Spectrum Representation (2B)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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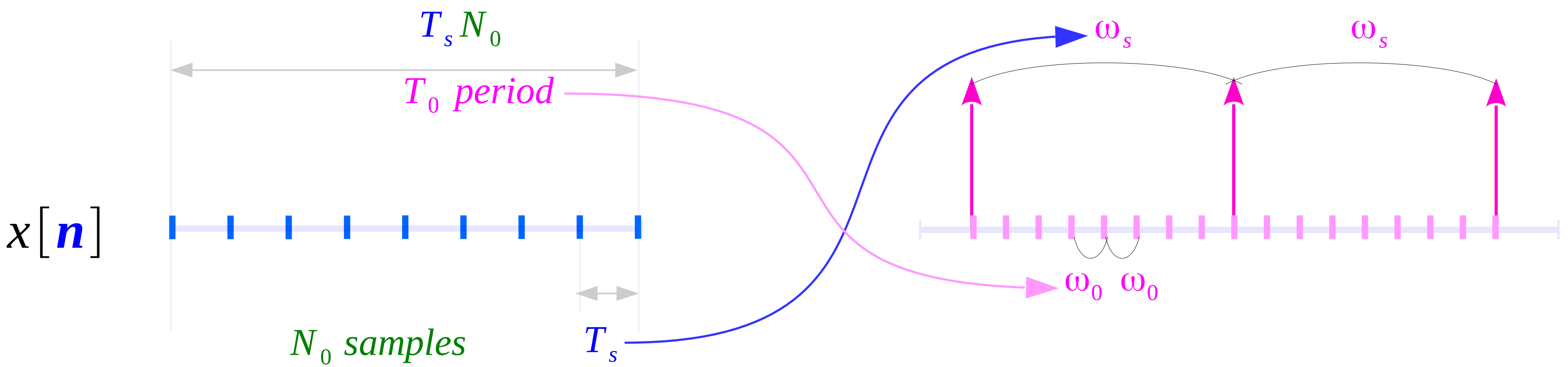
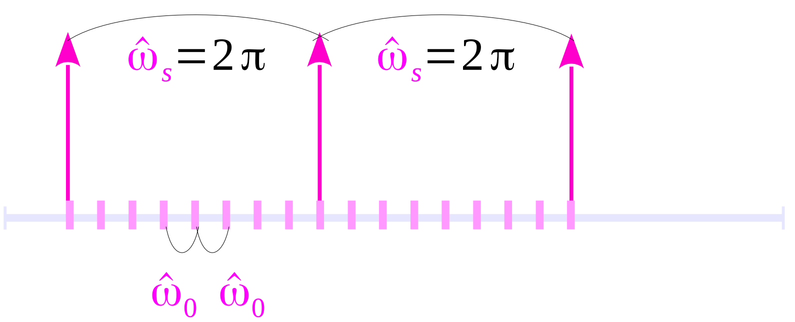
ω_s and ω_0





$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$$\hat{\omega}_s = \frac{2\pi}{1}$$

Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \iff x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \iff x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \iff x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Discrete Time Fourier **Transform**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \iff x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

Computation at $k\omega_0$

CTFS — **Periodic x(t)**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \qquad k\omega_0$$

$$@ \quad k\omega_0 = k\left(\frac{2\pi}{T}\right) \quad rad/sec$$

CTFT — **Aperiodic x(t)**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \qquad \omega \leftarrow k\omega_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

DTFS — **Periodic x[n]**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \qquad k\hat{\omega}_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

DTFT — **Aperiodic x[n]**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \qquad \hat{\omega} \leftarrow k\hat{\omega}_0$$

$$@ \quad k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right) \quad rad/sec$$

Computations using DFT

CTFS — Periodic x(t)

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k \approx \frac{1}{N}\, \mathbf{DFT}\{x(nT_s)\}$$

$k\omega_0$

$$@\ \ k\omega_0 = k\left(\frac{2\pi}{T}\right)\ \ rad/sec$$

CTFT — Aperiodic x(t)

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(jk\omega_0) \approx T_s\, \mathbf{DFT}\{x(nT_s)\}$$

$\omega \leftarrow k\omega_0$

$$@\ \ k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right)\ \ rad/sec$$

DTFS — Periodic x[n]

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\gamma[k] = \frac{1}{N}\, \mathbf{DFT}\{x[n]\}$$

$k\hat{\omega}_0$

$$@\ \ k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right)\ \ rad/sec$$

DTFT — Aperiodic x[n]

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$X(jk\hat{\omega}_0) \approx \mathbf{DFT}\{x[n]\}$$

$\hat{\omega} \leftarrow k\hat{\omega}_0$

$$@\ \ k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{NT_s}\right)\ \ rad/sec$$

Computations using DFT

CTFS — **Periodic x(t)**

$$C_k \approx \frac{1}{N} \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx N\, \boldsymbol{IDFT}\{C_k\}$$

CTFT — **Aperiodic x(t)**

$$X(jk\omega_0) \approx T_s\, \boldsymbol{DFT}\{x(nT_s)\}$$

$$x(nT_s) \approx \frac{1}{T_s} \boldsymbol{IDFT}\{X(jk\omega_0)\}$$

DTFS — **Periodic x[n]**

$$\gamma[k] = \frac{1}{N} \boldsymbol{DFT}\{x[n]\}$$

$$x[n] = N\, \boldsymbol{IDFT}\{\gamma_k\}$$

DTFT — **Aperiodic x[n]**

$$X(jk\hat{\omega}_0) \approx \boldsymbol{DFT}\{x[n]\}$$

$$x[n] \approx \boldsymbol{IDFT}\{X(jk\hat{\omega}_0)\}$$

FFT Amplitude and Power Spectrum

Two-Sided Amplitude Spectrum

$$A_k = \frac{1}{N}|X[k]| \qquad (V/Hz^{-1/2})$$

$$= \frac{1}{N}\sqrt{\Re^2\{X[k]\}+\Im^2\{X[k]\}}$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

One-Sided Amplitude Spectrum

$$\bar{A}_0 = \frac{1}{N}|X[0]| \qquad k=0$$

$$\bar{A}_k = \frac{2}{N}|X[k]| \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f = k\frac{f_s}{N}$$

Two-Sided Power Spectrum

$$P_k = \frac{1}{N^2}|X[k]|^2 \qquad (V^2/Hz^{-1})$$

$$= \frac{1}{N^2}\left(\Re^2\{X[k]\}+\Im^2\{X[k]\}\right)$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

One-Sided Power Spectrum

$$\bar{P}_0 = \frac{1}{N^2}|X[0]|^2 \qquad k=0$$

$$\bar{P}_k = \frac{2}{N^2}|X[k]|^2 \qquad k=1,2,\cdots,N/2$$

Frequency Bin

$$f = k\frac{f_s}{N}$$

FFT Amplitude and Phase Spectrum

Two-Sided Amplitude Spectrum

$$A_k = \frac{1}{N}|X[k]|$$

$$= \frac{1}{N}\sqrt{\Re^2\{X[k]\}+\Im^2\{X[k]\}}$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

Two-Sided Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im\{X[k]\}}{\Re\{X[k]\}}\right)$$

$$k = 0,1,2,\cdots,N/2,N/2+1,\cdots,N-1$$

Frequency Bin

$$f = k\frac{f_s}{N}$$

CTFS and Power Spectrum

Two-Sided Power Spectrum

$$\frac{1}{N^2}|X[k]|^2 = |C_k|^2 = \frac{1}{4}(a_k^2 + b_k^2) = \frac{1}{4}|g_k|^2$$

Single-Sided Power Spectrum

$$\frac{2}{N^2}|X[k]|^2 = 2|C_k|^2 = \frac{1}{2}|g_k|^2 = |g_{k,rms}|^2$$

$$C_k = \frac{1}{2} g_{+k}\, e^{+j\phi_k} \quad (k > 0)$$

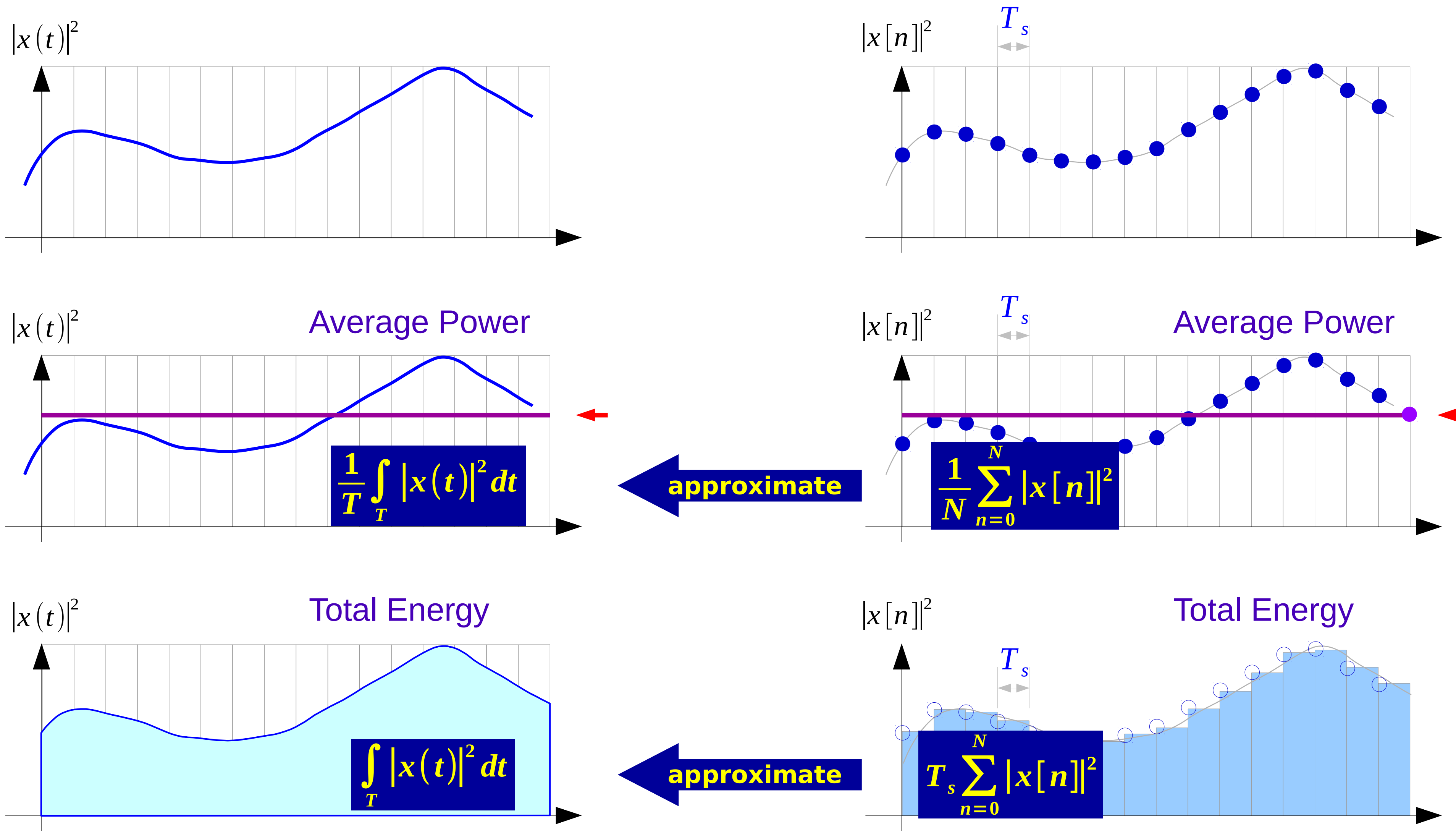
$$C_k = \frac{1}{2} g_{-k}\, e^{-j\phi_k} \quad (k < 0)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

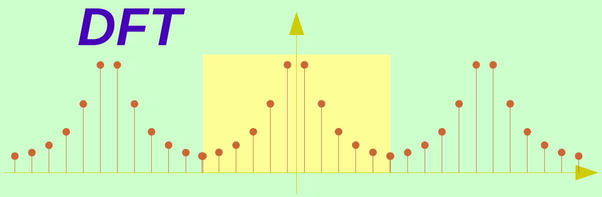
g_k *each sinusoid's amplitude*

$g_{k,rms}$ *each sinusoid's amplitude rms value*

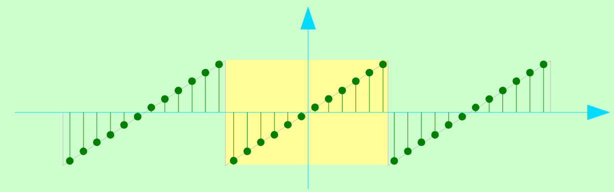
Average Power and Total Energy



Parseval's Theorem for DFT



$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$



$$\sum_{n=0}^{N-1}|x[n]|^2$$

Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1}P_{xx}[k] = \frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2$$

Average Power

$$\frac{1}{N^2}\sum_{k=0}^{N-1}|X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$$

Periodogram

$$P_{xx}[k] = \frac{1}{N}|X[k]|^2 = \left|\frac{X[k]}{\sqrt{N}}\right|^2$$

Parseval's Theorem

$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2 = \sum_{n=0}^{N-1}|x[n]|^2$$

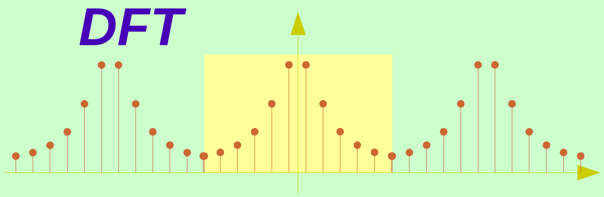
Total Energy

$$T_s\sum_{k=0}^{N-1}P_{xx}[k] = \frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2$$

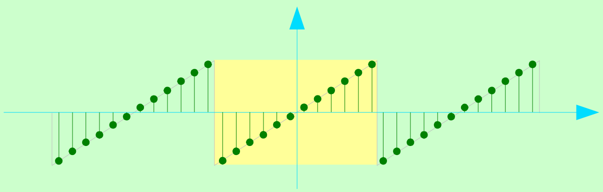
Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2 = T_s\sum_{n=0}^{N-1}|x[n]|^2$$

Periodogram as a frequency domain samples



$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2$$



$$\sum_{n=0}^{N-1}|x[n]|^2$$

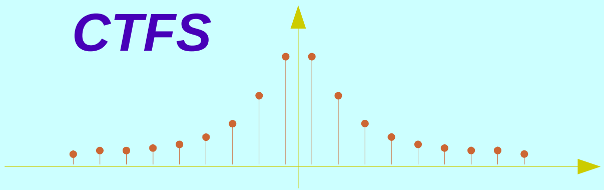
Average Power $\quad \frac{1}{N}\sum_{k=0}^{N-1} P_{xx}[k] = \frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2 \quad$ Average Power

freq-domain samples $\qquad$ time-domain samples

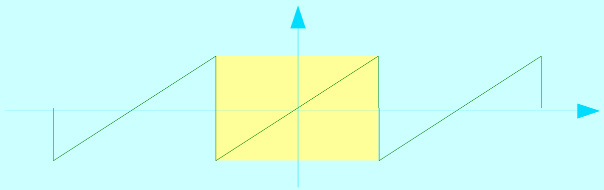
Total Energy $\quad T_s\sum_{k=0}^{N-1} P_{xx}[k] = T_s\sum_{n=0}^{N-1}|x[n]|^2 \quad$ Total Energy

Parseval's Theorem

CTFS



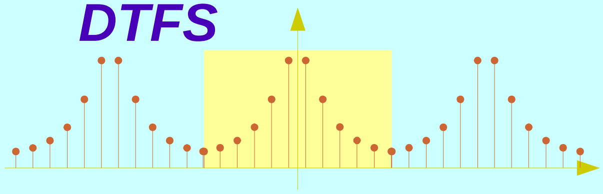
$$\sum_{k=-\infty}^{+\infty} |C_k|^2$$



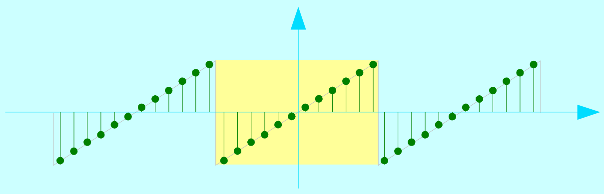
Average Power

$$\frac{1}{T}\int_T |x(t)|^2 \, dt$$

DTFS



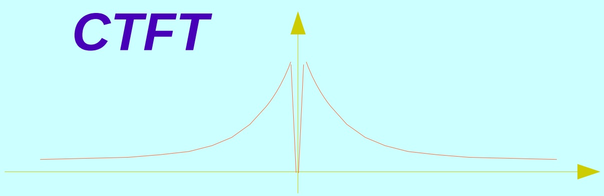
$$\sum_{k=0}^{N-1} |\gamma[k]|^2$$



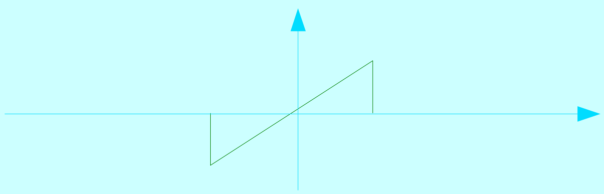
Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

CTFT



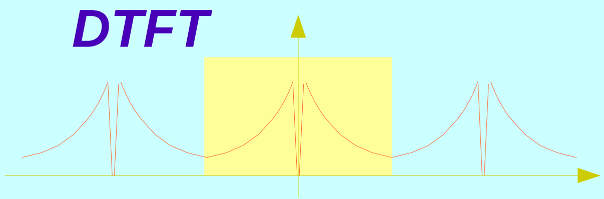
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega$$



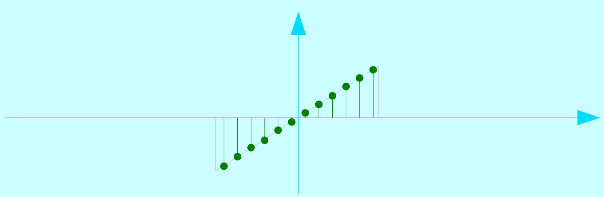
Total Energy

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

DTFT



$$\frac{T_s}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2 \, d\hat{\omega}$$

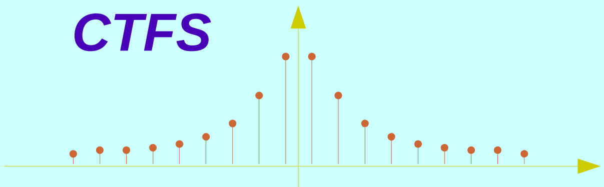


Total Energy

$$T_s \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

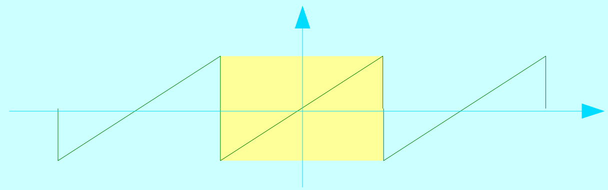
Approximate **CTFS** Parseval's Theorem

CTFS



Average Power

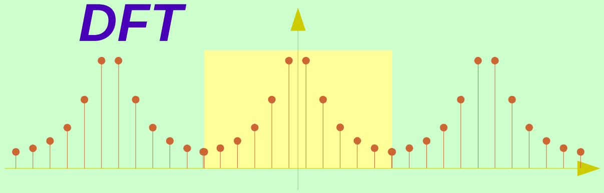
$$\sum_{k=-\infty}^{+\infty} |C_k|^2$$



Average Power

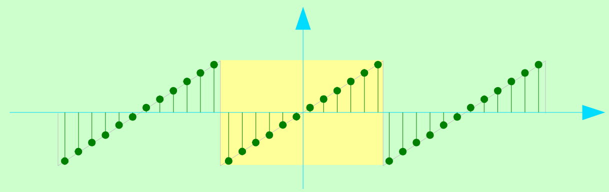
$$\frac{1}{T}\int_T |x(t)|^2\,dt$$

DFT



Average Power

$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$



Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$C_k \approx \frac{X[k]}{N}$$

$$\sum_{k=0}^{N-1} |C_k|^2 \;\Rightarrow\; \sum_{k=0}^{N-1} \left|\frac{X[k]}{N}\right|^2 = \frac{T_s}{T}\sum_{n=0}^{N-1} |x[n]|^2 \;\Leftarrow\; \frac{1}{T}\int_T |x(t)|^2\,dt$$

1

$$C_k \approx \frac{1}{N}\,\boldsymbol{DFT}\{x(nT_s)\}$$

$$\frac{T_s}{T} = \frac{T_s}{N\,T_s}$$

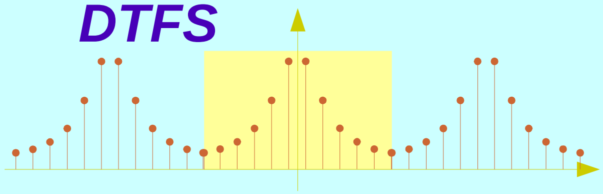
$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

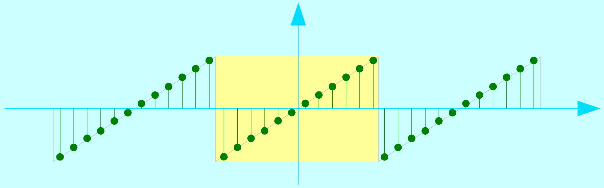
DTFS Parseval's Theorem

DTFS



Average Power

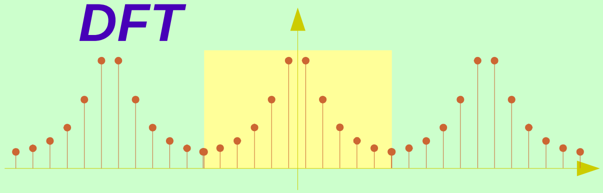
$$\sum_{k=0}^{N-1} |\gamma[k]|^2$$



Average Power

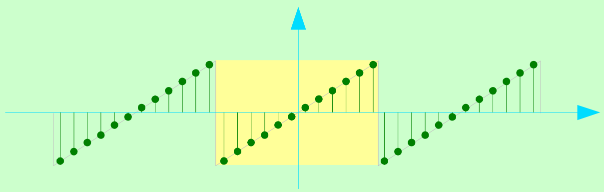
$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

DFT



Average Power

$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$



Average Power

$$\frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

$$\gamma[k] = \frac{X[k]}{N}$$

$$\sum_{k=0}^{N-1} |\gamma[k]|^2 \quad \Rightarrow \quad \sum_{k=0}^{N-1} \left|\frac{X[k]}{N}\right|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2 \quad \Leftarrow \quad \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

2

$$\gamma[k] = \frac{1}{N} \boldsymbol{DFT}\{x[n]\}$$

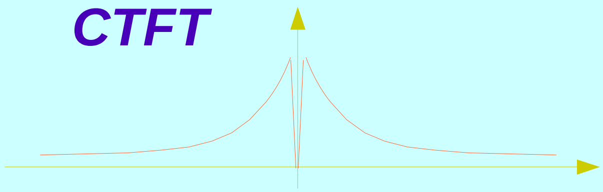
$$\frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |x[n]|^2$$

Average Power

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

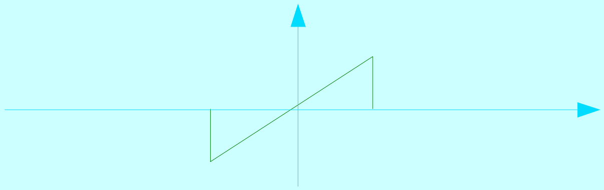
Approximate **CTFT** Parseval's Theorem

CTFT



Total Energy

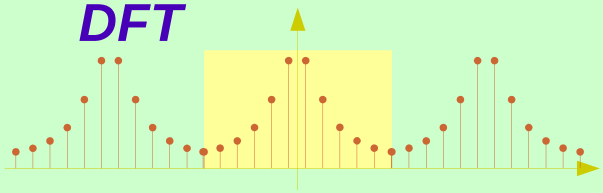
$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2\,d\omega$$



Total Energy

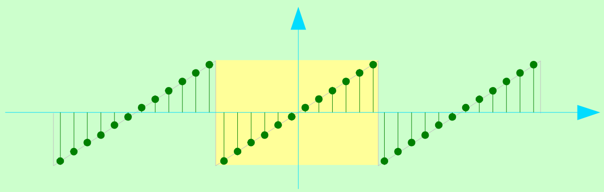
$$\int_{-\infty}^{+\infty}|x(t)|^2\,dt$$

DFT



Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2$$



Total Energy

$$T_s\sum_{n=0}^{N-1}|x[n]|^2$$

$$X(jk\omega_0)\approx T_s X[k]$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2\,d\omega \quad\Rightarrow\quad \frac{1}{N\,T_s}\sum_{k=0}^{N-1}|T_s X[k]|^2 = T_s\sum_{n=0}^{N-1}|x[n]|^2 \quad\Leftarrow\quad \int_{-\infty}^{+\infty}|x(t)|^2\,dt$$

$$\frac{1}{2\pi}\omega_0 = \frac{1}{T_0}$$

3 $X(jk\omega_0)\approx T_s\,\boldsymbol{DFT}\{x(nT_s)\}$

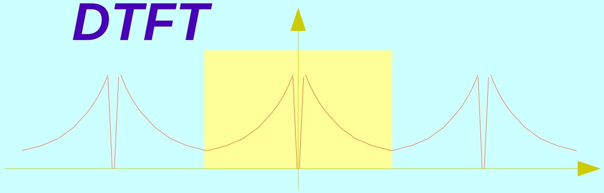
$$\frac{T_s}{N}\sum_{k=0}^{N-1}|X[k]|^2 = T_s\sum_{n=0}^{N-1}|x[n]|^2$$

Total Energy

$$\frac{1}{N}\sum_{k=0}^{N-1}|X[k]|^2 = \sum_{n=0}^{N-1}|x[n]|^2$$

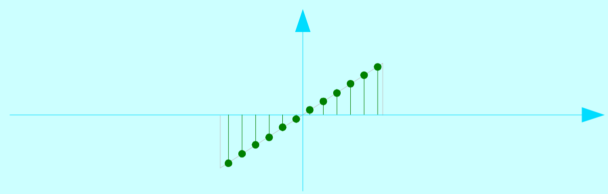
Approximate **DTFT** Parseval's Theorem

DTFT



Total Energy

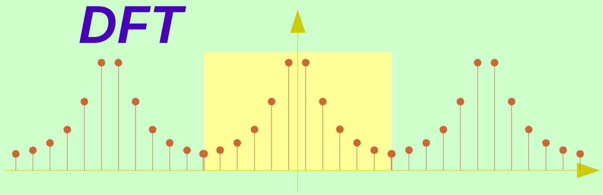
$$\frac{T_s}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2 \, d\hat{\omega}$$



Total Energy

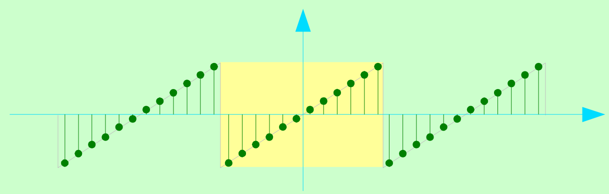
$$T_s \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

DFT



Total Energy

$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2$$



Total Energy

$$T_s \sum_{n=0}^{N-1} |x[n]|^2$$

$$X(jk\hat{\omega}_0) \approx X[k]$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\hat{\omega})|^2 \, d\hat{\omega} \quad \Rightarrow \quad \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2 \quad \Leftarrow \quad \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

$$\frac{1}{2\pi}\hat{\omega}_0 = \frac{1}{N}$$

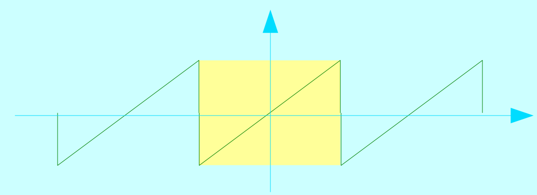
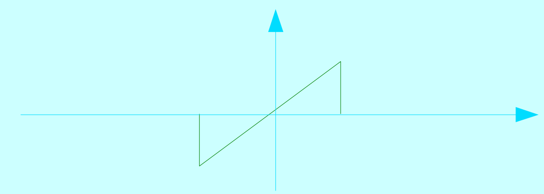
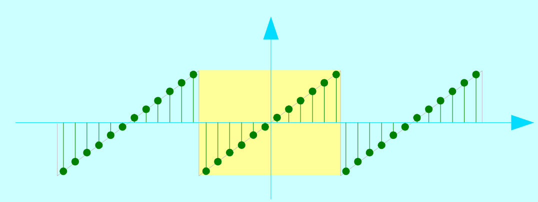
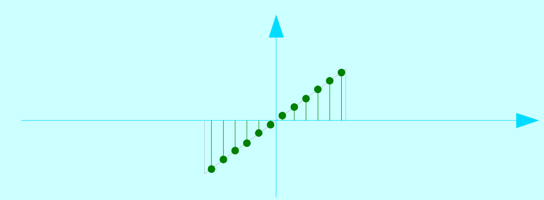
4 $X(jk\hat{\omega}_0) \approx \mathbf{DFT}\{x[n]\}$

$$\frac{T_s}{N}\sum_{k=0}^{N-1} |X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Total Energy

$$\frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2 = \sum_{n=0}^{N-1} |x[n]|^2$$

Average Power and Total Energy

	Periodic Signals	Aperiodic Signals
	Average Power	**Total Energy**
Continuous Time	 Average Power $\frac{1}{T}\int_{T} \lvert x(t)\rvert^2 dt$	 Total Energy $\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$
Discrete Time	 Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$	 Total Energy $T_s \sum_{n=-\infty}^{+\infty} \lvert x[n]\rvert^2$

Parseval's Theorem

	Periodic Signals	Aperiodic Signals
	Average Power	Total Energy
Continuous Time	***CTFS*** Average Power $\frac{1}{T}\int_T \lvert x(t)\rvert^2 dt = \sum_{k=-\infty}^{+\infty} \lvert C_k\rvert^2$	***CTFT*** Total Energy $\int_T \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\omega)\rvert^2 d\omega$
Discrete Time	***DTFS*** Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \sum_{k=0}^{N-1} \lvert y[k]\rvert^2$	***DTFT*** Total Energy $T_s \sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{T_s}{2\pi}\int_{2\pi} \lvert X(j\hat{\omega})\rvert^2 d\hat{\omega}$

Average Power and Total Energy

Periodic Signals	Aperiodic Signals
Average Power	Total Energy

	Periodic	Aperiodic
Continuous Time	***CTFS*** Average Power $\frac{1}{T}\int_{T} \lvert x(t)\rvert^2 dt$ $\xrightarrow{\cdot T}$	***CTFT*** Total Energy $\int_{T} \lvert x(t)\rvert^2 dt$
Discrete Time	***DTFS*** Average Power $\frac{1}{N}\sum_{n=0}^{N-1} \lvert x[n]\rvert^2$ $\xrightarrow{\cdot T}$ $T = N \cdot T_s$	***DTFT*** Total Energy $T_s \sum_{n=0}^{N-1} \lvert x[n]\rvert^2$

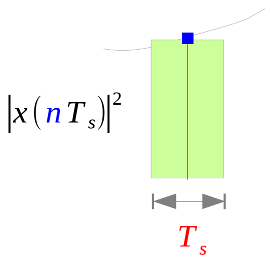
Average Power

Total Energy

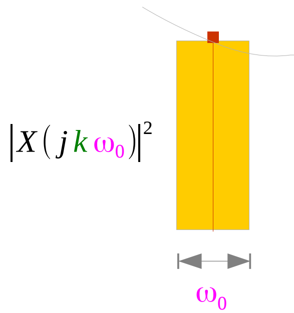
DFT Approximation of Parseval's Theorem

	Periodic Signals	Aperiodic Signals
	Average Power	Total Energy
Continuous Time	CTFS Average Power $\frac{1}{N^2}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$	CTFT Total Energy $\frac{T_s}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = T_s\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$
Discrete Time	DTFS Average Power $\frac{1}{N^2}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = \frac{1}{N}\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$	DTFT Total Energy $\frac{T_s}{N}\sum_{k=0}^{N-1}\lvert X[k]\rvert^2 = T_s\sum_{n=0}^{N-1}\lvert x[n]\rvert^2$

Integration Approximation



$$dt \leftarrow T_s$$



$$d\omega \leftarrow \omega_0$$

$$d\hat{\omega} \leftarrow \hat{\omega}_0$$

$$\int_T |x(t)|^2 \, dt$$

$$\sum_{n=0}^{N-1} |x(nT_s)|^2 T_s \quad = T_s \sum_{n=0}^{N-1} |x(nT_s)|^2$$

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 \, d\omega \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{N T_s}$$

$$\frac{1}{2\pi} \sum_{k=0}^{N-1} |X(jk\omega_0)|^2 \omega_0 \quad = \frac{1}{N T_s} \sum_{k=0}^{N-1} |X(jk\omega_0)|^2$$

$$\frac{1}{T} \int_T |x(t)|^2 \, dt \qquad T_0 = N T_s$$

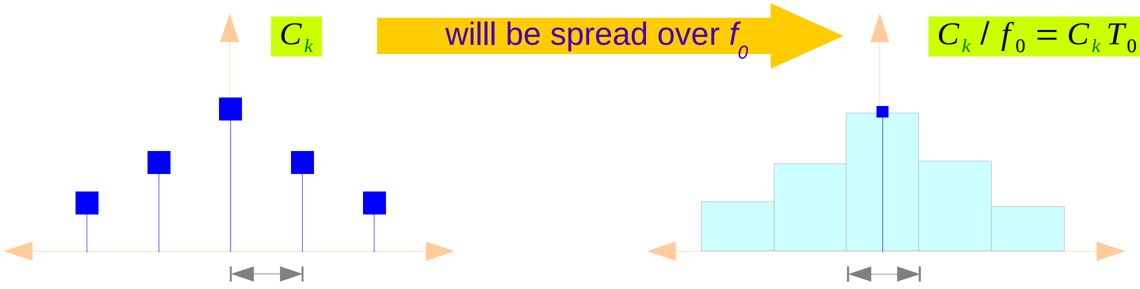
$$\frac{1}{T_0} \sum_{n=0}^{N-1} |x(nT_s)|^2 T_s \quad = \frac{1}{N} \sum_{n=0}^{N-1} |x(nT_s)|^2$$

$$\frac{T_s}{2\pi} \int_{2\pi} |X(j\hat{\omega})|^2 \, d\hat{\omega} \qquad \hat{\omega}_0 = \frac{2\pi}{T_0} T_s = \frac{2\pi}{N}$$

$$\frac{T_s}{2\pi} \sum_{k=0}^{N-1} |X(jk\hat{\omega}_0)|^2 \hat{\omega}_0 \quad = \frac{T_s}{N} \sum_{k=0}^{N-1} |X(jk\hat{\omega}_0)|^2 \hat{\omega}_0$$

Fourier Series Coefficients

	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\omega_0 = \dfrac{2\pi}{N T_s}$	$\omega_0 = \dfrac{2\pi}{N T_s} \quad \left(\hat{\omega}_0 = \dfrac{2\pi}{N}\right)$
Amplitude Spectral Density	$\dfrac{1}{N} X[k] = C_k$	$\dfrac{T_0}{N} X[k] = T_s X[k] = X(j k \omega_0)$
Frequency Bin	$k\,\omega_0 = k\left(\dfrac{2\pi}{N T_s}\right)$	$k\,\omega_0 = k\,\hat{\omega}_0 f_s = k\left(\dfrac{2\pi}{N T_s}\right)$



$$f_0 = \frac{1}{T_0} = \frac{1}{N T_s}$$

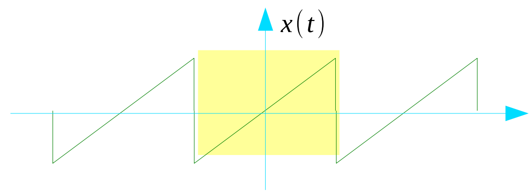
$$f_0 = \frac{1}{T_0} = \frac{1}{N T_s}$$

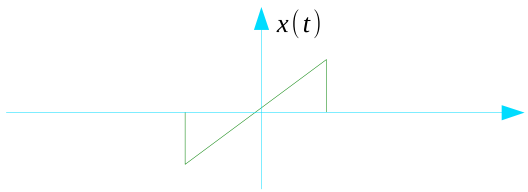
One-sided Fourier Series Coefficients

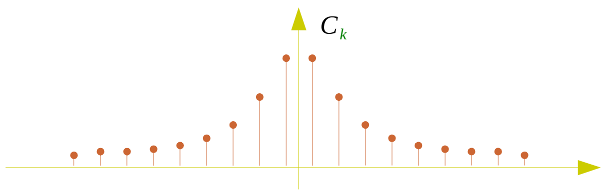
	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\omega_0 = \frac{2\pi}{N T_s} = \frac{2\pi}{T_0}$	$\omega_0 = \frac{2\pi}{N T_s} \quad \left(\hat{\omega}_0 = \frac{2\pi}{N}\right)$
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(j k \omega_0)$
One Sided F.S. Coefficient	$\frac{1}{N} X[k] \quad k=0, \frac{N}{2}$ $\frac{2}{N} X[k] \quad k=1,\cdots,\frac{N}{2}-1$	$\frac{T_0}{N} X[k] \quad k=0, \frac{N}{2}$ $\frac{2T_0}{N} X[k] \quad k=1,\cdots,\frac{N}{2}-1$
Frequency Bin	$k\omega_0 = k\left(\frac{2\pi}{N T_s}\right)$	$k\omega_0 = k\hat{\omega}_0 f_s = k\left(\frac{2\pi}{N T_s}\right)$
	Average Power	Total Energy

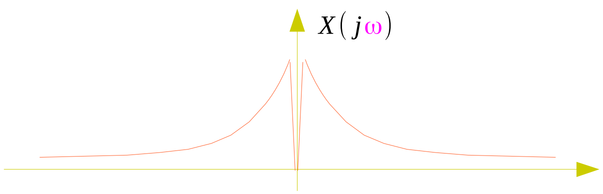
Spectral Density Functions

	Periodic Signals	**Aperiodic Signals**
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T} \int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
	Average Power	Total Energy









Amplitude Spectral Density (**ASD**)

C_k

Power Spectral Density (**PSD**)

$\sum_{k=-\infty}^{+\infty} \lvert C_k \rvert^2 \delta(f - k f_0)$

Amplitude Spectral Density (**ASD**)

$X(j\omega)$

Energy Spectral Density (**ESD**)

$\lvert X(f) \rvert^2$

DFT Approximation

	Periodic Signals	Aperiodic Signals
Two Sided F.S. Coefficient	$\frac{1}{N} X[k] = C_k$	$\frac{T_0}{N} X[k] = X(jk\omega_0)$
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T} \int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
		$\cdot\ \omega_0 \left(= \frac{2\pi}{T_0} \right)$
Approximation Of Integration	$\sum_{k=0}^{N-1} \left\lvert \frac{X[k]}{N} \right\rvert^2 = \frac{T_s}{T} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Average Power	$\frac{1}{N T_s} \sum_{k=0}^{N-1} \lvert T_s X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Total Energy
DFT Approximation	$\frac{1}{N^2} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{T_s}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$

Using Periodograms

	Periodic Signals	Aperiodic Signals
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T} \int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
DFT Approximation	$\frac{1}{N^2} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{T_s}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$
Approximation By DFT's	$\frac{1}{N} \sum_{k=0}^{N-1} \left\{ \frac{1}{N} \lvert X[k] \rvert^2 \right\} = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Averaging Operation	$T_s \sum_{k=0}^{N-1} \left\{ \frac{1}{N} \lvert X[k] \rvert^2 \right\} = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$ Integrating Operation
	Average Power	Total Energy
Using Periodograms	$\frac{1}{N} \sum_{k=0}^{N-1} P_{xx}[k] = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$T_s \sum_{k=0}^{N-1} P_{xx}[k] = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$

Using PSD & ESD

	Periodic Signals	Aperiodic Signals
Parseval's Theorem	$\sum_{k=0}^{N-1} \lvert C_k \rvert^2 = \frac{1}{T} \int_T \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\omega) \rvert^2 d\omega = \int_{-\infty}^{+\infty} \lvert x(t) \rvert^2 dt$
DFT Approximation	$\frac{1}{N^2} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = \frac{1}{N} \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{T_s}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2 = T_s \sum_{n=0}^{N-1} \lvert x[n] \rvert^2$
Average Power & Total Energy	$\sum_{k=0}^{N-1} \mathbf{PSD}[k] = \textit{Average Power}$	$\sum_{k=0}^{N-1} \mathbf{ESD}[k] = \textit{Total Energy}$
Approximated PSD & ESD	$\mathbf{PSD}[k] = \frac{1}{N^2} \lvert X[k] \rvert^2$	$\mathbf{ESD}[k] = \frac{T_s}{N} \lvert X[k] \rvert^2$

Using PSD, ESD, and Periodograms

	Periodic Signals		Aperiodic Signals	
ASD	$\frac{1}{N} X[k] = C_k$		$\frac{T_0}{N} X[k] = X(jk\omega_0)$	
Approximated PSD & ESD	$\frac{\lvert X[k]\rvert^2}{N^2} = \boldsymbol{PSD}[k]$		$\frac{T_s}{N} \lvert X[k]\rvert^2 = \boldsymbol{ESD}[k]$	
Using PSD & ESD	$\sum_{k=0}^{N-1} \boldsymbol{PSD}[k]$	Sum(**PSD**[k])	$\sum_{k=0}^{N-1} \boldsymbol{ESD}[k]$	Sum(**ESD**[k])
Average Power, Total Enegy	$\sum_{k=0}^{N-1} \frac{1}{N^2} \lvert X[k]\rvert^2$	Average Power	$\sum_{k=0}^{N-1} \frac{T_s}{N} \lvert X[k]\rvert^2$	Total Energy
Using Periodograms	$\frac{1}{N} \sum_{k=0}^{N-1} P_{xx}[k]$	Averaging	$T_s \sum_{k=0}^{N-1} P_{xx}[k]$	Integrating
Approximated Periodograms	$\frac{1}{N} \lvert X[k]\rvert^2 = P_{xx}[k]$		$\frac{1}{N} \lvert X[k]\rvert^2 = P_{xx}[k]$	

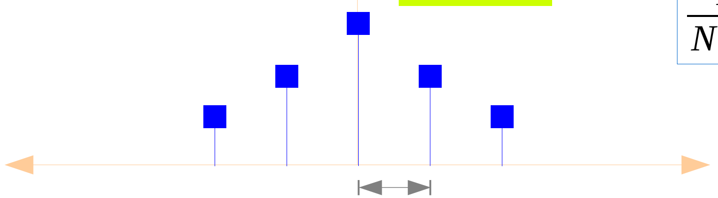
FS Coefficients of Periodic and Aperiodic Signals

	Periodic Signals	Aperiodic Signals
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$
Two Sided F.S. Coefficient	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$
One Sided F.S. Coefficient	$\frac{1}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2}{N} X(k) \quad k=1, \cdots, \frac{N}{2}-1$	$\frac{\Delta t}{N} X(k) \quad k=0, \frac{N}{2}$ $\frac{2 \Delta t}{N} X(k) \quad k=1, \cdots, \frac{N}{2}-1$
Frequency Bin	$k \Delta f$	$k \Delta f$
	Average Power	Total Energy

Power Spectrum and Power Spectral Density

Power Spectrum

$$\frac{1}{N}|X[k]|^2$$



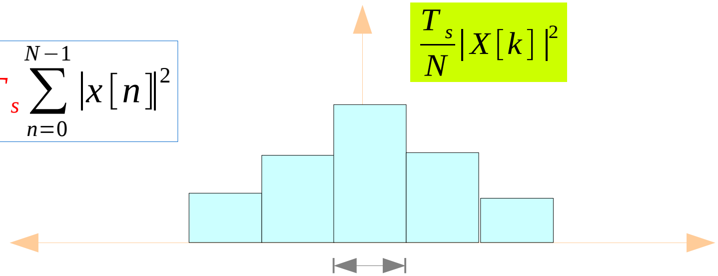
$$\Delta f = \frac{1}{N} f_s = \frac{1}{N T}$$

$$\frac{1}{N T_s} \sum_{k=0}^{N-1} |T_s X[k]|^2 = T_s \sum_{n=0}^{N-1} |x[n]|^2$$

Power Spectral Density

Hz

$$\frac{T_s}{N}|X[k]|^2$$



$$\Delta f = \frac{1}{N} f_s = \frac{1}{N T_s}$$

$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$\Rightarrow \sum_{k=0}^{N-1} S[k] \Delta f$$

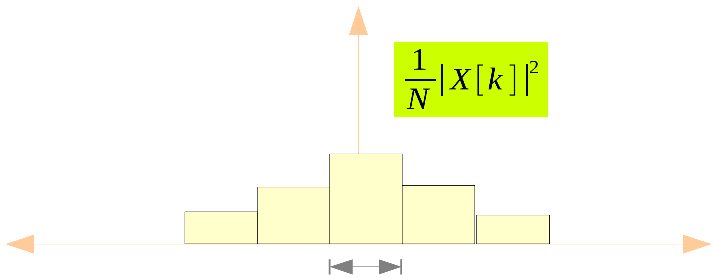
$$= \frac{1}{N T_s} \sum_{k=0}^{N-1} S[k] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$S[k] = \frac{T_s}{N} |X[k]|^2$$

Power Spectral Density

Hz ▪ sec

$$\frac{1}{N}|X[k]|^2$$



$$\frac{\Delta f}{f_s} = \frac{1}{N}$$

Normalized Frequency

Periodogram → Power Spectral Density

FS Coefficients of Random Signals

Random Signals

Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$
Two Sided Power Spectral Density	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided Power Spectral Density	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$ $S_1(k) = 2S(k) \quad k = 0, \frac{N}{2}$ $S_1(k) = S(k) \quad k = 1, \cdots, \frac{N}{2} - 1$
Frequency Bin	$k \Delta f$

$$\frac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j(2\pi/N)kn}$$

X = fft(*x*)

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{+j(2\pi/N)kn}$$

x = ifft(X)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

CTFS

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k\, e^{+jk\omega_0 t}$$

DTFS

fc = fft(*x*)/N = X/N

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$

Approximated
Fourier Coefficients

x = ifft(fc)*N

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$

Approximated
Power Spectrum

Periodogram using FFT

$$C_k \approx \gamma_k = \frac{X[k]}{N}$$ **Approximated Fourier Coefficients**

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2}$$ **Approximated Power Spectrum**

$$\frac{1}{N}\sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2}\sum_{k=0}^{N-1} |X[k]|^2$$ **Average Power**

$$\Rightarrow \left(\sqrt{\frac{\sum_{k=0}^{N-1} \frac{|X[k]|^2}{N}}{N}} \right)^2$$ **RMS of sq root Periodogram**

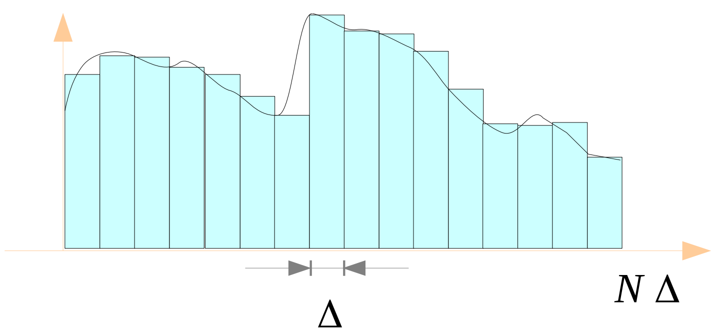
$$\frac{|X[k]|^2}{N} \quad k = 0, 1, \dots, N-1$$ **Approximated Periodogram**

$$\frac{|X[k]|}{\sqrt{N}} \quad k = 0, 1, \dots, N-1$$ **Square root Periodogram**

RMS in continuous time

$$\frac{1}{T}\int_0^T g^2(t)\, dt$$

RMS in discrete time



$$\frac{1}{N\Delta}\sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N}\sum_{k=0}^{N-1} |g[k]|^2$$

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

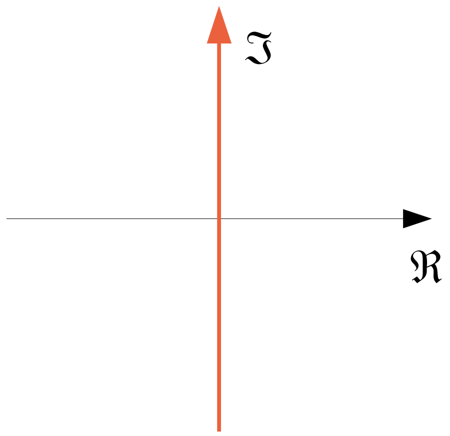
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$f(t)\, e^{-j\omega t}$

$-\infty < \omega < +\infty$

Not so useful in transient analysis

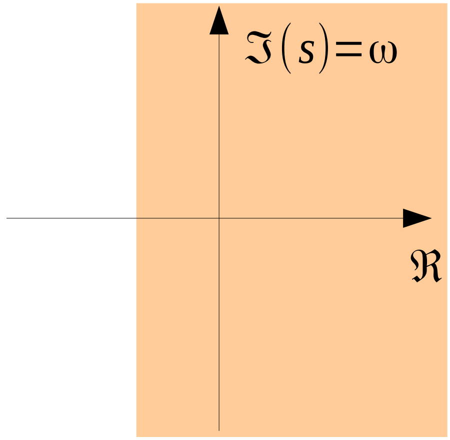


Laplace Transform

$f(t)\, e^{-st} = f(t) e^{-(\sigma + j\omega)t}$

Linear Time Domain Analysis

Initial Condition



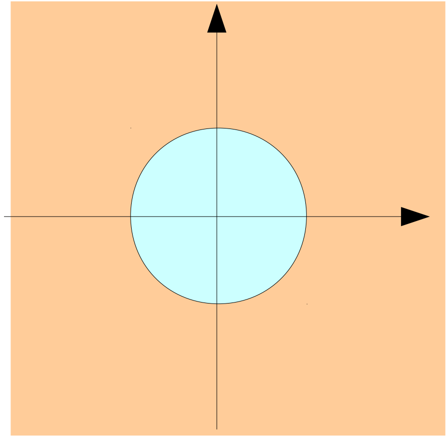
z Transform

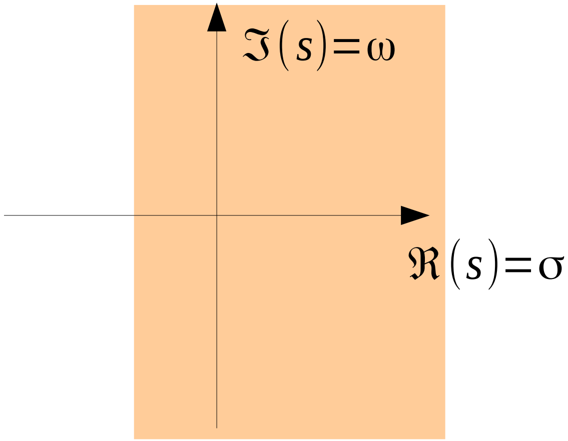
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M.J. Roberts, Fundamentals of Signals and Systems
[4] S.J. Orfanidis, Introduction to Signal Processing
[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings

[6] A “graphical interpretation” of the DFT and FFT, by Steve Mann