Spectrum Representation (2A)

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1/2/17

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Fourier Series with real coefficients

1

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

a

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

b

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\left\{ X_k e^{+jk\omega_0 t} \right\}$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

Phasor Representation X_k via g_k, ϕ_k

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \,\Re\left\{e^{+j(k\omega_0 t + \phi_k)}\right\}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \Re\left\{g_k \cdot e^{+j\phi_k} \cdot e^{+jk\omega_0 t}\right\}$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

Spectrum

Real Single-tone Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$= \Re\{X e^{j\omega_0 t}\}$$

$$= \left\{\frac{X}{2} e^{+j\omega_0 t} + \frac{X^*}{2} e^{-j\omega_0 t}\right\}$$

$$X = A e^{j\phi}$$

$$A\cos(\omega_0 t + \phi)$$

$$= \frac{A}{2}\left(e^{+j(\omega_0 t + \phi)} + e^{-j(\omega_0 t + \phi)}\right)$$

$$= \frac{A}{2}\left(e^{+j\phi} e^{+j\omega_0 t} + e^{-j\phi} e^{-j\omega_0 t}\right)$$

$$= \frac{\{A e^{+j\phi}\}}{2} e^{+j\omega_0 t} + \frac{\{A e^{+j\phi}\}^*}{2} e^{-j\omega_0 t}$$

Spectrum

Real Single-tone Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi)$$

$$= \Re\{X e^{j\omega_0 t}\}$$

$$= \left\{ \frac{X}{2} e^{+j\omega_0 t} + \frac{X^*}{2} e^{-j\omega_0 t} \right\}$$

$$X = A e^{j\phi}$$

Real Multi-tone Sinusoidal Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \phi_k)$$

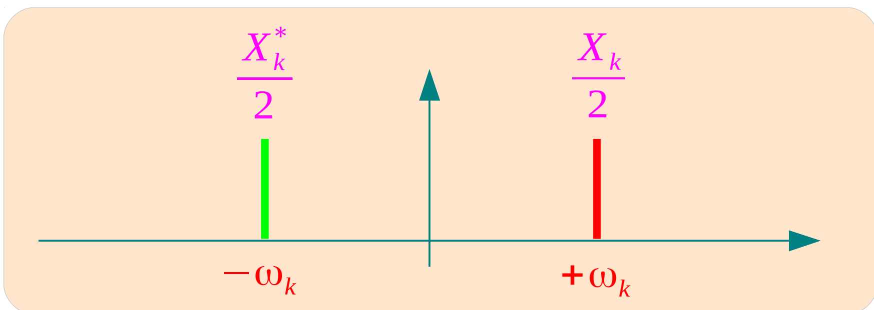
$$= X_0 + \Re\left\{ \sum_{k=1}^{N} X_k e^{j\omega_k t} \right\}$$

$$= X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{+j\omega_k t} + \frac{X_k^*}{2} e^{-j\omega_k t} \right\}$$

$$X_k = A_k e^{j\phi_k}$$ the phasor of angular frequency ω_k

$$X_0 = A_0$$

$$\frac{X_k^*}{2} e^{-j\omega_k t} \qquad \frac{X_k}{2} e^{+j\omega_k t}$$



only magnitude is display

Real Multi-tone Sinusoidal Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \phi_k)$$

$$= X_0 + \Re\left\{\sum_{k=1}^{N} X_k e^{j\omega_k t}\right\}$$

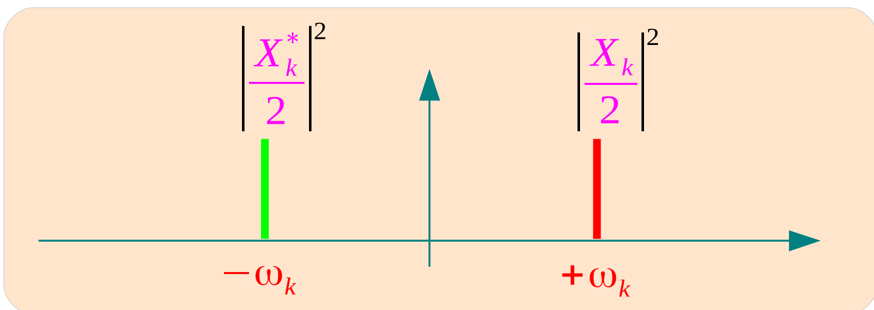
$$= X_0 + \sum_{k=1}^{N}\left\{\frac{X_k}{2} e^{+j\omega_k t} + \frac{X_k^*}{2} e^{-j\omega_k t}\right\}$$

$X_k = A_k e^{j\phi_k}$ the phasor of angular frequency ω_k

$$X_0 = A_0$$

Power Spectrum

$$\left| \frac{X_k^*}{2} e^{-j\omega_k t} \right|^2 \qquad \left| \frac{X_k}{2} e^{+j\omega_k t} \right|^2$$



only magnitude is display

$$\left|\frac{X_k^*}{2}\right|^2 = \left|\frac{X_k}{2}\right|^2 = \left|\frac{A_k}{2}\right|^2 = \frac{|g_k|^2}{4}$$

Real Multi-tone Sinusoidal Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(\omega_k t + \phi_k)$$

$$= X_0 + \Re\left\{ \sum_{k=1}^{N} X_k e^{j\omega_k t} \right\}$$

$$= X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{+j\omega_k t} + \frac{X_k^*}{2} e^{-j\omega_k t} \right\}$$

$X_k = A_k e^{j\phi_k}$ the phasor of angular frequency ω_k

$$X_0 = A_0$$

Fourier Series with complex coefficients

4

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

a

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

b

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re\{X_k e^{+jk\omega_0 t}\}$$

$$C_k = \frac{1}{2} g_{+k} \, e^{+j\phi_k} \quad (k > 0) \qquad C_k = \frac{1}{2} g_{-k} \, e^{-j\phi_k} \quad (k < 0)$$

Real Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt$$

$$a_k = \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)\,dt$$

$a_0,\ a_1,\ a_2,\ \ldots$
$b_0,\ b_1,\ b_2,\ \ldots$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k} \right)$$

$g_0,\ g_1,\ g_2,\ \ldots$
$\phi_0,\ \phi_1,\ \phi_2,\ \ldots$

Real Coefficients $a_k, \; b_k$ via $g_k, \; \phi_k$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t) \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k) \cos(k\omega_0 t) - g_k \sin(\phi_k) \sin(k\omega_0 t)$$

$$= a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

$$a_0 = g_0$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

⟺

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

Complex Coefficient C_k *via* g_k, ϕ_k

4

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

$$\cdots, C_{-1}, C_0, C_{+1}, \cdots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cdot \cos(k\omega_0 t + \phi_k)$$

$$= g_0 + \sum_{k=1}^{\infty} g_k \cdot \frac{1}{2}\left(e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)}\right)$$

$$= g_0 + \sum_{k=1}^{\infty} \left(\left[\frac{1}{2} g_k \, e^{+j\phi_k}\right] e^{+jk\omega_0 t} + \left[\frac{1}{2} g_k \, e^{-j\phi_k}\right] e^{-jk\omega_0 t}\right)$$

$$= g_0 + \sum_{k=1}^{\infty} \left(\left[C_k\right] e^{+jk\omega_0 t} + \left[C_{-k}\right] e^{-jk\omega_0 t}\right)$$

$$C_k = \frac{1}{2} g_{+k} \, e^{+j\phi_k} \quad (k > 0) \qquad C_k = \frac{1}{2} g_{-k} \, e^{-j\phi_k} \quad (k < 0)$$

$$g_0, g_1, g_2, \cdots$$

Fourier Coefficients Relationship

1 $k = 1, 2, \ldots$

$$a_0 = g_0$$

$$a_k = g_k \cos(\phi_k)$$

$$b_k = g_k \sin(\phi_k)$$

a $k = 1, 2, \ldots$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1}\left(-\frac{b_k}{a_k}\right)$$

4 $k = 0, \pm 1, \pm 2, \ldots$

$$C_k = \tfrac{1}{2} g_{+k}\, e^{+j\phi_k} \quad (k > 0)$$

$$C_k = \tfrac{1}{2} g_{-k}\, e^{-j\phi_k} \quad (k < 0)$$

b $k = 1, 2, \ldots$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

Two-Sided & One-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - j b_k) & (k > 0) \\ \frac{1}{2}(a_k + j b_k) & (k < 0) \end{cases}$$

$$C_k = \begin{cases} g_0 & (k = 0) \\ \frac{1}{2} g_{+k}\, e^{+j\phi_k} & (k > 0) \\ \frac{1}{2} g_{-k}\, e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} \tan^{-1}(-b_k / a_k) & (k > 0) \\ \tan^{-1}(+b_k / a_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} g_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$Arg(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$|C_k|^2 = |C_{-k}|^2 = \tfrac{1}{4}|g_k|^2 = \tfrac{1}{4}(a_k^2 + b_k^2)$$

Periodogram *One-Sided*

$$\tfrac{1}{2}|g_k|^2 = 2 \cdot |C_k|^2 = \tfrac{1}{2}(a_k^2 + b_k^2)$$

Power Spectrum

4

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+j k \omega_0 t}$$

$$C_k = \frac{1}{2} g_{+k} \, e^{+j\phi_k} \quad (k > 0)$$

$$C_k = \frac{1}{2} g_{-k} \, e^{-j\phi_k} \quad (k < 0)$$

Two-Sided Power Spectrum

$$|C_k|^2 = \tfrac{1}{4}\left(a_k^2 + b_k^2\right)$$

$$= \left(\tfrac{1}{2}\sqrt{a_k^2 + b_k^2}\right)^2$$

$$= \left(\tfrac{1}{2}|g_k|\right)^2 = \tfrac{1}{4}|g_k|^2$$

$$\cdots, |C_{-1}|^2, |C_0|^2, |C_{+1}|^2, \cdots$$

$$\cdots, \tfrac{1}{4}|g_1|^2, |g_0|^2, \tfrac{1}{4}|g_1|^2, \cdots$$

Single-Sided Power Spectrum

$$2\,|C_k|^2 = 2\cdot\tfrac{1}{4}|g_k|^2 = \left(\tfrac{1}{\sqrt{2}}|g_k|\right)^2 = |g_{k,rms}|^2$$

$$|g_0|^2, \tfrac{1}{2}|g_1|^2, \cdots$$

Periodogram

a

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

Periodogram

One-Sided

$$\left(\tfrac{1}{\sqrt{2}}|g_k|\right)^2 = \tfrac{1}{2}|g_k|^2 = \tfrac{1}{2}(a_k^2 + b_k^2)$$

$$|g_0|^2,\ \tfrac{1}{2}|g_1|^2,\ \tfrac{1}{2}|g_2|^2,\ \cdots$$

$$|g_k| = \sqrt{a_k^2 + b_k^2}$$

$$k = 0,\ 1,\ 2,\ \cdots$$

In Matlab, periodogram(x) is a
one-sided PSD (Power Spectral Density) estimate for **real x**
two-sided PSD (Power Spectral Density) estimate for **complex x**

Example Signal

$$x(t) = g_0 + g_1 \cos(1 \cdot 2\pi t) + g_2 \cos(2 \cdot 2\pi t) + g_3 \cos(3 \cdot 2\pi t)$$
$$= 5 \cos(2\pi t) - 7 \cos(4\pi t) + 11 \cos(6\pi t)$$

$$T_0 = 1 \qquad f = 1$$
$$\omega_0 = 2\pi f = 2\pi$$

$$g_0 = 0 \qquad 0 \cdot \omega_0 = 0$$
$$g_1 = 5 \qquad 1 \cdot \omega_0 = 2\pi$$
$$g_2 = -7 \qquad 2 \cdot \omega_0 = 4\pi$$
$$g_3 = 11 \qquad 3 \cdot \omega_0 = 6\pi$$
$$g_4 = 0 \qquad 4 \cdot \omega_0 = 8\pi$$
$$\cdots = 0$$

Real Coefficients $(a_k, \ b_k), \ (g_k, \ \phi_k)$

$$x(t) = g_0 + g_1 \cos(1 \cdot 2\pi t) + g_2 \cos(2 \cdot 2\pi t) + g_3 \cos(3 \cdot 2\pi t)$$
$$= 5 \cos(2\pi t) - 7 \cos(4\pi t) + 11 \cos(6\pi t)$$

$$g_k \cos(k\omega_0 t + \phi_k) = g_k \cos(\phi_k) \cos(k\omega_0 t) - g_k \sin(\phi_k) \sin(k\omega_0 t)$$
$$= a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_k^2 + b_k^2 = g_k^2$$
$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

$$a_0 = g_0 \qquad a_k = g_k \qquad -b_k = 0$$

$$a_0 = g_0 \qquad a_k = g_k \cos(0) \qquad -b_k = g_k \sin(0)$$

$\Leftrightarrow$

$$g_0 = a_0 \qquad g_k = \sqrt{a_k^2 + 0^2} \qquad \phi_k = \tan^{-1}\left(-\frac{0}{a_k}\right)$$

$$g_0 = a_0 \qquad g_k = a_k \qquad \phi_k = 0$$

Complex Coefficients C_k via g_k, ϕ_k

$$x(t) = g_0 + g_1 \cos(1\cdot 2\pi t) + g_2 \cos(2\cdot 2\pi t) + g_3 \cos(3\cdot 2\pi t)$$
$$= 5\cos(2\pi t) - 7\cos(4\pi t) + 11\cos(6\pi t)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cdot \cos(k\omega_0 t + \phi_k)$$
$$= g_0 + \textstyle\sum_{k=1}^{\infty} g_k \cdot \frac{1}{2}\left(e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)}\right)$$
$$= g_0 + \textstyle\sum_{k=1}^{\infty}\left(\frac{1}{2} g_k\, e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{1}{2} g_k\, e^{-j\phi_k} e^{-jk\omega_0 t}\right)$$
$$= g_0 + \sum_{k=1}^{\infty}\left(C_k\, e^{+jk\omega_0 t} + C_{-k}\, e^{-jk\omega_0 t}\right)$$

$$C_1 = C_{-1} = \frac{1}{2} g_1$$
$$C_2 = C_{-2} = \frac{1}{2} g_2$$
$$C_3 = C_{-3} = \frac{1}{2} g_3$$

$$C_k = \frac{1}{2} g_{+k}\, e^{+j0} \quad (k > 0) \qquad C_k = \frac{1}{2} g_{-k}\, e^{-j0} \quad (k < 0)$$

$g_0,\ g_1,\ g_2,\ \cdots$

Periodogram Examples

$$x(t) = g_0 + g_1 \cos(1\cdot 2\pi t) + g_2 \cos(2\cdot 2\pi t) + g_3 \cos(3\cdot 2\pi t)$$
$$= 5\cos(2\pi t) - 7\cos(4\pi t) + 11\cos(6\pi t)$$

Periodogram

One-Sided

$$\left(\frac{1}{\sqrt{2}}|g_k|\right)^2 = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$\left(\frac{1}{\sqrt{2}}|g_1|\right)^2 = \frac{1}{2}|5|^2 = \frac{25}{2}$$

$$\left(\frac{1}{\sqrt{2}}|g_2|\right)^2 = \frac{1}{2}|-7|^2 = \frac{49}{2}$$

$$\left(\frac{1}{\sqrt{2}}|g_3|\right)^2 = \frac{1}{2}|11|^2 = \frac{121}{2}$$

Power Spectrum Examples

$$x(t) = g_0 + g_1 \cos(1 \cdot 2\pi t) + g_2 \cos(2 \cdot 2\pi t) + g_3 \cos(3 \cdot 2\pi t)$$
$$= 5 \cos(2\pi t) - 7 \cos(4\pi t) + 11 \cos(6\pi t)$$

Power Spectrum

$$|C_k|^2 = \tfrac{1}{4}(a_k^2 + b_k^2)$$
$$= \left(\tfrac{1}{2}\sqrt{a_k^2 + b_k^2}\right)^2$$
$$= \left(\tfrac{1}{2}|g_k|\right)^2$$

Two-Sided

$$\cdots, C_{-1}, C_0, C_{+1}, \cdots$$

$$|C_1|^2 = |C_{-1}|^2 = \frac{1}{4}|g_1|^2 = \frac{25}{4}$$

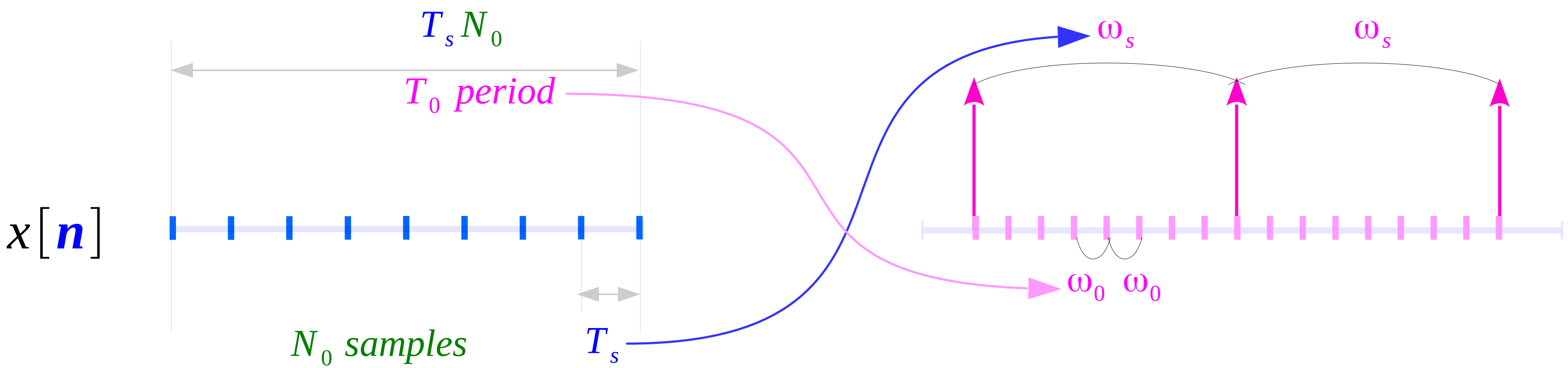
$$|C_2|^2 = |C_{-2}|^2 = \frac{1}{4}|g_2|^2 = \frac{49}{4}$$

$$|C_3|^2 = |C_{-3}|^2 = \frac{1}{4}|g_3|^2 = \frac{121}{4}$$

ω_s and ω_0

		$T_s = 1 \cdot T_s$	$T_0 = N_0 \cdot T_s$
		replication frequency	*frequency resolution*
$\omega = \dfrac{2\pi}{T}$	**Continuous Time**	$\omega_s = \dfrac{2\pi}{T_s}$	$\omega_0 = \dfrac{2\pi}{T_0}$
$\omega = \dfrac{\hat{\omega}}{T_s}$	**Discrete Time**	$\hat{\omega}_s = \dfrac{2\pi}{1}$	$\hat{\omega}_0 = \dfrac{2\pi}{N_0}$
		normalized	*normalized*

ω_s and ω_0

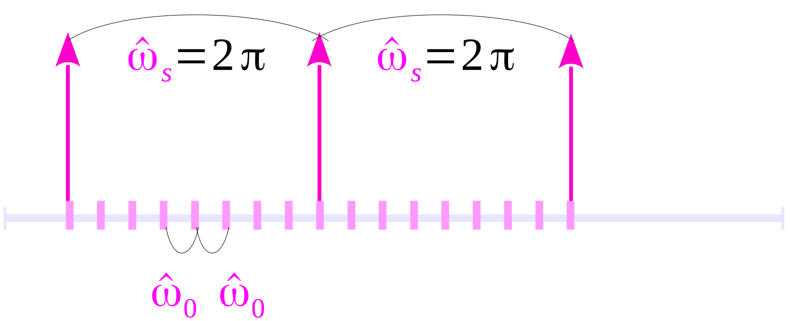


$$\omega_0 = \frac{2\pi}{T_0}$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$

$$\hat{\omega}_s = \frac{2\pi}{1}$$

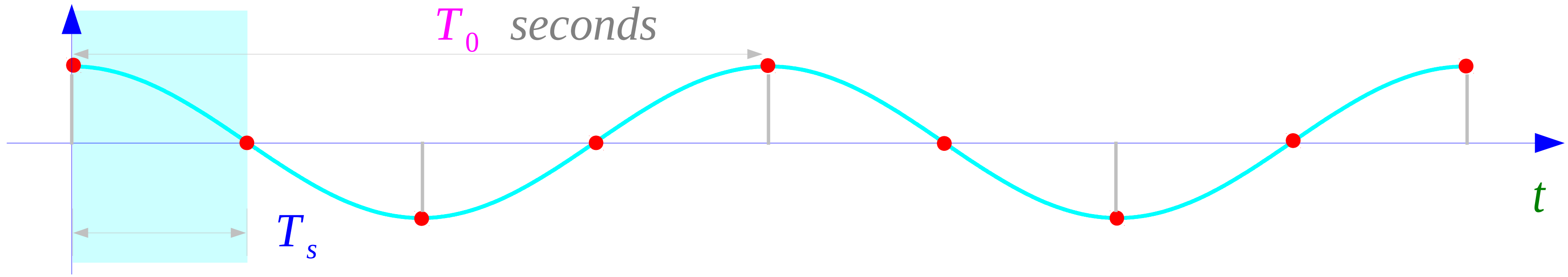


Frequency and Digital Frequency

Continuous Time

$x(t) = \cos(\omega_0 t)$

$$\omega_0 = \frac{2\pi}{T_0}$$ *rad/sec*



$x[n] = x(nT_s)$

$= \cos(n\omega_0 T_s)$

$= \cos(n\hat{\omega}_0)$

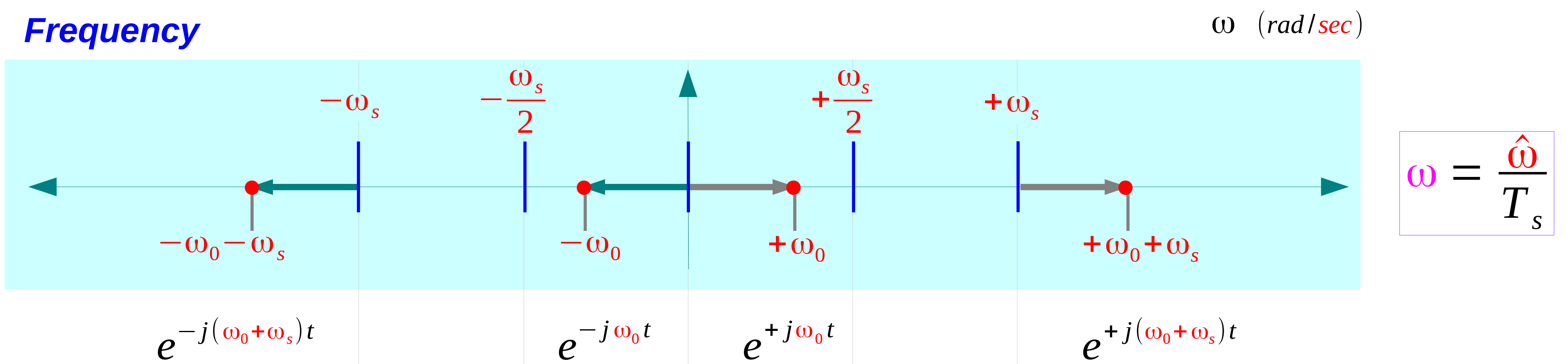
Discrete Time

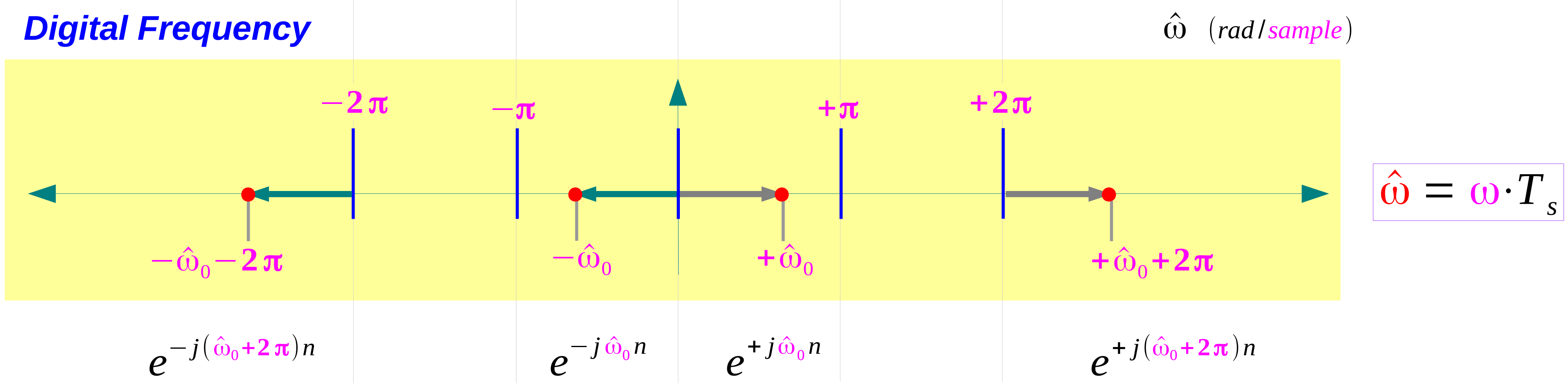
$$\hat{\omega}_0 = \frac{2\pi}{N_0}$$ *rad/sample*

$$\omega_0 = \frac{2\pi}{N_0 T_s}$$

$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{f_s}$$

Frequency and Digital Frequency





Power Spectrum Definitions (1)

$$\sum_{n=0}^{N} |x[n]|^2$$ **Sum** squared amplitude

$$\frac{1}{T}\int_T |x(t)|^2 dt \approx \frac{1}{N}\sum_{n=0}^{N} |x[n]|^2$$ **Mean** squared amplitude

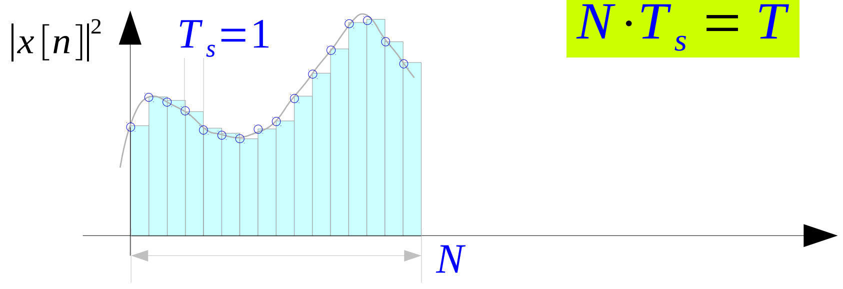
$$\int_T |x(t)|^2 dt \approx T_s \sum_{n=0}^{N} |x[n]|^2$$ **Time-integral** squared amplitude

Power Spectrum Definitions (2)

$N \cdot T_s = T$

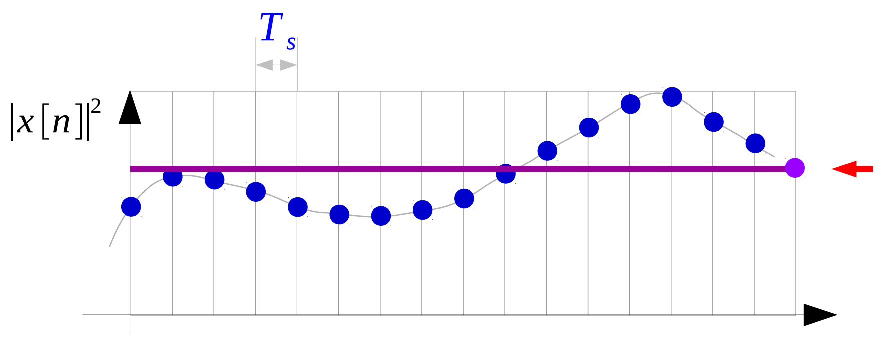
Sum squared amplitude

$$\sum_{n=0}^{N} |x[n]|^2$$



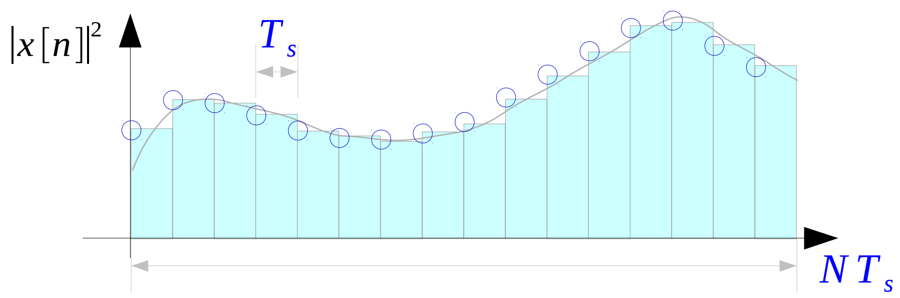
Mean squared amplitude

$$\frac{1}{T}\int_T |x(t)|^2 dt \approx \frac{1}{N}\sum_{n=0}^{N} |x[n]|^2$$

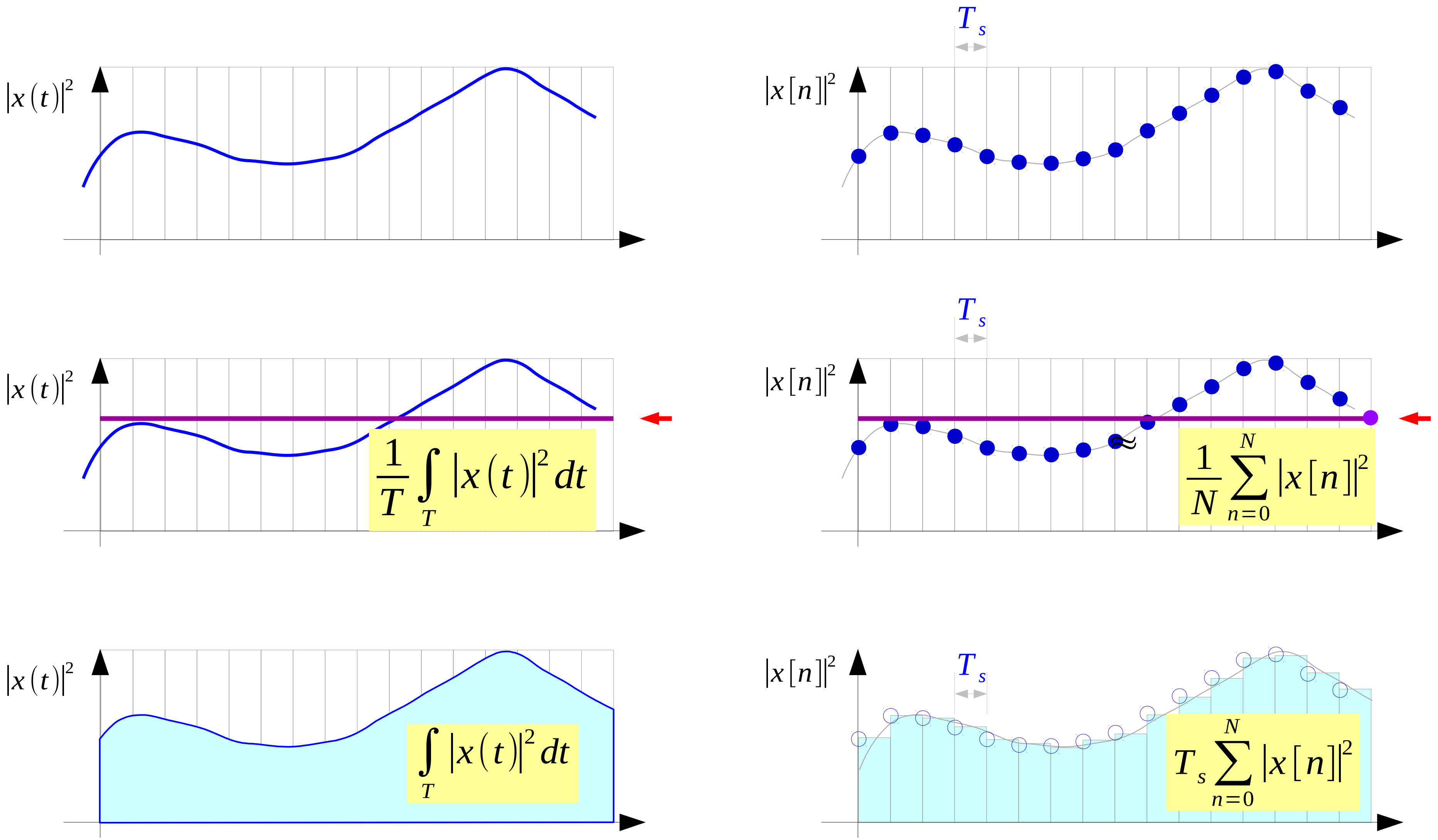


Time-integral squared amplitude

$$\int_T |x(t)|^2 dt \approx T_s \sum_{n=0}^{N} |x[n]|^2$$



Power Spectrum – Continuous & Discrete Signals



Power Spectrum & Average Power

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t}$$

Power Spectrum

$$|C_k|^2 = \tfrac{1}{4}(a_k^2 + b_k^2)$$

$$C_k = \frac{1}{T} \int_0^T x(t) \, e^{-jk\omega_0 t} \, dt$$

$$C_k^* = \frac{1}{T} \int_0^T x^*(t) \, e^{+jk\omega_0 t} \, dt$$

Average Power

$$P = \frac{1}{T} \int_T |x(t)|^2 dt \quad = \frac{1}{T} \int_T x(t) x^*(t) dt$$

$$= \frac{1}{T} \int_T x^*(t) \sum_{k=-\infty}^{+\infty} C_k \, e^{+jk\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} C_k \int_T x^*(t) \, e^{+jk\omega_0 t} dt$$

$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} C_k \, T \, C_k^* \quad = |C_k|^2$$

Energy Spectrum & Total Energy

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Energy Spectrum

$$|X(j\omega)|^2$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X^*(j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{+j\omega t}\, dt$$

Total Energy

$$E = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t) x^*(t)\, dt$$

$$= \int_{-\infty}^{+\infty} x^*(t) \left[\frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega \right] dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) \left[\int_{-\infty}^{+\infty} x^*(t)\, e^{+j\omega t}\, dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) X^*(j\omega)\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

Fourier Transform Types

Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$

Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Leftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$

Continuous Time Fourier **Transform**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

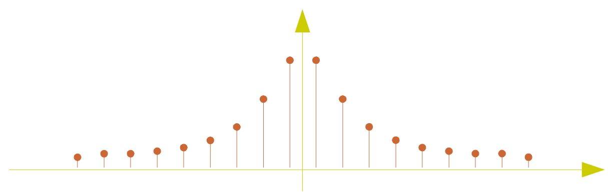
Discrete Time Fourier **Transform**

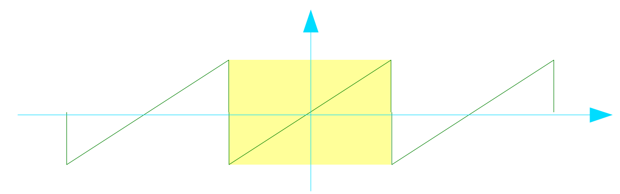
$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n} \quad \Leftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega} n}\, d\hat{\omega}$$

Continuous Time – **CTFS**

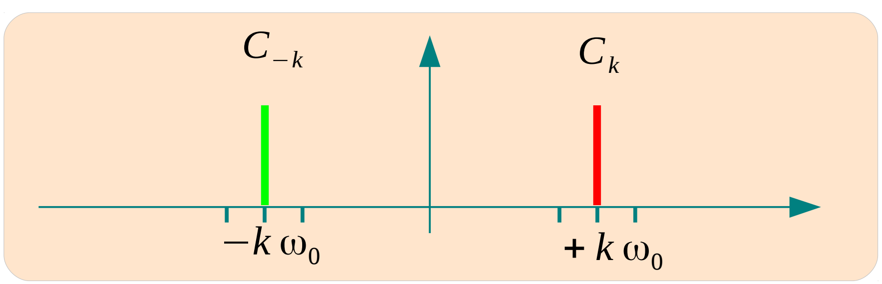
Continuous Time Fourier **Series**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{+\infty} C_k\, e^{+jk\omega_0 t}$$





$$\sum_{k=-\infty}^{+\infty} |C_k|^2 \qquad \frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

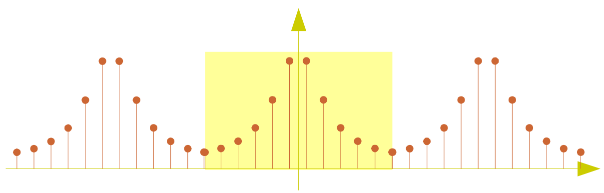


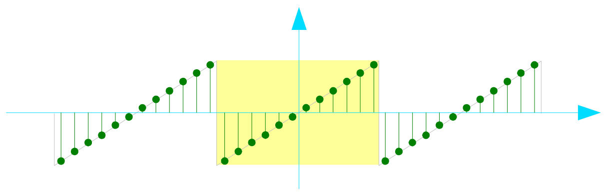
$$C_{-k}\, e^{-jk\omega_0 t} \qquad C_k\, e^{+jk\omega_0 t}$$

Discrete Time – **DTFS**

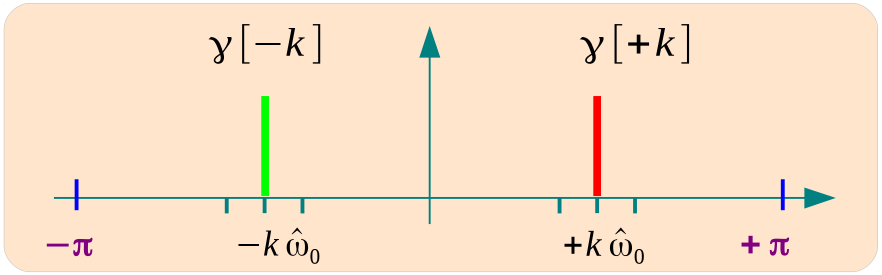
Discrete Time Fourier **Series**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n} \quad \Longleftrightarrow \quad x[n] = \sum_{k=0}^{N-1} \gamma[k]\, e^{+jk\hat{\omega}_0 n}$$





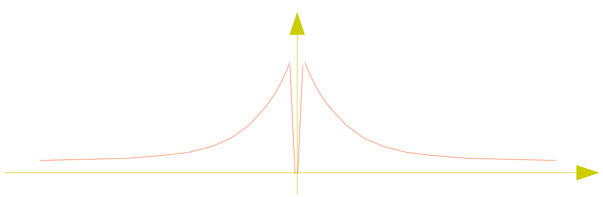
$$\sum_{k=\langle N_0\rangle} |\gamma[k]|^2 \qquad\qquad \frac{1}{N_0}\sum_{n=0}^{N_0} |x[n]|^2$$

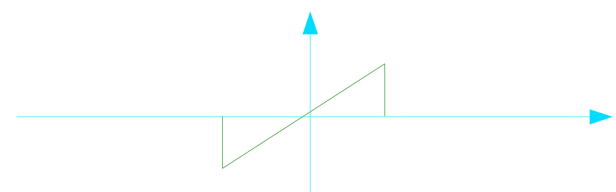


$$\gamma[-k]\, e^{-jk\hat{\omega}_0 n} \qquad\qquad \gamma[+k]\, e^{+jk\hat{\omega}_0 n}$$

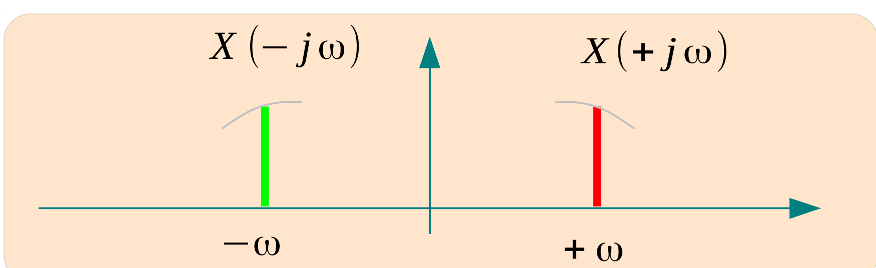
Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$





$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega \qquad\qquad \int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

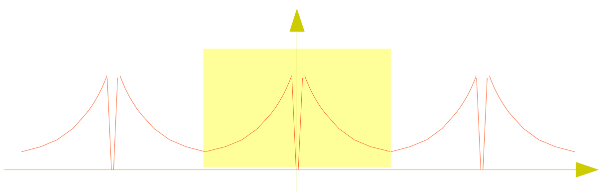


$$X(-j\omega)\, e^{-j\omega t} \qquad\qquad X(+j\omega)\, e^{+j\omega t}$$

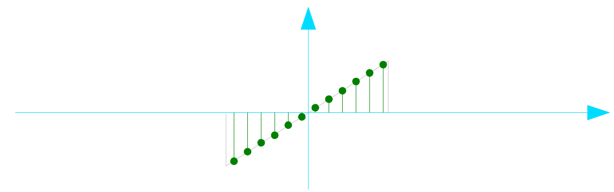
Discrete Time – **DTFT**

Discrete Time Fourier Transform

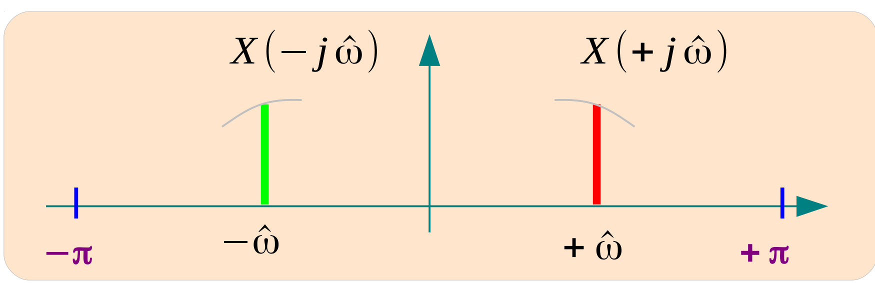
$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega}n} \quad \Longleftrightarrow \quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(j\hat{\omega})\, e^{+j\hat{\omega}n}\, d\hat{\omega}$$



$$\frac{1}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2\, d\hat{\omega}$$



$$\sum_{n=-\infty}^{+\infty} |x[n]|^2$$

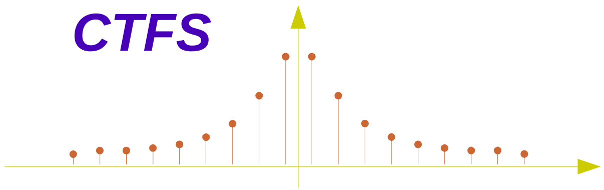


$X(e^{-j\hat{\omega}})e^{-j\hat{\omega}n}$

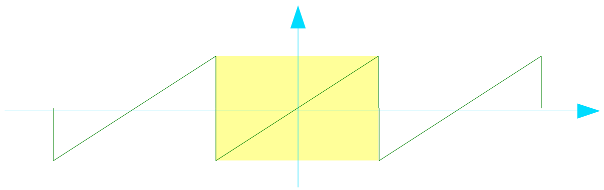
$X(e^{+j\hat{\omega}})e^{+j\hat{\omega}n}$

***

CTFS

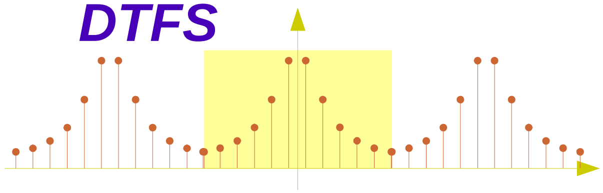


$$\sum_{k=-\infty}^{+\infty} |C_k|^2$$

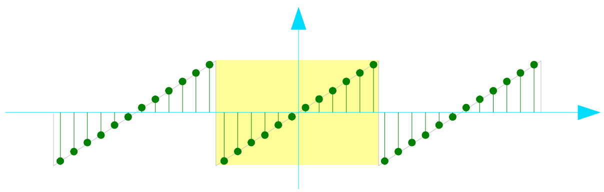


$$\frac{1}{T_0}\int_{T_0} |x(t)|^2\, dt$$

DTFS

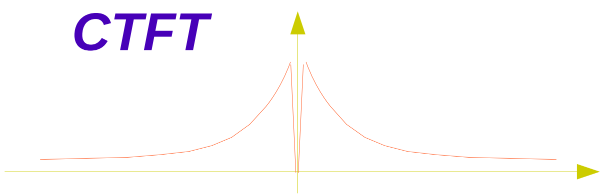


$$\sum_{k=<N_0>} |\gamma[k]|^2$$

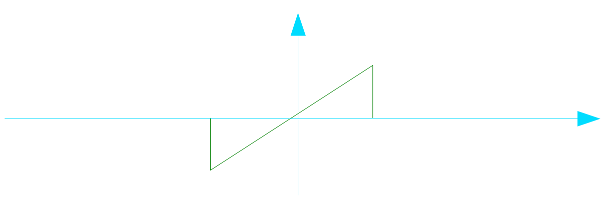


$$\frac{1}{N_0}\sum_{n=0}^{N_0} |x[n]|^2$$

CTFT

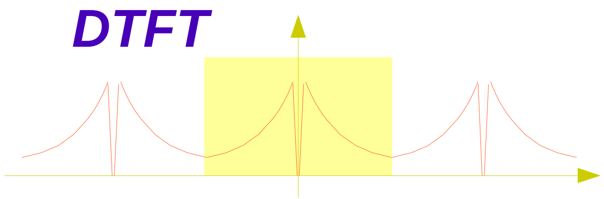


$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2\, d\omega$$

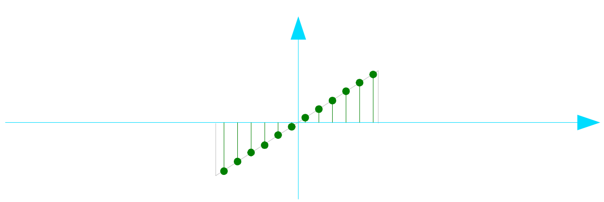


$$\int_{-\infty}^{+\infty} |x(t)|^2\, dt$$

DTFT



$$\frac{1}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2\, d\hat{\omega}$$



$$\sum_{n=-\infty}^{+\infty} |x[n]|^2$$

Parseval's Theorem

CTFS **Periodic x(t)**

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |C_k|^2$$

CTFT **Aperiodic x(t)**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

DTFS **Periodic x[n]**

$$\gamma[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\hat{\omega}_0 n}$$

$$\frac{1}{N_0}\sum_{n=0}^{N_0} |x[n]|^2 = \sum_{k=<N_0>} |\gamma[k]|^2$$

DTFT **Aperiodic x[n]**

$$X(j\hat{\omega}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(j\hat{\omega})|^2 d\hat{\omega}$$

Parseval's Theorem

Convolution

$$u(t) * v(t) = \int_{-\infty}^{+\infty} u(\tau) v(t-\tau) d\tau$$

Convolution Theorem

$$w(t) = u(t) \cdot v(t) \qquad W(f) = U(f) * V(f)$$

$$w(t) = u(t) * v(t) \qquad W(f) = U(f) \cdot V(f)$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} u^*(t) v(t) dt = \int_{-\infty}^{+\infty} U^*(f) V(f) df$$

http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/00700_TransformParseval.pdf

Energy Spectrum

Energy Spectral Density

$$|U(f)|^2 \quad (energy/Hz)$$

Parseval's Theorem

$$\int_{-\infty}^{+\infty} u^*(t)v(t)\,dt = \int_{-\infty}^{+\infty} U^*(f)V(f)\,df$$

$$\int_{-\infty}^{+\infty} u^*(t)u(t)\,dt = \int_{-\infty}^{+\infty} U^*(f)U(f)\,df$$

$$\int_{-\infty}^{+\infty} |u(t)|^2\,dt = \int_{-\infty}^{+\infty} |U(f)|^2\,df$$

http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/00700_TransformParseval.pdf

Energy Spectrum

Energy Spectral Density

$$|U(f)|^2 \quad (energy/Hz)$$

$$\int_{-\infty}^{+\infty} |U(f)|^2 df \quad (total\ energy)$$

Energy Spectrum

$$the\ graph\ of\ |U(f)|^2$$

How the energy of w(t) is distributed over frequencies

$$\frac{|U(f)|^2}{\int_{-\infty}^{+\infty} |U(f)|^2 df}$$

Like a probability distribution function
Integrates to one

http://www.ee.ic.ac.uk/hp/staff/dmb/courses/E1Fourier/00700_TransformParseval.pdf

References

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[5] K. Shin, et al., Fundamentals of Signal Processing for Sound and Vibration Engineerings

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