{\tan\theta}{1+\sqrt{1+\tan^2\theta}}$ $\text{for } \quad \theta \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$	$\cot\frac{\theta}{2} = \csc\theta + \cot\theta = \pm\sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

The fact that the triple-angle formula for sine and cosine only involves powers of a single function allows one to relate the geometric problem of a compass and straightedge construction of angle trisection to the algebraic problem of solving a cubic equation, which allows one to prove that this is in general impossible using the given tools, by field theory.

A formula for computing the trigonometric identities for the third-angle exists, but it requires finding the zeroes of the cubic equation $x^3 - \frac{3x+d}{4} = 0$, where x is the value of the sine function at some angle and d is the known value of the sine function at the triple angle. However, the discriminant of this equation is negative, so this equation has three real roots (of which only one is the solution within the correct third-circle) but none of these solutions is reducible to a real algebraic expression, as they use intermediate complex numbers under the cube roots, (which may be expressed in terms of real-only functions only if using hyperbolic functions).

Sine, cosine, and tangent of multiple angles

For specific multiples, these follow from the angle addition formulas, while the general formula was given by 16th century French mathematician Vieta.

$$\sin n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta \, \sin^{n-k}\theta \, \sin\left(\frac{1}{2}(n-k)\pi\right)$$
$$\cos n\theta = \sum_{k=0}^{n} \binom{n}{k} \cos^k\theta \, \sin^{n-k}\theta \, \cos\left(\frac{1}{2}(n-k)\pi\right)$$

In each of these two equations, the first parenthesized term is a binomial coefficient, and the final trigonometric function equals one or minus one or zero so that half the entries in each of the sums are removed. tan $n\theta$ can be written in terms of tan θ using the recurrence relation:

$$\tan\,(n{+}1)\theta = \frac{\tan n\theta + \tan\theta}{1 - \tan n\theta\,\tan\theta}.$$

cot $n\theta$ can be written in terms of cot θ using the recurrence relation:

$$\cot\,(n{+}1)\theta = \frac{\cot n\theta\,\cot\theta - 1}{\cot n\theta + \cot\theta}.$$

Chebyshev method

The Chebyshev method is a recursive algorithm for finding the n^{th} multiple angle formula knowing the $(n-1)^{\text{th}}$ and $(n-2)^{\text{th}}$ formulae.[18]

The cosine for nx can be computed from the cosine of $(n-1)x$ and $(n-2)x$ as follows:

$$\cos nx = 2 \cdot \cos x \cdot \cos((n-1)x) - \cos((n-2)x)$$

Similarly $\sin(nx)$ can be computed from the sines of $(n-1)x$ and $(n-2)x$

$$\sin nx = 2 \cdot \cos x \cdot \sin((n-1)x) - \sin((n-2)x)$$

For the tangent, we have:

$$\tan nx = \frac{H + K \tan x}{K - H \tan x}$$

where $H/K = \tan(n-1)x$.

Tangent of an average

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\sin\alpha + \sin\beta}{\cos\alpha + \cos\beta} = -\frac{\cos\alpha - \cos\beta}{\sin\alpha - \sin\beta}$$

Setting either α or β to 0 gives the usual tangent half-angle formulæ.

Viète's infinite product

$$\cos\left(\frac{\theta}{2}\right) \cdot \cos\left(\frac{\theta}{4}\right) \cdot \cos\left(\frac{\theta}{8}\right) \cdots = \prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right) = \frac{\sin\theta}{\theta} = \operatorname{sinc}\theta.$$

Power-reduction formula

Obtained by solving the second and third versions of the cosine double-angle formula.

Sine	Cosine	Other
$\sin^2\theta = \frac{1-\cos 2\theta}{2}$	$\cos^2\theta = \frac{1+\cos 2\theta}{2}$	$\sin^2\theta\cos^2\theta = \frac{1-\cos 4\theta}{8}$
$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$	$\cos^3\theta = \frac{3\cos\theta + \cos 3\theta}{4}$	$\sin^3\theta\cos^3\theta = \frac{3\sin 2\theta - \sin 6\theta}{32}$
$\sin^4\theta = \frac{3 - 4\cos 2\theta + \cos 4\theta}{8}$	$\cos^4\theta = \frac{3 + 4\cos 2\theta + \cos 4\theta}{8}$	$\sin^4\theta\cos^4\theta = \frac{3 - 4\cos 4\theta + \cos 8\theta}{128}$
$\sin^5\theta = \frac{10\sin\theta - 5\sin 3\theta + \sin 5\theta}{16}$	$\cos^5\theta = \frac{10\cos\theta + 5\cos 3\theta + \cos 5\theta}{16}$	$\sin^5\theta\cos^5\theta = \frac{10\sin 2\theta - 5\sin 6\theta + \sin 10\theta}{512}$

and in general terms of powers of $\sin\theta$ or $\cos\theta$ the following is true, and can be deduced using De Moivre's formula, Euler's formula and binomial theorem.

	Cosine	Sine
if n is odd	$\cos^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos\left((n-2k)\theta\right)$	$\sin^n\theta = \frac{2}{2^n}\sum_{k=0}^{\frac{n-1}{2}}(-1)^{\left(\frac{n-1}{2}-k\right)}\binom{n}{k}\sin\left((n-2k)\theta\right)$
if n is even	$\cos^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos\left((n-2k)\theta\right)$	$\sin^n\theta = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{2}{2^n}\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\left(\frac{n}{2}-k\right)}\binom{n}{k}\cos\left((n-2k)\theta\right)$

Product-to-sum and sum-to-product identities

The product-to-sum identities or prosthaphaeresis formulas can be proven by expanding their right-hand sides using the angle addition theorems. See beat (acoustics) and phase detector for applications of the sum-to-product formulæ.

Product-to-sum[19]
$\cos\theta\cos\varphi = \frac{\cos(\theta-\varphi)+\cos(\theta+\varphi)}{2}$
$\sin\theta\sin\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{2}$
$\sin\theta\cos\varphi = \frac{\sin(\theta+\varphi)+\sin(\theta-\varphi)}{2}$
$\cos\theta\sin\varphi = \frac{\sin(\theta+\varphi)-\sin(\theta-\varphi)}{2}$
$\tan\theta\tan\varphi = \frac{\cos(\theta-\varphi)-\cos(\theta+\varphi)}{\cos(\theta-\varphi)+\cos(\theta+\varphi)}$
$\prod_{k=1}^{n}\cos\theta_k = \frac{1}{2^n}\sum_{e\in S}\cos(e_1\theta_1+\cdots+e_n\theta_n)$ where $S=\{1,-1\}^n$

Sum-to-product[20]
$\sin\theta\pm\sin\varphi = 2\sin\left(\frac{\theta\pm\varphi}{2}\right)\cos\left(\frac{\theta\mp\varphi}{2}\right)$
$\cos\theta+\cos\varphi = 2\cos\left(\frac{\theta+\varphi}{2}\right)\cos\left(\frac{\theta-\varphi}{2}\right)$
$\cos\theta-\cos\varphi = -2\sin\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta-\varphi}{2}\right)$

Other related identities

- If $x+y+z=\pi=$ half circle,

  $$\text{then }\ \sin(2x)+\sin(2y)+\sin(2z)=4\sin(x)\sin(y)\sin(z).$$

- (Triple tangent identity) If $x+y+z=\pi=$ half circle,

  $$\text{then }\ \tan(x)+\tan(y)+\tan(z)=\tan(x)\tan(y)\tan(z).$$

  In particular, the formula holds when x, y, and z are the three angles of any triangle.

  (If any of x, y, z is a right angle, one should take both sides to be ∞. This is neither +∞ nor −∞; for present purposes it makes sense to add just one point at infinity to the real line, that is approached by tan(θ) as tan(θ) either increases through positive values or decreases through negative values.

This is a one-point compactification of the real line.)

- (Triple cotangent identity) If $x+y+z=\frac{\pi}{2}=\text{quarter circle,}$

$$\text{then } \cot(x)+\cot(y)+\cot(z)=\cot(x)\cot(y)\cot(z).$$

Hermite's cotangent identity

Main article: Hermite's cotangent identity

Charles Hermite demonstrated the following identity.[21] Suppose $a_1, \ldots, a_n$ are complex numbers, no two of which differ by an integer multiple of π. Let

$$A_{n,k}=\prod_{\substack{1\le j\le n\\ j\ne k}}\cot(a_k-a_j)$$

(in particular, $A_{1,1}$, being an empty product, is 1). Then

$$\cot(z-a_1)\cdots\cot(z-a_n)=\cos\frac{n\pi}{2}+\sum_{k=1}^{n}A_{n,k}\cot(z-a_k).$$

The simplest non-trivial example is the case $n = 2$:

$$\cot(z-a_1)\cot(z-a_2)=-1+\cot(a_1-a_2)\cot(z-a_1)+\cot(a_2-a_1)\cot(z-a_2).$$

Ptolemy's theorem

$$\begin{aligned}&\text{If } w+x+y+z=\pi=\text{half circle,}\\ &\qquad\text{then } \sin(w+x)\sin(x+y)\\ &\qquad\qquad=\sin(x+y)\sin(y+z)\\ &\qquad\qquad=\sin(y+z)\sin(z+w)\\ &\qquad\qquad=\sin(z+w)\sin(w+x)=\sin(w)\sin(y)+\sin(x)\sin(z).\end{aligned}$$

(The first three equalities are trivial; the fourth is the substance of this identity.) Essentially this is Ptolemy's theorem adapted to the language of modern trigonometry.

Linear combinations

For some purposes it is important to know that any linear combination of sine waves of the same period or frequency but different phase shifts is also a sine wave with the same period or frequency, but a different phase shift. This is useful in sinusoid data fitting, because the measured or observed data are linearly related to the *a* and *b* unknowns of the in-phase and quadrature components basis below, resulting in a simpler Jacobian, compared to that of *c* and φ. In the case of a non-zero linear combination of a sine and cosine wave[22] (which is just a sine wave with a phase shift of π/2), we have

$$a\sin x+b\cos x=c\cdot\sin(x+\varphi)$$

where

$$c=\sqrt{a^2+b^2},$$

and (using the atan2 function)

$$\varphi=\operatorname{atan2}(b,a).$$

More generally, for an arbitrary phase shift, we have

$$a\sin x+b\sin(x+\alpha)=c\sin(x+\beta)$$

where

$$c=\sqrt{a^2+b^2+2ab\cos\alpha},$$

and

$$\beta = \arctan\left(\frac{b\sin\alpha}{a + b\cos\alpha}\right) + \begin{cases} 0 & \text{if } a + b\cos\alpha \ge 0, \\ \pi & \text{if } a + b\cos\alpha < 0. \end{cases}$$

The general case readsWikipedia:Citation needed

$$\sum_i a_i \sin(x + \delta_i) = a\sin(x + \delta),$$

where

$$a^2 = \sum_{i,j} a_i a_j \cos(\delta_i - \delta_j)$$

and

$$\tan\delta = \frac{\sum_i a_i \sin\delta_i}{\sum_i a_i \cos\delta_i}.$$

See also Phasor addition.

Lagrange's trigonometric identities

These identities, named after Joseph Louis Lagrange, are:

$$\sum_{n=1}^{N} \sin n\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(N + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$
$$\sum_{n=1}^{N} \cos n\theta = -\frac{1}{2} + \frac{\sin(N + \frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}$$

A related function is the following function of x, called the Dirichlet kernel.

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left(\left(n + \frac{1}{2}\right)x\right)}{\sin(x/2)}.$$

Other sums of trigonometric functions

Sum of sines and cosines with arguments in arithmetic progression:[23] if $\alpha \neq 0$, then

$$\sin\varphi + \sin(\varphi + \alpha) + \sin(\varphi + 2\alpha) + \cdots$$
$$\cdots + \sin(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \sin\left(\varphi + \frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}} \quad \text{and}$$

$$\cos\varphi + \cos(\varphi + \alpha) + \cos(\varphi + 2\alpha) + \cdots$$
$$\cdots + \cos(\varphi + n\alpha) = \frac{\sin\left(\frac{(n+1)\alpha}{2}\right) \cdot \cos\left(\varphi + \frac{n\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

For any *a* and *b*:

$$a\cos(x) + b\sin(x) = \sqrt{a^2 + b^2}\cos(x - \operatorname{atan2}(b, a))$$

where atan2(y, x) is the generalization of arctan(y/x) that covers the entire circular range.

$$\tan(x) + \sec(x) = \tan\left(\frac{x}{2} + \frac{\pi}{4}\right).$$

The above identity is sometimes convenient to know when thinking about the Gudermannian function, which relates the circular and hyperbolic trigonometric functions without resorting to complex numbers.

If x, y, and z are the three angles of any triangle, i.e. if $x + y + z = \pi$, then

$$\cot(x)\cot(y) + \cot(y)\cot(z) + \cot(z)\cot(x) = 1.$$

Certain linear fractional transformations

If $f(x)$ is given by the linear fractional transformation

$$f(x) = \frac{(\cos\alpha)x - \sin\alpha}{(\sin\alpha)x + \cos\alpha},$$

and similarly

$$g(x) = \frac{(\cos\beta)x - \sin\beta}{(\sin\beta)x + \cos\beta},$$

then

$$f(g(x)) = g(f(x)) = \frac{(\cos(\alpha+\beta))x - \sin(\alpha+\beta)}{(\sin(\alpha+\beta))x + \cos(\alpha+\beta)}.$$

More tersely stated, if for all α we let f_α be what we called f above, then

$$f_\alpha \circ f_\beta = f_{\alpha+\beta}.$$

If x is the slope of a line, then $f(x)$ is the slope of its rotation through an angle of $-\alpha$.

Inverse trigonometric functions

$$\arcsin(x) + \arccos(x) = \pi/2$$
$$\arctan(x) + \operatorname{arccot}(x) = \pi/2.$$
$$\arctan(x) + \arctan(1/x) = \begin{cases} \pi/2, & \text{if } x > 0 \\ -\pi/2, & \text{if } x < 0 \end{cases}$$

Compositions of trig and inverse trig functions

$\sin[\arccos(x)] = \sqrt{1-x^2}$	$\tan[\arcsin(x)] = \frac{x}{\sqrt{1-x^2}}$
$\sin[\arctan(x)] = \frac{x}{\sqrt{1+x^2}}$	$\tan[\arccos(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arctan(x)] = \frac{1}{\sqrt{1+x^2}}$	$\cot[\arcsin(x)] = \frac{\sqrt{1-x^2}}{x}$
$\cos[\arcsin(x)] = \sqrt{1-x^2}$	$\cot[\arccos(x)] = \frac{x}{\sqrt{1-x^2}}$

Relation to the complex exponential function

$e^{ix} = \cos(x) + i\sin(x)$ [24] (Euler's formula),

$e^{-ix} = \cos(-x) + i\sin(-x) = \cos(x) - i\sin(x)$

$e^{i\pi} = -1$ (Euler's identity),

$e^{2\pi i} = 1$

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$ [25]

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$ [26]

and hence the corollary:

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

where $i^2 = -1$.

Infinite product formulae

For applications to special functions, the following infinite product formulae for trigonometric functions are useful:[27][28]

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 n^2}\right) \qquad \cos x = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\pi^2 (n - \frac{1}{2})^2}\right)$$

$$\sinh x = x \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 n^2}\right) \qquad \cosh x = \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{\pi^2 (n - \frac{1}{2})^2}\right)$$

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right) \qquad |\sin x| = \frac{1}{2} \prod_{n=0}^{\infty} \sqrt[2^{n+1}]{|\tan(2^n x)|}$$

Identities without variables

The curious identity

$$\cos 20^\circ \cdot \cos 40^\circ \cdot \cos 80^\circ = \frac{1}{8}$$

is a special case of an identity that contains one variable:

$$\prod_{j=0}^{k-1} \cos(2^j x) = \frac{\sin(2^k x)}{2^k \sin(x)}.$$

Similarly:

$$\sin 20^\circ \cdot \sin 40^\circ \cdot \sin 80^\circ = \frac{\sqrt{3}}{8}.$$

The same cosine identity in radians is

$$\cos\frac{\pi}{9} \cos\frac{2\pi}{9} \cos\frac{4\pi}{9} = \frac{1}{8},$$

Similarly:

$$\tan 50^\circ \cdot \tan 60^\circ \cdot \tan 70^\circ = \tan 80^\circ.$$

$$\tan 40^\circ \cdot \tan 30^\circ \cdot \tan 20^\circ = \tan 10^\circ.$$

The following is perhaps not as readily generalized to an identity containing variables (but see explanation below):

$$\cos 24^\circ + \cos 48^\circ + \cos 96^\circ + \cos 168^\circ = \frac{1}{2}.$$

Degree measure ceases to be more felicitous than radian measure when we consider this identity with 21 in the denominators:

$$\cos\left(\frac{2\pi}{21}\right) + \cos\left(2\cdot\frac{2\pi}{21}\right) + \cos\left(4\cdot\frac{2\pi}{21}\right)$$

$$+ \cos\left(5\cdot\frac{2\pi}{21}\right) + \cos\left(8\cdot\frac{2\pi}{21}\right) + \cos\left(10\cdot\frac{2\pi}{21}\right) = \frac{1}{2}.$$

The factors 1, 2, 4, 5, 8, 10 may start to make the pattern clear: they are those integers less than 21/2 that are relatively prime to (or have no prime factors in common with) 21. The last several examples are corollaries of a basic fact about the irreducible cyclotomic polynomials: the cosines are the real parts of the zeroes of those polynomials; the sum of the zeroes is the Möbius function evaluated at (in the very last case above) 21; only half of the zeroes are present above. The two identities preceding this last one arise in the same fashion with 21 replaced by 10 and 15, respectively.

Many of those curious identities stem from more general facts like the following:[29]

$$\prod_{k=1}^{n-1} \sin\left(\frac{k\pi}{n}\right) = \frac{n}{2^{n-1}}$$

and

$$\prod_{k=1}^{n-1} \cos\left(\frac{k\pi}{n}\right) = \frac{\sin(\pi n/2)}{2^{n-1}}$$

Combining these gives us

$$\prod_{k=1}^{n-1} \tan\left(\frac{k\pi}{n}\right) = \frac{n}{\sin(\pi n/2)}$$

If n is an odd number ($n = 2m + 1$) we can make use of the symmetries to get

$$\prod_{k=1}^{m} \tan\left(\frac{k\pi}{2m+1}\right) = \sqrt{2m+1}$$

The transfer function of the Butterworth low pass filter can be expressed in terms of polynomial and poles. By setting the frequency as the cutoff frequency, the following identity can be proved:

$$\prod_{k=1}^{n} \sin\left(\frac{(2k-1)\,\pi}{4n}\right) = \prod_{k=1}^{n} \cos\left(\frac{(2k-1)\,\pi}{4n}\right) = \frac{\sqrt{2}}{2^n}$$

Computing π

An efficient way to compute π is based on the following identity without variables, due to Machin:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

or, alternatively, by using an identity of Leonhard Euler:

$$\frac{\pi}{4} = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}.$$

A useful mnemonic for certain values of sines and cosines

For certain simple angles, the sines and cosines take the form $\sqrt{n}/2$ for $0 \le n \le 4$, which makes them easy to remember.

$$\begin{matrix} \sin 0 & = & \sin 0^\circ & = & \sqrt{0}/2 & = & \cos 90^\circ & = & \cos\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{6}\right) & = & \sin 30^\circ & = & \sqrt{1}/2 & = & \cos 60^\circ & = & \cos\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{4}\right) & = & \sin 45^\circ & = & \sqrt{2}/2 & = & \cos 45^\circ & = & \cos\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{3}\right) & = & \sin 60^\circ & = & \sqrt{3}/2 & = & \cos 30^\circ & = & \cos\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{2}\right) & = & \sin 90^\circ & = & \sqrt{4}/2 & = & \cos 0^\circ & = & \cos 0 \end{matrix}$$

Miscellany

With the golden ratio φ:

$$\cos\left(\frac{\pi}{5}\right) = \cos 36^\circ = \tfrac{1}{4}(\sqrt{5}+1) = \tfrac{1}{2}\varphi$$
$$\sin\left(\frac{\pi}{10}\right) = \sin 18^\circ = \tfrac{1}{4}(\sqrt{5}-1) = \tfrac{1}{2}\varphi^{-1}$$

Also see exact trigonometric constants.

An identity of Euclid

Euclid showed in Book XIII, Proposition 10 of his *Elements* that the area of the square on the side of a regular pentagon inscribed in a circle is equal to the sum of the areas of the squares on the sides of the regular hexagon and the regular decagon inscribed in the same circle. In the language of modern trigonometry, this says:

$$\sin^2(18^\circ) + \sin^2(30^\circ) = \sin^2(36^\circ).$$

Ptolemy used this proposition to compute some angles in his table of chords.

Composition of trigonometric functions

This identity involves a trigonometric function of a trigonometric function:[30]

$$\cos(t\sin(x)) = J_0(t) + 2\sum_{k=1}^{\infty} J_{2k}(t)\cos(2kx)$$

where J_0 and J_{2k} are Bessel functions.

Calculus

In calculus the relations stated below require angles to be measured in radians; the relations would become more complicated if angles were measured in another unit such as degrees. If the trigonometric functions are defined in terms of geometry, along with the definitions of arc length and area, their derivatives can be found by verifying two limits. The first is:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1,$$

verified using the unit circle and squeeze theorem. The second limit is:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0,$$

verified using the identity tan(x/2) = (1 − cos x)/sin x. Having established these two limits, one can use the limit definition of the derivative and the addition theorems to show that (sin x)′ = cos x and (cos x)′ = −sin x. If the sine and cosine functions are defined by their Taylor series, then the derivatives can be found by differentiating the power series term-by-term.

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

The rest of the trigonometric functions can be differentiated using the above identities and the rules of differentiation:[31][32]

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\arcsin x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\arccos x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccot} x = \frac{-1}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \tan x \sec x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\csc x \cot x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\operatorname{arccsc} x = \frac{-1}{|x|\sqrt{x^2-1}}$$

The integral identities can be found in "list of integrals of trigonometric functions". Some generic forms are listed below.

$$\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{\mathrm{d}u}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{\mathrm{d}u}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Implications

The fact that the differentiation of trigonometric functions (sine and cosine) results in linear combinations of the same two functions is of fundamental importance to many fields of mathematics, including differential equations and Fourier transforms.

Some differential equations satisfied by the sine function

Let $i = \sqrt{-1}$ be the imaginary unit and let $\circ$ denote composition of differential operators. Then for every **odd** positive integer n,

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\mathrm{d}}{\mathrm{d}x}-\sin x\right)\circ\left(\frac{\mathrm{d}}{\mathrm{d}x}-\sin x+i\right)\circ\cdots\circ\left(\frac{\mathrm{d}}{\mathrm{d}x}-\sin x+(k-1)i\right)(\sin x)^{n-k} = 0.$$

(When $k = 0$, then the number of differential operators being composed is 0, so the corresponding term in the sum above is just $(\sin x)^n$.) This identity was discovered as a by-product of research in medical imaging.[33]

Exponential definitions

Function	Inverse function[34]
$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$	$\operatorname{arcsin} x = -i \ln\left(ix + \sqrt{1-x^2}\right)$
$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$	$\operatorname{arccos} x = i \ln\left(x - i\sqrt{1-x^2}\right)$
$\tan\theta = \frac{e^{i\theta} - e^{-i\theta}}{i(e^{i\theta} + e^{-i\theta})}$	$\operatorname{arctan} x = \frac{i}{2} \ln\left(\frac{i+x}{i-x}\right)$
$\csc\theta = \frac{2i}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccsc} x = -i \ln\left(\tfrac{i}{x} + \sqrt{1-\tfrac{1}{x^2}}\right)$
$\sec\theta = \frac{2}{e^{i\theta} + e^{-i\theta}}$	$\operatorname{arcsec} x = -i \ln\left(\tfrac{1}{x} + \sqrt{1-\tfrac{i}{x^2}}\right)$
$\cot\theta = \frac{i(e^{i\theta} + e^{-i\theta})}{e^{i\theta} - e^{-i\theta}}$	$\operatorname{arccot} x = \frac{i}{2} \ln\left(\frac{x-i}{x+i}\right)$
$\operatorname{cis}\theta = e^{i\theta}$	$\operatorname{arccis} x = \frac{\ln x}{i} = -i \ln x = \arg x$

Miscellaneous

Dirichlet kernel

The **Dirichlet kernel** $D_n(x)$ is the function occurring on both sides of the next identity:

$$1 + 2\cos(x) + 2\cos(2x) + 2\cos(3x) + \cdots + 2\cos(nx) = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}.$$

The convolution of any integrable function of period 2π with the Dirichlet kernel coincides with the function's nth-degree Fourier approximation. The same holds for any measure or generalized function.

Tangent half-angle substitution

Main article: Tangent half-angle substitution

If we set

$$t = \tan\left(\frac{x}{2}\right),$$

then[35]

$$\sin(x) = \frac{2t}{1+t^2} \text{ and } \cos(x) = \frac{1-t^2}{1+t^2} \text{ and } e^{ix} = \frac{1+it}{1-it}$$

where $e^{ix} = \cos(x) + i\sin(x)$, sometimes abbreviated to $\operatorname{cis}(x)$.

When this substitution of t for $\tan(x/2)$ is used in calculus, it follows that $\sin(x)$ is replaced by $2t/(1 + t^2)$, $\cos(x)$ is replaced by $(1 - t^2)/(1 + t^2)$ and the differential dx is replaced by $(2\,dt)/(1 + t^2)$. Thereby one converts rational functions of $\sin(x)$ and $\cos(x)$ to rational functions of t in order to find their antiderivatives.

Notes

[1] Schaumberger, N. "A Classroom Theorem on Trigonometric Irrationalities." Two-Year College Math. J. 5, 73-76, 1974. also see Weisstein, Eric W. "Niven's Theorem." From MathWorld--A Wolfram Web Resource. http://mathworld.wolfram.com/NivensTheorem.html
[2] Abramowitz and Stegun, p. 73, 4.3.45
[3] Abramowitz and Stegun, p. 78, 4.3.147
[4] Abramowitz and Stegun, p. 72, 4.3.13–15
[5] The Elementary Identities (http://jwbales.home.mindspring.com/precal/part5/part5.1.html)
[6] Abramowitz and Stegun, p. 72, 4.3.9
[7] Abramowitz and Stegun, p. 72, 4.3.7–8
[8] Abramowitz and Stegun, p. 72, 4.3.16
[9] Abramowitz and Stegun, p. 72, 4.3.17
[10] Abramowitz and Stegun, p. 72, 4.3.18
[11] Abramowitz and Stegun, p. 80, 4.4.42
[12] Abramowitz and Stegun, p. 80, 4.4.43
[13] Abramowitz and Stegun, p. 80, 4.4.36
[14] Abramowitz and Stegun, p. 74, 4.3.48
[15] Abramowitz and Stegun, p. 72, 4.3.24–26
[16] Abramowitz and Stegun, p. 72, 4.3.27–28
[17] Abramowitz and Stegun, p. 72, 4.3.20–22
[18] Ken Ward's Mathematics Pages, http://www.trans4mind.com/personal_development/mathematics/trigonometry/multipleAnglesRecursiveFormula.htm
[19] Abramowitz and Stegun, p. 72, 4.3.31–33
[20] Abramowitz and Stegun, p. 72, 4.3.34–39
[21] Warren P. Johnson, "Trigonometric Identities à la Hermite", *American Mathematical Monthly*, volume 117, number 4, April 2010, pages 311–327
[22] Proof at http://pages.pacificcoast.net/~cazelais/252/lc-trig.pdf
[23] Michael P. Knapp, Sines and Cosines of Angles in Arithmetic Progression (http://evergreen.loyola.edu/mpknapp/www/papers/knapp-sv.pdf)
[24] Abramowitz and Stegun, p. 74, 4.3.47
[25] Abramowitz and Stegun, p. 71, 4.3.2
[26] Abramowitz and Stegun, p. 71, 4.3.1
[27] Abramowitz and Stegun, p. 75, 4.3.89–90
[28] Abramowitz and Stegun, p. 85, 4.5.68–69
[29] Weisstein, Eric W., " Sine (http://mathworld.wolfram.com/Sine.html)" from MathWorld
[30] Milton Abramowitz and Irene Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, New York, 1972, formula 9.1.42
[31] Abramowitz and Stegun, p. 77, 4.3.105–110
[32] Abramowitz and Stegun, p. 82, 4.4.52–57
[33] Peter Kuchment and Sergey Lvin, "Identities for sin *x* that Came from Medical Imaging", *American Mathematical Monthly*, volume 120, August–September, 2013, pages 609–621.
[34] Abramowitz and Stegun, p. 80, 4.4.26–31
[35] Abramowitz and Stegun, p. 72, 4.3.23

References

• Abramowitz, Milton; Stegun, Irene A., eds. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover Publications. ISBN 978-0-486-61272-0.

External links

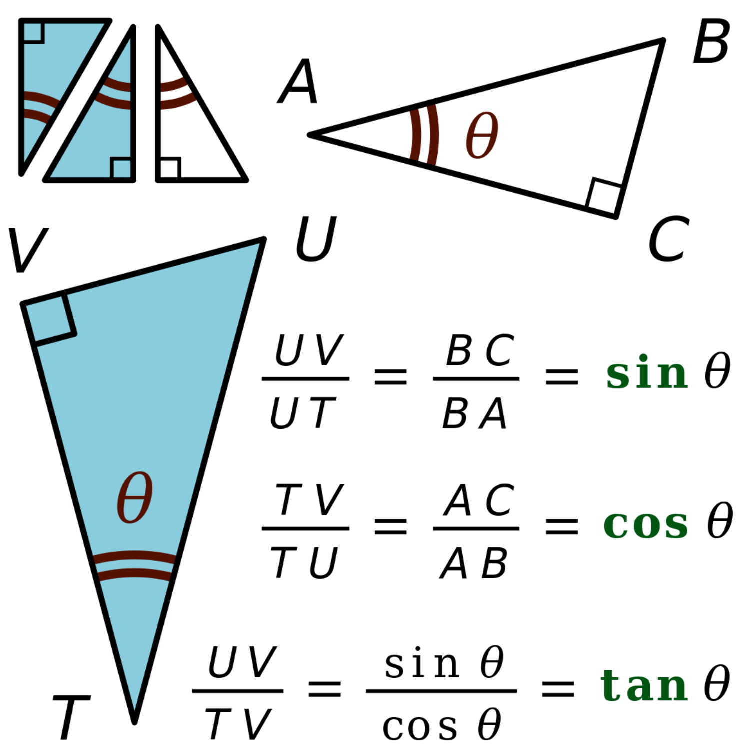
• Values of Sin and Cos, expressed in surds, for integer multiples of 3° and of 5⅝° (http://www.jdawiseman.com/papers/easymath/surds_sin_cos.html), and for the same angles Csc and Sec (http://www.jdawiseman.com/papers/easymath/surds_csc_sec.html) and Tan (http://www.jdawiseman.com/papers/easymath/surds_tan.html).

Trigonometric functions

"Cosine" redirects here. For the similarity measure, see Cosine similarity.

Trigonometry
• Outline • History • Usage • Functions (inverse) • Generalized trigonometry
Reference
• Identities • Exact constants • Tables
Laws and theorems
• Sines • Cosines • Tangents • Cotangents • Pythagorean theorem
Calculus
• Trigonometric substitution • Integrals (inverse functions) • Derivatives
• v • t • e [1]

In mathematics, the **trigonometric functions** (also called the **circular functions**) are functions of an angle. They relate the angles of a triangle to the lengths of its sides. Trigonometric functions are important in the study of triangles and modeling periodic phenomena, among many other applications.



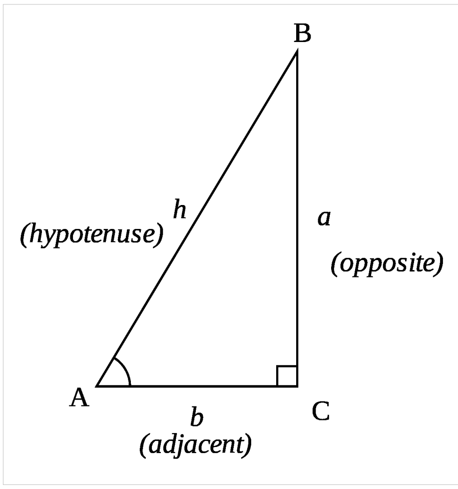
Base of trigonometry: if two right triangles have equal acute angles, they are similar, so their side lengths are proportional. Proportionality constants are written within the image: $\sin\theta$, $\cos\theta$, $\tan\theta$, where θ is the common measure of five acute angles.

The most familiar trigonometric functions are the sine, cosine, and tangent. In the context of the standard unit circle (a circle with radius 1 unit), where a triangle is formed by a ray originating at the origin and making some angle with the x-axis, the sine of the angle gives the length of the y-component (the opposite to the angle or the rise) of the triangle, the cosine gives the length of the x-component (the adjacent of the angle or the run), and the tangent function gives the slope (y-component divided by the x-component). More precise definitions are detailed below. Trigonometric functions are commonly defined as ratios of two sides of a right triangle containing the angle, and can equivalently be defined as the lengths of various line segments from a unit circle. More modern definitions express them as infinite series or as solutions of certain differential equations, allowing their extension to arbitrary positive and negative values and even to complex numbers.

Trigonometric functions have a wide range of uses including computing unknown lengths and angles in triangles (often right triangles). In this use, trigonometric functions are used, for instance, in navigation, engineering, and physics. A common use in elementary physics is resolving a vector into Cartesian coordinates. The sine and cosine functions are also commonly used to model periodic function phenomena such as sound and light waves, the position and velocity of harmonic oscillators, sunlight intensity and day length, and average temperature variations through the year.

In modern usage, there are six basic trigonometric functions, tabulated here with equations that relate them to one another. Especially with the last four, these relations are often taken as the *definitions* of those functions, but one can define them equally well geometrically, or by other means, and then derive these relations.

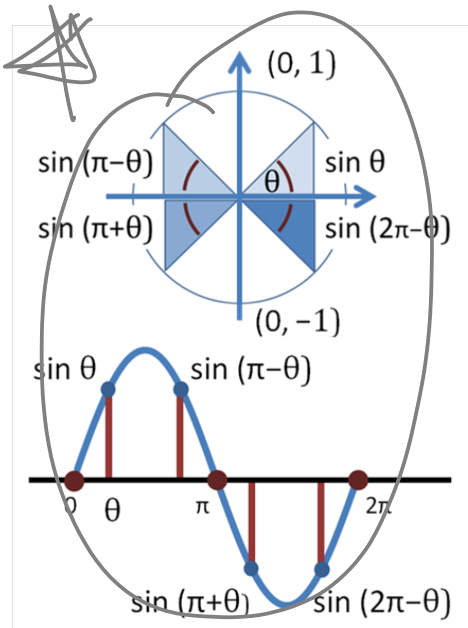
Right-angled triangle definitions



The notion that there should be some standard correspondence between the lengths of the sides of a triangle and the angles of the triangle comes as soon as one recognizes that similar triangles maintain the same ratios between their sides. That is, for any similar triangle the ratio of the hypotenuse (for example) and another of the sides remains the same. If the hypotenuse is twice as long, so are the sides. It is these ratios that the trigonometric functions express.

To define the trigonometric functions for the angle A, start with any right triangle that contains the angle A. The three sides of the triangle are named as follows:

- The *hypotenuse* is the side opposite the right angle, in this case side **h**. The hypotenuse is always the longest side of a right-angled triangle.
- The *opposite side* is the side opposite to the angle we are interested in (angle A), in this case side **a**.
- The *adjacent side* is the side having both the angles of interest (angle A and right-angle C), in this case side **b**.



(Top): Trigonometric function sinθ for selected angles θ, $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ in the four quadrants. (Bottom) Graph of sine function versus angle. Angles from the top panel are identified.

In ordinary Euclidean geometry, according to the triangle postulate, the inside angles of every triangle total 180° (π radians). Therefore, in a right-angled triangle, the two non-right angles total 90° (π/2 radians), so each of these angles must be in the range of (0°,90°) as expressed in interval notation. The following definitions apply to angles in this 0° – 90° range. They can be extended to the full set of real arguments by using the unit circle, or by requiring certain symmetries and that they be periodic functions. For example, the figure shows $\sin\theta$ for angles θ, $\pi - \theta$, $\pi + \theta$, and $2\pi - \theta$ depicted on the unit circle (top) and as a graph (bottom). The value of the sine repeats itself apart from sign in all four quadrants, and if the range of θ is extended to additional rotations, this behavior repeats periodically with a period 2π.

Rigorously, in metric space, one should express angle, defined as scaled arc length, as a function of triangle sides. It leads to inverse trigonometric functions first and usual trigonometric functions can be defined by inverting them back.

The trigonometric functions are summarized in the following table and described in more detail below. The angle θ is the angle between the hypotenuse and the adjacent line – the angle at A in the accompanying diagram.

Function	Abbreviation	Description	Identities (using radians)
sine	sin	opposite / hypotenuse	$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\csc\theta}$
cosine	cos	adjacent / hypotenuse	$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sec\theta}$
tangent	tan (or tg)	opposite / adjacent	$\tan\theta = \frac{\sin\theta}{\cos\theta} = \cot\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\cot\theta}$
cotangent	cot (or cotan or cotg or ctg or ctn)	adjacent / opposite	$\cot\theta = \frac{\cos\theta}{\sin\theta} = \tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\tan\theta}$
secant	sec	hypotenuse / adjacent	$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\cos\theta}$
cosecant	csc (or cosec)	hypotenuse / opposite	$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) = \frac{1}{\sin\theta}$

Sine, cosine and tangent

The **sine** of an angle is the ratio of the length of the opposite side to the length of the hypotenuse. (The word comes from the Latin *sinus* for gulf or bay,[1] since, given a unit circle, it is the side of the triangle on which the angle *opens*.) In our case

$$\sin A = \frac{\textrm{opposite}}{\textrm{hypotenuse}} = \frac{a}{h}.$$

This ratio does not depend on the size of the particular right triangle chosen, as long as it contains the angle *A*, since all such triangles are similar.

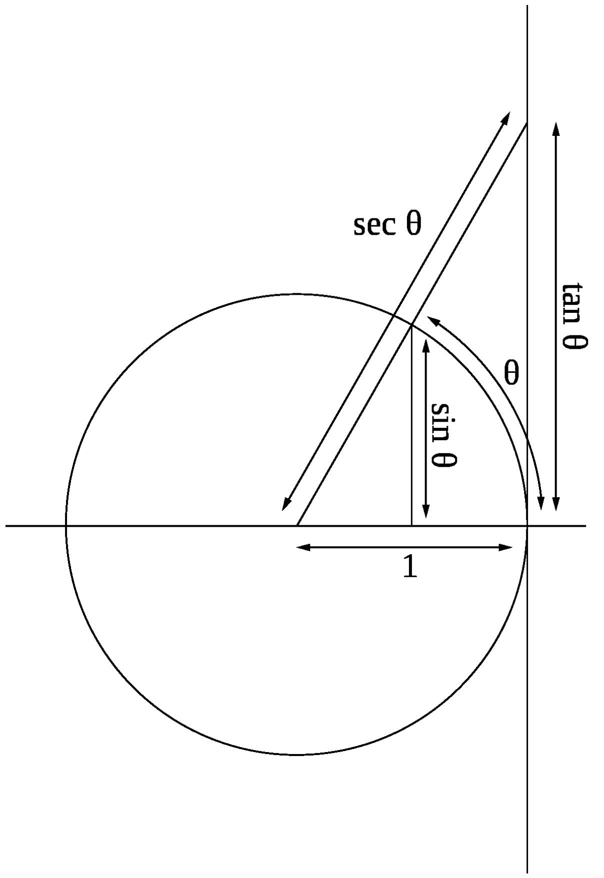
The **cosine** of an angle is the ratio of the length of the adjacent side to the length of the hypotenuse: so called because it is the sine of the complementary or co-angle.[2] In our case

$$\cos A = \frac{\textrm{adjacent}}{\textrm{hypotenuse}} = \frac{b}{h}.$$

The **tangent** of an angle is the ratio of the length of the opposite side to the length of the adjacent side: so called because it can be represented as a line segment tangent to the circle, that is the line that touches the circle, from Latin *linea tangens* or touching line (cf. *tangere*, to touch).[3] In our case

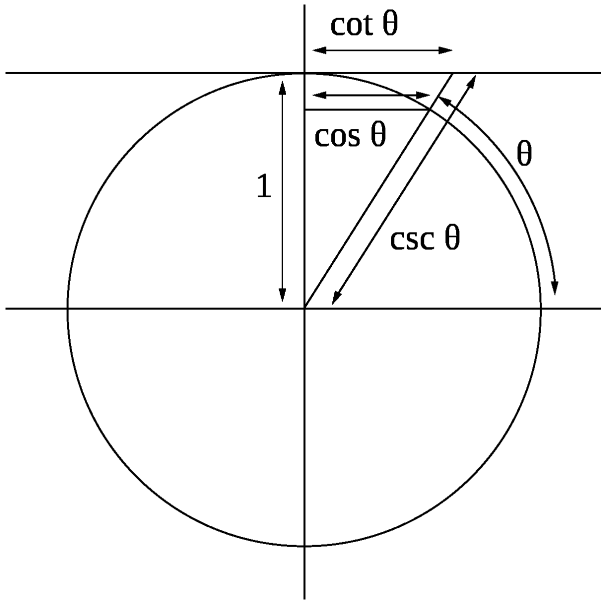
$$\tan A = \frac{\textrm{opposite}}{\textrm{adjacent}} = \frac{a}{b}.$$

The acronyms "SOHCAHTOA" ("Soak-a-toe", "Sock-a-toa", "So-kah-toa") and "OHSAHCOAT" are commonly used mnemonics for these ratios.



The sine, tangent, and secant functions of an angle constructed geometrically in terms of a unit circle. The number θ is the length of the curve; thus angles are being measured in radians. The secant and tangent functions rely on a fixed vertical line and the sine function on a moving vertical line. ("Fixed" in this context means not moving as θ changes; "moving" means depending on θ.) Thus, as θ goes from 0 up to a right angle, sin θ goes from 0 to 1, tan θ goes from 0 to ∞, and sec θ goes from 1 to ∞.

Reciprocal functions



The cosine, cotangent, and cosecant functions of an angle *θ* constructed geometrically in terms of a unit circle. The functions whose names have the prefix *co-* use horizontal lines where the others use vertical lines.

The remaining three functions are best defined using the above three functions.

The **cosecant** csc(A), or cosec(A), is the reciprocal of sin(A); i.e. the ratio of the length of the hypotenuse to the length of the opposite side:

$$\csc A = \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{h}{a}.$$

The **secant** sec(A) is the reciprocal of cos(A); i.e. the ratio of the length of the hypotenuse to the length of the adjacent side:

$$\sec A = \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{h}{b}.$$

It is so called because it represents the line that *cuts* the circle (from Latin: *secare*, to cut).[4]

The **cotangent** cot(A) is the reciprocal of tan(A); i.e. the ratio of the length of the adjacent side to the length of the opposite side:

$$\cot A = \frac{1}{\tan A} = \frac{\text{adjacent}}{\text{opposite}} = \frac{b}{a}.$$

Slope definitions

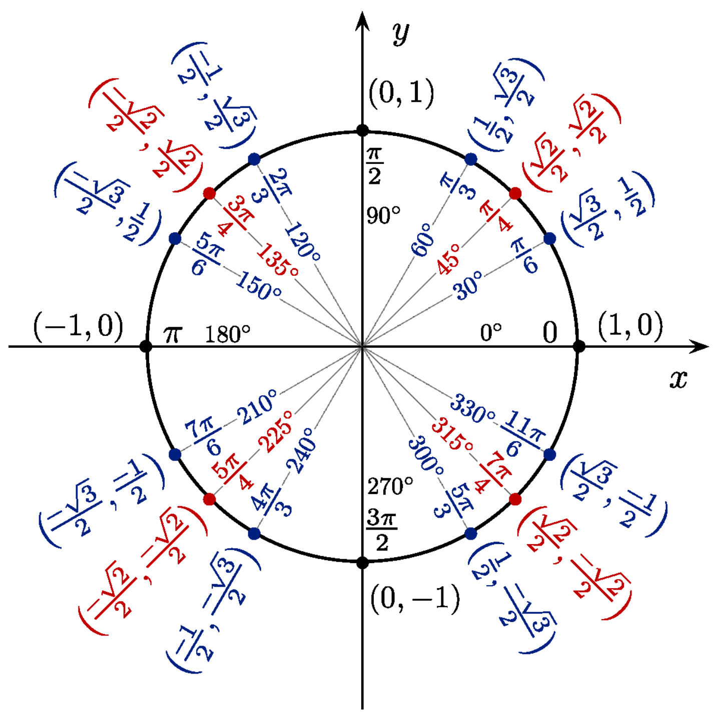
Equivalent to the right-triangle definitions, the trigonometric functions can also be defined in terms of the *rise*, *run*, and *slope* of a line segment relative to horizontal. The slope is commonly taught as "rise over run" or $^{\text{rise}}/_{\text{run}}$. The three main trigonometric functions are commonly taught in the order sine, cosine, tangent. With a line segment length of 1 (as in a unit circle), the following mnemonic devices show the correspondence of definitions:

1. "Sine is first, rise is first" meaning that Sine takes the angle of the line segment and tells its vertical rise when the length of the line is 1.
2. "Cosine is second, run is second" meaning that Cosine takes the angle of the line segment and tells its horizontal run when the length of the line is 1.
3. "Tangent combines the rise and run" meaning that Tangent takes the angle of the line segment and tells its slope; or alternatively, tells the vertical rise when the line segment's horizontal run is 1.

This shows the main use of tangent and arctangent: converting between the two ways of telling the slant of a line, *i.e.,* angles and slopes. (The arctangent or "inverse tangent" is not to be confused with the *cotangent,* which is cosine divided by sine.)

While the length of the line segment makes no difference for the slope (the slope does not depend on the length of the slanted line), it does affect rise and run. To adjust and find the actual rise and run when the line does not have a length of 1, just multiply the sine and cosine by the line length. For instance, if the line segment has length 5, the run at an angle of 7° is 5 cos(7°)

Unit-circle definitions



The unit circle

The six trigonometric functions can also be defined in terms of the unit circle, the circle of radius one centered at the origin. The unit circle definition provides little in the way of practical calculation; indeed it relies on right triangles for most angles.

The unit circle definition does, however, permit the definition of the trigonometric functions for all positive and negative arguments, not just for angles between 0 and π/2 radians.

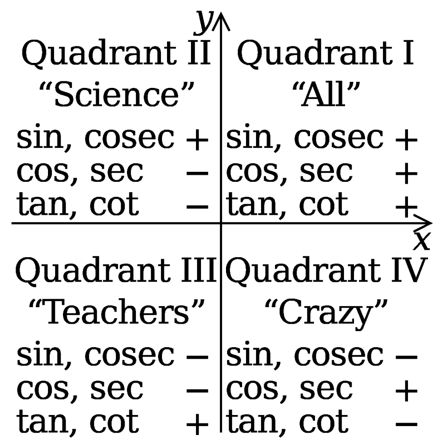
It also provides a single visual picture that encapsulates at once all the important triangles. From the Pythagorean theorem the equation for the unit circle is:

$$x^2 + y^2 = 1.$$

In the picture, some common angles, measured in radians, are given. Measurements in the counterclockwise direction are positive angles and measurements in the clockwise direction are negative angles.

Let a line through the origin, making an angle of θ with the positive half of the x-axis, intersect the unit circle. The x- and y-coordinates of this point of intersection are equal to cos θ and sin θ, respectively.

The triangle in the graphic enforces the formula; the radius is equal to the hypotenuse and has length 1, so we have sin $\theta = y/1$ and cos $\theta = x/1$. The unit circle can be thought of as a way of looking at an infinite number of triangles by varying the lengths of their legs but keeping the lengths of their hypotenuses equal to 1.



Signs of trigonometric functions in each quadrant. The mnemonic "**All** **S**cience **T**eachers (are) **C**razy" lists the functions which are positive from quadrants I to IV.[5] This is a variation on the mnemonic "All Students Take Calculus".

These values (sin 0°, sin 30°, sin 45°, sin 60° and sin 90°) can be expressed in the form

$$\frac{1}{2}\sqrt{0},\quad \frac{1}{2}\sqrt{1},\quad \frac{1}{2}\sqrt{2},\quad \frac{1}{2}\sqrt{3},\quad \frac{1}{2}\sqrt{4},$$

but the angles are not equally spaced.

The values for 15°, 18°, 36°, 54°, 72°, and 75° are derived as follows:

$$\sin 15^\circ = \cos 75^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

$$\sin 18^\circ = \cos 72^\circ = \frac{\sqrt{5}-1}{4}$$
$$\sin 36^\circ = \cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$
$$\sin 54^\circ = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$
$$\sin 72^\circ = \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$
$$\sin 75^\circ = \cos 15^\circ = \frac{\sqrt{6}+\sqrt{2}}{4}.$$

From these, the values for all multiples of 3° can be analytically computed. For example:

$$\sin 3^\circ = \cos 87^\circ = \frac{\sqrt{30}+\sqrt{10}+\sqrt{20+4\sqrt{5}}-\sqrt{6}-\sqrt{2}-\sqrt{60+12\sqrt{5}}}{16}$$
$$\sin 6^\circ = \cos 84^\circ = \frac{\sqrt{30-6\sqrt{5}}-\sqrt{5}-1}{8}$$
$$\sin 9^\circ = \cos 81^\circ = \frac{\sqrt{90}+\sqrt{18}+\sqrt{10}+\sqrt{2}-\sqrt{20-4\sqrt{5}}-\sqrt{180-36\sqrt{5}}}{32}$$
$$\sin 84^\circ = \cos 6^\circ = \frac{\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{8}$$
$$\sin 87^\circ = \cos 3^\circ = \frac{\sqrt{60+12\sqrt{5}}+\sqrt{20+4\sqrt{5}}+\sqrt{30}+\sqrt{2}-\sqrt{6}-\sqrt{10}}{16}$$

Though a complex task, the analytical expression of sin 1° can be obtained by analytically solving the cubic equation

$$\sin 3^\circ = 3\sin 1^\circ - 4\sin^3 1^\circ$$

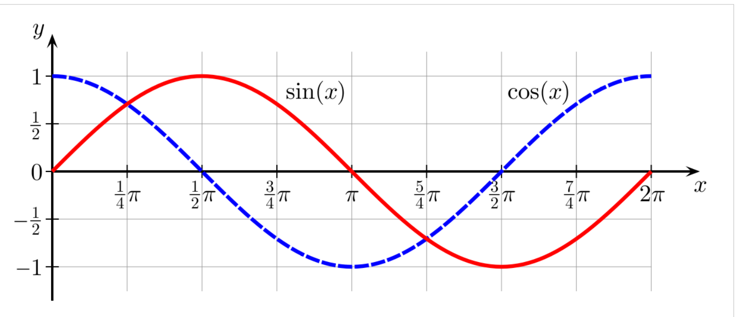
from whose solution one can analytically derive trigonometric functions of all angles of integer degrees.

For angles greater than 2π or less than −2π, simply continue to rotate around the circle; sine and cosine are periodic functions with period 2π:

$$\sin\theta = \sin\left(\theta + 2\pi k\right),$$
$$\cos\theta = \cos\left(\theta + 2\pi k\right),$$

for any angle θ and any integer *k*.

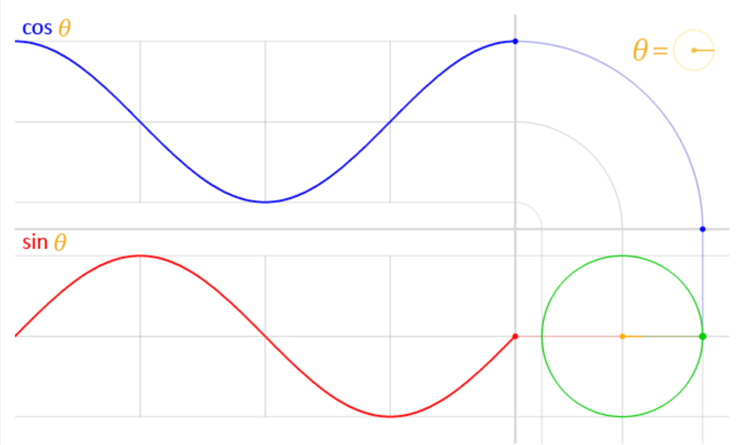
The *smallest* positive period of a periodic function is called the *primitive period* of the function.



The sine and cosine functions graphed on the Cartesian plane.

The primitive period of the sine or cosine is a full circle, i.e. 2π radians or 360 degrees.

Above, only sine and cosine were defined directly by the unit circle, but other trigonometric functions can be defined by:



Animation showing the relationship between the unit circle and the sine and cosine functions.

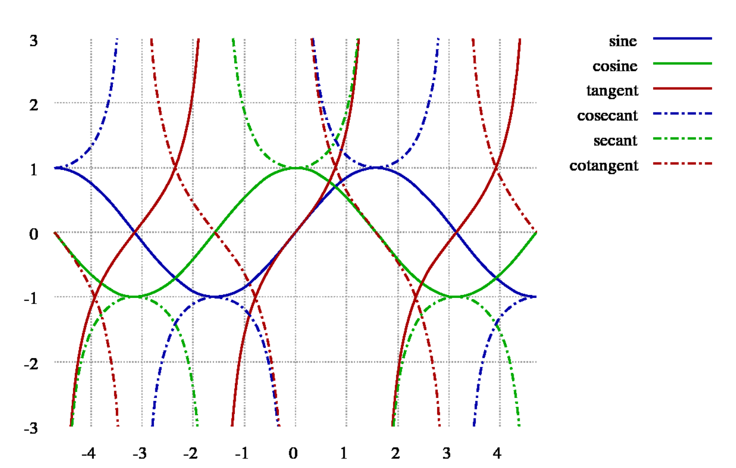
$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta}$$
$$\sec\theta = \frac{1}{\cos\theta},\ \csc\theta = \frac{1}{\sin\theta}$$

So :

- The primitive period of the secant, or cosecant is also a full circle, i.e. 2π radians or 360 degrees.
- The primitive period of the tangent or cotangent is only a half-circle, i.e. π radians or 180 degrees.

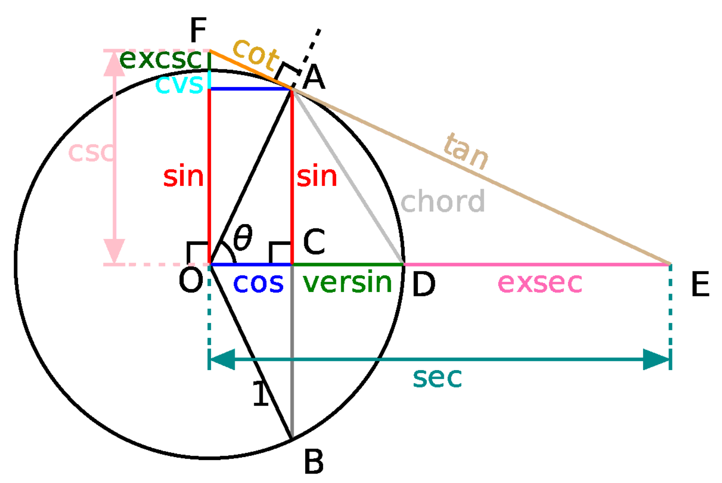
The image at right includes a graph of the tangent function.

- Its θ-intercepts correspond to those of $\sin(\theta)$ while its undefined values correspond to the θ-intercepts of $\cos(\theta)$.
- The function changes slowly around angles of $k\pi$, but changes rapidly at angles close to $(k + 1/2)\pi$.
- The graph of the tangent function also has a vertical asymptote at $\theta = (k + 1/2)\pi$, the θ-intercepts of the cosine function, because the function approaches infinity as θ approaches $(k + 1/2)\pi$ from the left and minus infinity as it approaches $(k + 1/2)\pi$ from the right.



Trigonometric functions: Sine, Cosine, Tangent, Cosecant (dotted), Secant (dotted), Cotangent (dotted)

Alternatively, *all* of the basic trigonometric functions can be defined in terms of a unit circle centered at O (as shown in the picture to the right), and similar such geometric definitions were used historically.

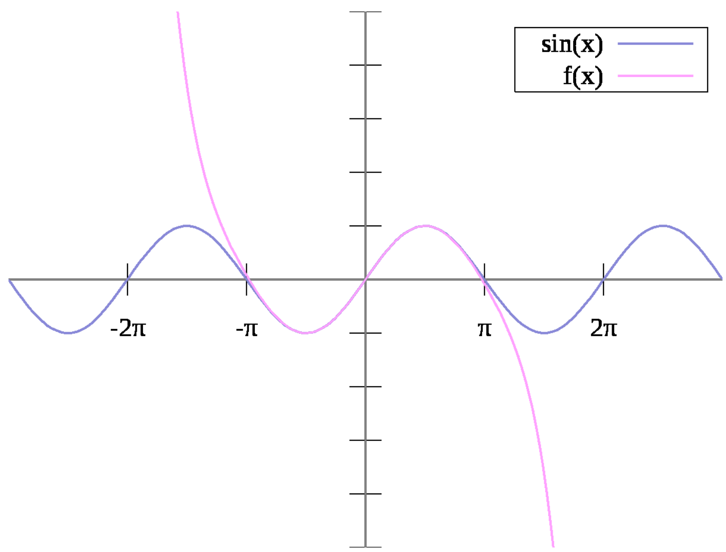


All of the trigonometric functions of the angle θ can be constructed geometrically in terms of a unit circle centered at O.

- In particular, for a chord AB of the circle, where θ is half of the subtended angle, $\sin(\theta)$ is AC (half of the chord), a definition introduced in India (see history).
- $\cos(\theta)$ is the horizontal distance OC, and $\text{versin}(\theta) = 1 - \cos(\theta)$ is CD.
- $\tan(\theta)$ is the length of the segment AE of the tangent line through A, hence the word *tangent* for this function. $\cot(\theta)$ is another tangent segment, AF.
- $\sec(\theta) = OE$ and $\csc(\theta) = OF$ are segments of secant lines (intersecting the circle at two points), and can also be viewed as projections of OA along the tangent at A to the horizontal and vertical axes, respectively.
- DE is $\text{exsec}(\theta) = \sec(\theta) - 1$ (the portion of the secant outside, or *ex*, the circle).
- From these constructions, it is easy to see that the secant and tangent functions diverge as θ approaches $\pi/2$ (90 degrees) and that the cosecant and cotangent diverge as θ approaches zero. (Many similar constructions are possible, and the basic trigonometric identities can also be proven graphically.[6])

Series definitions

Trigonometric functions are analytic functions. Using only geometry and properties of limits, it can be shown that the derivative of sine is cosine and the derivative of cosine is the negative of sine. (Here, and generally in calculus, all angles are measured in radians; see also the significance of radians below.) One can then use the theory of Taylor series to show that the following identities hold for all real numbers x:[7]



The sine function (blue) is closely approximated by its Taylor polynomial of degree 7 (pink) for a full cycle centered on the origin.

$$\begin{aligned}\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.\end{aligned}$$

These identities are sometimes taken as the *definitions* of the sine and cosine function. They are often used as the starting point in a rigorous treatment of trigonometric functions and their applications (e.g., in Fourier series), since the theory of infinite series can be developed, independent of any geometric considerations, from the foundations of the real number system. The differentiability and continuity of these functions are then established from the series definitions alone. The value of π can be defined as the smallest positive number for which sin = 0.

Other series can be found.[8] For the following trigonometric functions:

U_n is the *n*th up/down number,

B_n is the *n*th Bernoulli number, and

E_n (below) is the *n*th Euler number.

Tangent

$$\begin{aligned}\tan x &= \sum_{n=0}^{\infty} \frac{U_{2n+1} x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n} x^{2n-1}}{(2n)!} \\ &= x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}.\end{aligned}$$

When this series for the tangent function is expressed in a form in which the denominators are the corresponding factorials, the numerators, called the "tangent numbers", have a combinatorial interpretation: they enumerate alternating permutations of finite sets of odd cardinality.[9]

Cosecant

$$\begin{aligned}\csc x &= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2 (2^{2n-1}-1) B_{2n} x^{2n-1}}{(2n)!} \\ &= x^{-1} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \cdots, \qquad \text{for } 0 < |x| < \pi.\end{aligned}$$

Secant

$$\begin{aligned}\sec x &= \sum_{n=0}^{\infty} \frac{U_{2n} x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} \\ &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots, \qquad \text{for } |x| < \frac{\pi}{2}.\end{aligned}$$

When this series for the secant function is expressed in a form in which the denominators are the corresponding factorials, the numerators, called the "secant numbers", have a combinatorial interpretation: they enumerate alternating permutations of finite sets of even cardinality.[10]

Cotangent

$$\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$
$$= x^{-1} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \cdots, \qquad \text{for } 0 < |x| < \pi.$$

From a theorem in complex analysis, there is a unique analytic continuation of this real function to the domain of complex numbers. They have the same Taylor series, and so the trigonometric functions are defined on the complex numbers using the Taylor series above.

There is a series representation as partial fraction expansion where just translated reciprocal functions are summed up, such that the poles of the cotangent function and the reciprocal functions match:

$$\pi \cdot \cot(\pi x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{x+n}.$$

This identity can be proven with the Herglotz trick.[11] By combining the $-n$ -th with the n -th term, it can be expressed as an absolutely convergent series:

$$\pi \cdot \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}.$$

Relationship to exponential function and complex numbers

It can be shown from the series definitions[12] that the sine and cosine functions are the imaginary and real parts, respectively, of the complex exponential function when its argument is purely imaginary:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This identity is called Euler's formula. In this way, trigonometric functions become essential in the geometric interpretation of complex analysis. For example, with the above identity, if one considers the unit circle in the complex plane, parametrized by e^{ix}, and as above, we can parametrize this circle in terms of cosines and sines, the relationship between the complex exponential and the trigonometric functions becomes more apparent.

Euler's formula can also be used to derive some trigonometric identities, by writing sine and cosine as:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$
$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Furthermore, this allows for the definition of the trigonometric functions for complex arguments *z*:

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = \frac{e^{iz} - e^{-iz}}{2i} = \frac{\sinh(iz)}{i}$$
$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = \frac{e^{iz} + e^{-iz}}{2} = \cosh(iz)$$

where $i^2 = -1$. The sine and cosine defined by this are entire functions. Also, for purely real *x*,

$$\cos x = \operatorname{Re}(e^{ix})$$
$$\sin x = \operatorname{Im}(e^{ix})$$

It is also sometimes useful to express the complex sine and cosine functions in terms of the real and imaginary parts of their arguments.

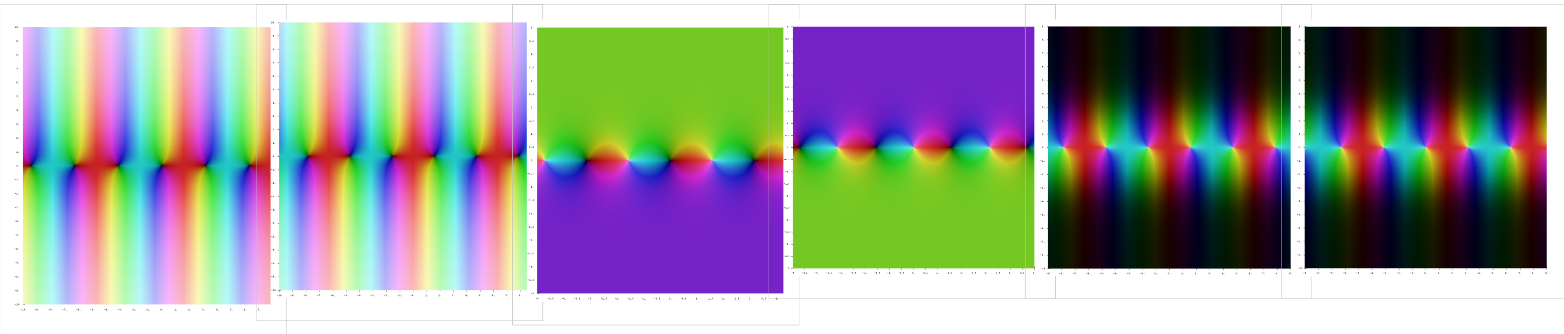
$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

This exhibits a deep relationship between the complex sine and cosine functions and their real (*sin*, *cos*) and hyperbolic real (*sinh*, *cosh*) counterparts.

Complex graphs

In the following graphs, the domain is the complex plane pictured, and the range values are indicated at each point by color. Brightness indicates the size (absolute value) of the range value, with black being zero. Hue varies with argument, or angle, measured from the positive real axis. (more)

Trigonometric functions in the complex plane



$\sin z$ $\cos z$ $\tan z$ $\cot z$ $\sec z$ $\csc z$

Definitions via differential equations

Both the sine and cosine functions satisfy the differential equation:

$$y'' = -y.$$

That is to say, each is the additive inverse of its own second derivative. Within the 2-dimensional function space *V* consisting of all solutions of this equation,

- the sine function is the unique solution satisfying the initial condition $(y'(0), y(0)) = (1, 0)$ and
- the cosine function is the unique solution satisfying the initial condition $(y'(0), y(0)) = (0, 1)$.

Since the sine and cosine functions are linearly independent, together they form a basis of *V*. This method of defining the sine and cosine functions is essentially equivalent to using Euler's formula. (See linear differential equation.) It turns out that this differential equation can be used not only to define the sine and cosine functions but also to prove the trigonometric identities for the sine and cosine functions.

Further, the observation that sine and cosine satisfies $y'' = -y$ means that they are eigenfunctions of the second-derivative operator.

The tangent function is the unique solution of the nonlinear differential equation

$$y' = 1 + y^2$$

satisfying the initial condition $y(0) = 0$. There is a very interesting visual proof that the tangent function satisfies this differential equation.

The significance of radians

Radians specify an angle by measuring the length around the path of the unit circle and constitute a special argument to the sine and cosine functions. In particular, only sines and cosines that map radians to ratios satisfy the differential equations that classically describe them. If an argument to sine or cosine in radians is scaled by frequency,

$$f(x) = \sin kx,$$

then the derivatives will scale by *amplitude*.

$$f'(x) = k \cos kx.$$

Here, k is a constant that represents a mapping between units. If x is in degrees, then

$$k = \frac{\pi}{180°}.$$

This means that the second derivative of a sine in degrees does not satisfy the differential equation

$$y'' = -y$$

but rather

$$y'' = -k^2 y.$$

The cosine's second derivative behaves similarly.

This means that these sines and cosines are different functions, and that the fourth derivative of sine will be sine again only if the argument is in radians.

Identities

Main article: List of trigonometric identities

Many identities interrelate the trigonometric functions. Among the most frequently used is the **Pythagorean identity,** which states that for any angle, the square of the sine plus the square of the cosine is 1. This is easy to see by studying a right triangle of hypotenuse 1 and applying the Pythagorean theorem. In symbolic form, the Pythagorean identity is written

$$\sin^2 x + \cos^2 x = 1$$

where $\sin^2 x + \cos^2 x = 1$ is standard notation for $(\sin x)^2 + (\cos x)^2 = 1$

Other key relationships are the **sum and difference formulas,** which give the sine and cosine of the sum and difference of two angles in terms of sines and cosines of the angles themselves. These can be derived geometrically, using arguments that date to Ptolemy. One can also produce them algebraically using Euler's formula.

Sum

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y,$$

Subtraction

$$\sin(x-y) = \sin x \cos y - \cos x \sin y,$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y.$$

These in turn lead to the following three-angle formulae:

$$\sin(x+y+z) = \sin x \cos y \cos z + \sin y \cos z \cos x + \sin z \cos y \cos x - \sin x \sin y \sin z,$$
$$\cos(x+y+z) = \cos x \cos y \cos z - \cos x \sin y \sin z - \cos y \sin x \sin z - \cos z \sin x \sin y,$$

When the two angles are equal, the sum formulas reduce to simpler equations known as the **double-angle formulae.**

$$\sin 2x = 2 \sin x \cos x,$$
$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x.$$

When three angles are equal, the three-angle formulae simplify to

$$\sin 3x = 3\sin x - 4\sin^3 x.$$
$$\cos 3x = 4\cos^3 x - 3\cos x.$$

These identities can also be used to derive the product-to-sum identities that were used in antiquity to transform the product of two numbers into a sum of numbers and greatly speed operations, much like the logarithm function.

Calculus

For integrals and derivatives of trigonometric functions, see the relevant sections of Differentiation of trigonometric functions, Lists of integrals and List of integrals of trigonometric functions. Below is the list of the derivatives and integrals of the six basic trigonometric functions. The number *C* is a constant of integration.

$f(x)$	$f'(x)$	$\int f(x)\,dx$
$\sin x$	$\cos x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sin x + C$
$\tan x$	$\sec^2 x = 1 + \tan^2 x$	$-\ln\lvert\cos x\rvert + C$
$\cot x$	$-\csc^2 x = -(1 + \cot^2 x)$	$\ln\lvert\sin x\rvert + C$
$\sec x$	$\sec x \tan x$	$\ln\lvert\sec x + \tan x\rvert + C$
$\csc x$	$-\csc x \cot x$	$-\ln\lvert\csc x + \cot x\rvert + C$

Definitions using functional equations

In mathematical analysis, one can define the trigonometric functions using functional equations based on properties like the difference formula. Taking as given these formulas, one can prove that only two real functions satisfy those conditions. Symbolically, we say that there exists exactly one pair of real functions — $\sin$ and $\cos$ — such that for all real numbers x and y, the following equation hold:

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

with the added condition that

$$0 < x\cos x < \sin x < x \text{ for } 0 < x < 1.$$

Other derivations, starting from other functional equations, are also possible, and such derivations can be extended to the complex numbers. As an example, this derivation can be used to define trigonometry in Galois fields.

Computation

The computation of trigonometric functions is a complicated subject, which can today be avoided by most people because of the widespread availability of computers and scientific calculators that provide built-in trigonometric functions for any angle. This section, however, describes details of their computation in three important contexts: the historical use of trigonometric tables, the modern techniques used by computers, and a few "important" angles where simple exact values are easily found.

The first step in computing any trigonometric function is range reduction—reducing the given angle to a "reduced angle" inside a small range of angles, say 0 to $\pi/2$, using the periodicity and symmetries of the trigonometric functions.

Main article: Generating trigonometric tables

Prior to computers, people typically evaluated trigonometric functions by interpolating from a detailed table of their values, calculated to many significant figures. Such tables have been available for as long as trigonometric functions have been described (see History below), and were typically generated by repeated application of the half-angle and angle-addition identities starting from a known value (such as $\sin(\pi/2) = 1$).

Modern computers use a variety of techniques.[13] One common method, especially on higher-end processors with floating point units, is to combine a polynomial or rational approximation (such as Chebyshev approximation, best uniform approximation, and Padé approximation, and typically for higher or variable precisions, Taylor and Laurent series) with range reduction and a table lookup—they first look up the closest angle in a small table, and then use the polynomial to compute the correction.[14] Devices that lack hardware multipliers often use an algorithm called CORDIC (as well as related techniques), which uses only addition, subtraction, bitshift, and table lookup. These methods are commonly implemented in hardware floating-point units for performance reasons.

For very high precision calculations, when series expansion convergence becomes too slow, trigonometric functions can be approximated by the arithmetic-geometric mean, which itself approximates the trigonometric function by the (complex) elliptic integral.

Main article: Exact trigonometric constants

Finally, for some simple angles, the values can be easily computed by hand using the Pythagorean theorem, as in the following examples. For example, the sine, cosine and tangent of any integer multiple of $\pi/60$ radians (3°) can be found exactly by hand.

Consider a right triangle where the two other angles are equal, and therefore are both $\pi/4$ radians (45°). Then the length of side b and the length of side a are equal; we can choose $a = b = 1$. The values of sine, cosine and tangent of an angle of $\pi/4$ radians (45°) can then be found using the Pythagorean theorem:

$$c = \sqrt{a^2 + b^2} = \sqrt{2}\,.$$

Therefore:

$$\sin\frac{\pi}{4} = \sin 45^\circ = \cos\frac{\pi}{4} = \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2},$$

$$\tan\frac{\pi}{4} = \tan 45^\circ = \frac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{1} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

To determine the trigonometric functions for angles of π/3 radians (60 degrees) and π/6 radians (30 degrees), we start with an equilateral triangle of side length 1. All its angles are π/3 radians (60 degrees). By dividing it into two, we obtain a right triangle with π/6 radians (30 degrees) and π/3 radians (60 degrees) angles. For this triangle, the shortest side = 1/2, the next largest side =(√3)/2 and the hypotenuse = 1. This yields:

$$\sin\frac{\pi}{6} = \sin 30^\circ = \cos\frac{\pi}{3} = \cos 60^\circ = \frac{1}{2}\,,$$

$$\cos\frac{\pi}{6} = \cos 30^\circ = \sin\frac{\pi}{3} = \sin 60^\circ = \frac{\sqrt{3}}{2}\,,$$

$$\tan\frac{\pi}{6} = \tan 30^\circ = \cot\frac{\pi}{3} = \cot 60^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\,.$$

Special values in trigonometric functions

There are some commonly used special values in trigonometric functions, as shown in the following table.

Function	$0\ (0^\circ)$	$\frac{\pi}{12}\ (15^\circ)$	$\frac{\pi}{8}\ (22.5^\circ)$	$\frac{\pi}{6}\ (30^\circ)$	$\frac{\pi}{4}\ (45^\circ)$	$\frac{\pi}{3}\ (60^\circ)$	$\frac{5\pi}{12}\ (75^\circ)$	$\frac{\pi}{2}\ (90^\circ)$
sin	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{2-\sqrt{2}}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1
cos	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0
tan	0	$2-\sqrt{3}$	$\sqrt{2}-1$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$2+\sqrt{3}$	∞ [15]
cot	∞	$2+\sqrt{3}$	$\sqrt{2}+1$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	$2-\sqrt{3}$	0
sec	1	$\sqrt{6}-\sqrt{2}$	$\sqrt{2}\sqrt{2-\sqrt{2}}$	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	$\sqrt{6}+\sqrt{2}$	∞
csc	∞	$\sqrt{6}+\sqrt{2}$	$\sqrt{2}\sqrt{2+\sqrt{2}}$	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	$\sqrt{6}-\sqrt{2}$	1

The symbol ∞ here represents the point at infinity on the real projective line, the limit on the extended real line is $+\infty$ on one side and $-\infty$ on the other.

Inverse functions

Main article: Inverse trigonometric functions

The trigonometric functions are periodic, and hence not injective, so strictly they do not have an inverse function. Therefore to define an inverse function we must restrict their domains so that the trigonometric function is bijective. In the following, the functions on the left are *defined* by the equation on the right; these are not proved identities. The principal inverses are usually defined as:

Function	Definition	Value Field
$\arcsin x = y$	$\sin y = x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\arccos x = y$	$\cos y = x$	$0 \le y \le \pi$
$\arctan x = y$	$\tan y = x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\operatorname{arccsc} x = y$	$\csc y = x$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$
$\operatorname{arcsec} x = y$	$\sec y = x$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$
$\operatorname{arccot} x = y$	$\cot y = x$	$0 < y < \pi$

The notations $\sin^{-1}$ and $\cos^{-1}$ are often used for arcsin and arccos, etc. When this notation is used, the inverse functions could be confused with the multiplicative inverses of the functions. The notation using the "arc-" prefix avoids such confusion, though "arcsec" can be confused with "arcsecond".

Just like the sine and cosine, the inverse trigonometric functions can also be defined in terms of infinite series. For example,

$$\arcsin z = z + \left(\frac{1}{2}\right)\frac{z^3}{3} + \left(\frac{1\cdot 3}{2\cdot 4}\right)\frac{z^5}{5} + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\frac{z^7}{7} + \cdots .$$

These functions may also be defined by proving that they are antiderivatives of other functions. The arcsine, for example, can be written as the following integral:

$$\arcsin z = \int_0^z (1-x^2)^{-1/2}\,dx, \quad |z| < 1.$$

Analogous formulas for the other functions can be found at Inverse trigonometric functions. Using the complex logarithm, one can generalize all these functions to complex arguments:

$$\arcsin z = -i \log\left(iz + \sqrt{1 - z^2}\right),$$

$$\arccos z = -i \log\left(z + \sqrt{z^2 - 1}\right),$$

$$\arctan z = \frac{1}{2} i \log\left(\frac{1 - iz}{1 + iz}\right).$$

Connection to the inner product

In an inner product space, the angle between two non-zero vectors is defined to be

$$\operatorname{angle}(x, y) = \arccos \frac{\langle x, y\rangle}{\|x\| \cdot \|y\|}.$$

Properties and applications

Main article: Uses of trigonometry

The trigonometric functions, as the name suggests, are of crucial importance in trigonometry, mainly because of the following two results.

Law of sines

The **law of sines** states that for an arbitrary triangle with sides *a*, *b*, and *c* and angles opposite those sides *A*, *B* and *C*:

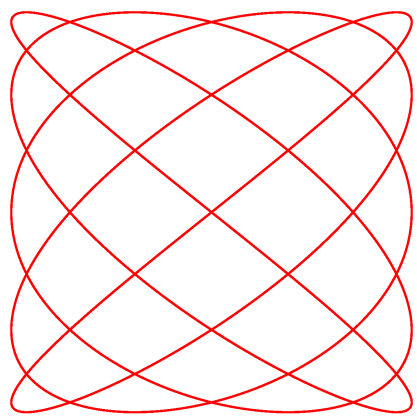
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc},$$

where Δ is the area of the triangle, or, equivalently,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R,$$

where *R* is the triangle's circumradius.

It can be proven by dividing the triangle into two right ones and using the above definition of sine. The law of sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known. This is a common situation occurring in *triangulation*, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.



A Lissajous curve, a figure formed with a trigonometry-based function.

Law of cosines

The **law of cosines** (also known as the cosine formula or cosine rule) is an extension of the Pythagorean theorem:

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

or equivalently,

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

In this formula the angle at *C* is opposite to the side *c*. This theorem can be proven by dividing the triangle into two right ones and using the Pythagorean theorem.

The law of cosines can be used to determine a side of a triangle if two sides and the angle between them are known. It can also be used to find the cosines of an angle (and consequently the angles themselves) if the lengths of all the sides are known.

Law of tangents

Main article: Law of tangents

The following all form the law of tangents[16]

$$\frac{\tan\frac{A-B}{2}}{\tan\frac{A+B}{2}} = \frac{a-b}{a+b}$$

$$\frac{\tan\frac{A-C}{2}}{\tan\frac{A+C}{2}} = \frac{a-c}{a+c}$$

$$\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}} = \frac{b-c}{b+c}$$

The explanation of the formulae in words would be cumbersome, but the patterns of sums and differences; for the lengths and corresponding opposite angles, are apparent in the theorem.

Law of cotangents

Main article: Law of cotangents

If

$$\zeta = \sqrt{\frac{1}{s}(s-a)(s-b)(s-c)}$$

(the radius of the inscribed circle for the triangle) and

$$s = \frac{a+b+c}{2}$$

(the semi-perimeter for the triangle), then the following all form the law of cotangents[17]

$$\cot\frac{A}{2} = \frac{s-a}{\zeta}$$

$$\cot\frac{B}{2} = \frac{s-b}{\zeta}$$

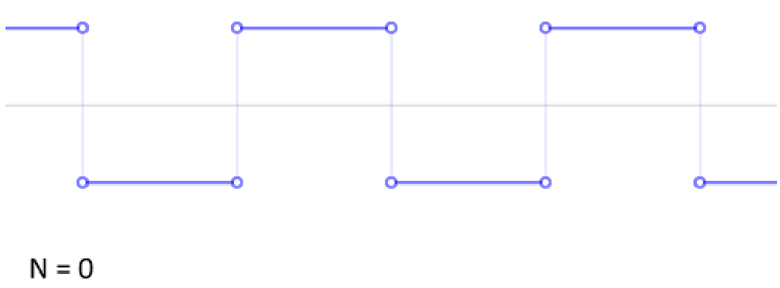
$$\cot\frac{C}{2} = \frac{s-c}{\zeta}$$

It follows that

$$\frac{\cot(A/2)}{s-a} = \frac{\cot(B/2)}{s-b} = \frac{\cot(C/2)}{s-c}.$$

In words the theorem is: the cotangent of a half-angle equals the ratio of the semi-perimeter minus the opposite side to the said angle, to the inradius for the triangle.

Periodic functions



An animation of the additive synthesis of a square wave with an increasing number of harmonics

The trigonometric functions are also important in physics. The sine and the cosine functions, for example, are used to describe simple harmonic motion, which models many natural phenomena, such as the movement of a mass attached to a spring and, for small angles, the pendular motion of a mass hanging by a string. The sine and cosine functions are one-dimensional projections of uniform circular motion.

Trigonometric functions also prove to be useful in the study of general periodic functions. The characteristic wave patterns of periodic functions are useful for modeling recurring phenomena such as sound or light waves.

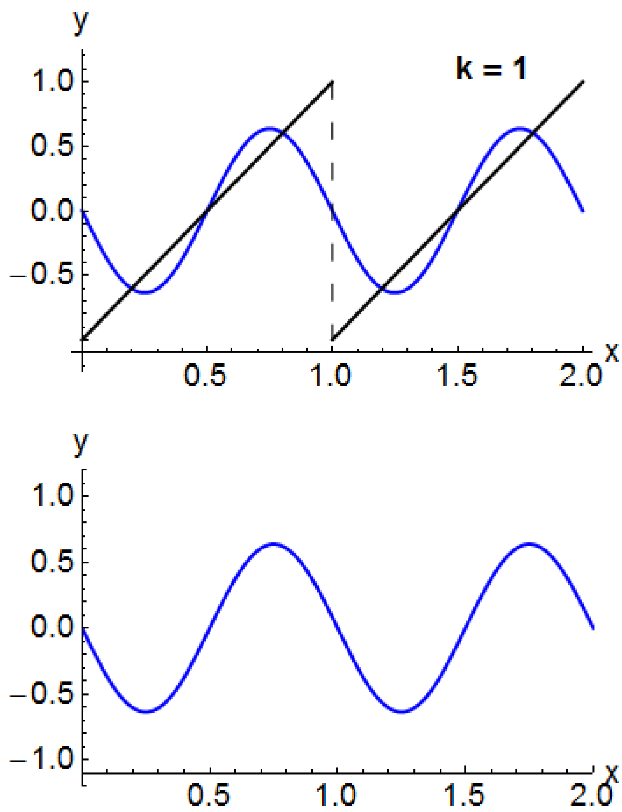
Under rather general conditions, a periodic function *f*(*x*) can be expressed as a sum of sine waves or cosine waves in a Fourier series.[18] Denoting the sine or cosine basis functions by φ_k, the expansion of the periodic function *f*(*t*) takes the form:

$$f(t) = \sum_{k=1}^{\infty} c_k \varphi_k(t).$$

For example, the square wave can be written as the Fourier series

$$f_{\text{square}}(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin\left((2k-1)t\right)}{2k-1}.$$

In the animation of a square wave at top right it can be seen that just a few terms already produce a fairly good approximation. The superposition of several terms in the expansion of a sawtooth wave are shown underneath.



Sinusoidal basis functions (bottom) can form a sawtooth wave (top) when added. All the basis functions have nodes at the nodes of the sawtooth, and all but the fundamental (k = 1) have additional nodes. The oscillation seen about the sawtooth when k is large is called the Gibbs phenomenon

History

Main article: History of trigonometric functions

While the early study of trigonometry can be traced to antiquity, the trigonometric functions as they are in use today were developed in the medieval period. The chord function was discovered by Hipparchus of Nicaea (180–125 BC) and Ptolemy of Roman Egypt (90–165 AD).

The functions sine and cosine can be traced to the *jyā* and *koti-jyā* functions used in Gupta period Indian astronomy (*Aryabhatiya*, *Surya Siddhanta*), via translation from Sanskrit to Arabic and then from Arabic to Latin.[19]

All six trigonometric functions in current use were known in Islamic mathematics by the 9th century, as was the law of sines, used in solving triangles. al-Khwārizmī produced tables of sines, cosines and tangents. They were studied by authors including Omar Khayyám, Bhāskara II, Nasir al-Din al-Tusi, Jamshīd al-Kāshī (14th century), Ulugh Beg (14th century), Regiomontanus (1464), Rheticus, and Rheticus' student Valentinus Otho.Wikipedia:Citation needed

Madhava of Sangamagrama (c. 1400) made early strides in the analysis of trigonometric functions in terms of infinite series.

The first published use of the abbreviations 'sin', 'cos', and 'tan' is by the 16th century French mathematician Albert Girard.

In a paper published in 1682, Leibniz proved that sin x is not an algebraic function of x.

Leonhard Euler's *Introductio in analysin infinitorum* (1748) was mostly responsible for establishing the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting "Euler's formula", as well as the near-modern abbreviations *sin., cos., tang., cot., sec.,* and *cosec.*[]

A few functions were common historically, but are now seldom used, such as the chord ($\mathrm{crd}(\theta) = 2\sin(\theta/2)$), the versine ($\mathrm{versin}(\theta) = 1 - \cos(\theta) = 2\sin^2(\theta/2)$) (which appeared in the earliest tables), the haversine ($\mathrm{haversin}(\theta) = \mathrm{versin}(\theta) / 2 = \sin^2(\theta/2)$), the exsecant ($\mathrm{exsec}(\theta) = \sec(\theta) - 1$) and the excosecant ($\mathrm{excsc}(\theta) = \mathrm{exsec}(\pi/2 - \theta) = \csc(\theta) - 1$). Many more relations between these functions are listed in the article about trigonometric identities.

Etymologically, the word *sine* derives from the Sanskrit word for half the chord, *jya-ardha*, abbreviated to *jiva*. This was transliterated in Arabic as *jiba*, written *jb*, vowels not being written in Arabic. Next, this transliteration was mis-translated in the 12th century into Latin as *sinus*, under the mistaken impression that *jb* stood for the word *jaib*, which means "bosom" or "bay" or "fold" in Arabic, as does *sinus* in Latin.[20] Finally, English usage converted the Latin word *sinus* to *sine*. The word *tangent* comes from Latin *tangens* meaning "touching", since the line *touches* the circle of unit radius, whereas *secant* stems from Latin *secans* — "cutting" — since the line *cuts* the circle.

Notes

[1] Oxford English Dictionary, sine, *n.*2

[2] Oxford English Dictionary, cosine, *n.*

[3] Oxford English Dictionary, tangent, *adj. and n.*

[4] Oxford English Dictionary, secant, *adj.* and *n.*

[5] Heng, Cheng and Talbert, "Additional Mathematics" (http://books.google.com/books?id=ZZoxLiJBwOUC&pg=PA228), page 228

[6] See Maor (1998)

[7] See Ahlfors, pages 43–44.

[8] Abramowitz; Weisstein.

[9] Stanley, Enumerative Combinatorics, Vol I., page 149

[10] Stanley, Enumerative Combinatorics, Vol I

[11] , Extract of page 327 (http://books.google.com/books?id=CC0dQxtYb6kC&pg=PA327)

[12] For a demonstration, see Euler's formula#Using power series

[13] Kantabutra.

[14] However, doing that while maintaining precision is nontrivial, and methods like Gal's accurate tables, Cody and Waite reduction, and Payne and Hanek reduction algorithms can be used.

[15] Abramowitz, Milton and Irene A. Stegun, p.74

[16] The Universal Encyclopaedia of Mathematics, Pan Reference Books, 1976, page 529. English version George Allen and Unwin, 1964. Translated from the German version Meyers Rechenduden, 1960.

[17] The Universal Encyclopaedia of Mathematics, Pan Reference Books, 1976, page 530. English version George Allen and Unwin, 1964. Translated from the German version Meyers Rechenduden, 1960.

[18] See for example,

[19] Boyer, Carl B. (1991). A History of Mathematics (Second ed.). John Wiley & Sons, Inc.. ISBN 0-471-54397-7, p. 210.

[20] See Maor (1998), chapter 3, regarding the etymology.

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- GonioLab (http://glab.trixon.se/): Visualization of the unit circle, trigonometric and hyperbolic functions
- Dave's draggable diagram. (http://www.clarku.edu/~djoyce/trig/) (Requires java browser plugin)

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Ric, Dan100, Dan1232112321, Danerrific, Danny, Dantheman531, Darth Panda, DatDoo, Davewild, David Levy, DavidStern, Davidkazuhiro, Dbtfz, DeadEyeArrow, Deathnroll, Deb 617, Debivort, Dekisugi, Dennisamilcar, DennyColt, Deor, DerHexer, Dhp1080, Dinosaursrule42, Dirac1933, Dirtyasianmen, Discospinster, Dmcq, Dmr2, Doc Tropics, DocRushing, Doctormatt, Donarreiskoffer, Donner60, Dougweller, Doulos Christos, Dr Smith, DrBob, Dratman, Dspradau, Dtgm, Dtreed, Dude1818, Duncan, Dusti, Dylan Lake, Dysfunktionall, E2eamon, Echion2, Edcolins, Edgar181, Editor plus, Eequor, Ehrenkater, Ela112, Enviroboy, Epbr123, Eric119, Ericoides, Eustress, Everyking, Excirial, Explicit, FJPB, Faithlessthewonderboy, Favonian, Fieldday-sunday, Fighterf4u, Flszen, Fnlayson, Fredrik, Frymaster, Full Shunyata, Funnyfarmofdoom, Furrykef, Fyyer, Fæ, GLaDOS, Gaff, Gandalf61, Gap9551, Gareth Griffith-Jones, Geekygator, Generaltater, Geoboe84, Georgebush3, Gfoley4, Gianluigi, Gibbsncis, Giftlite, Gilliam, Giro720, 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