Undersampling (2B)

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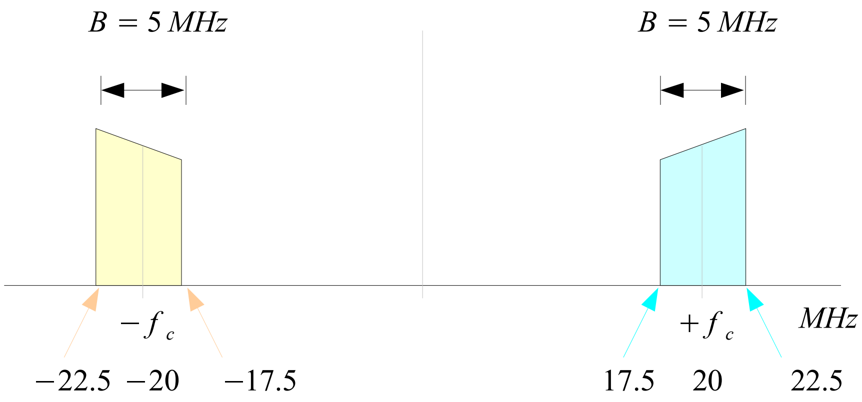
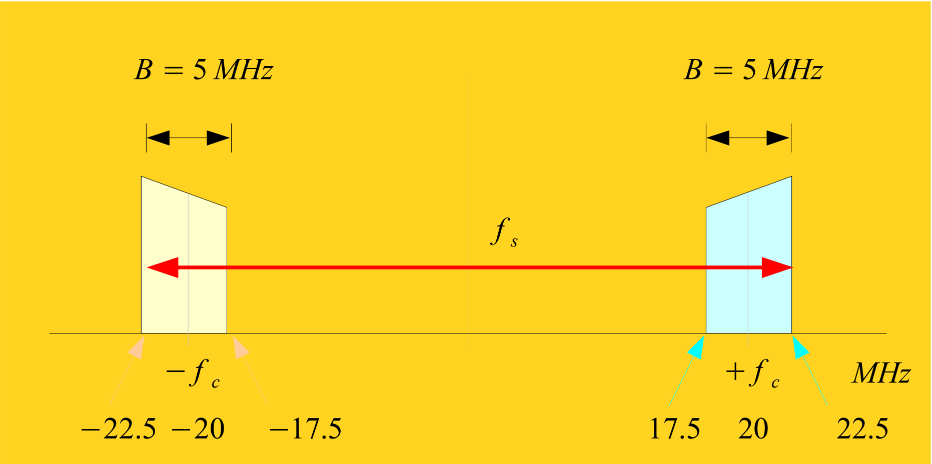
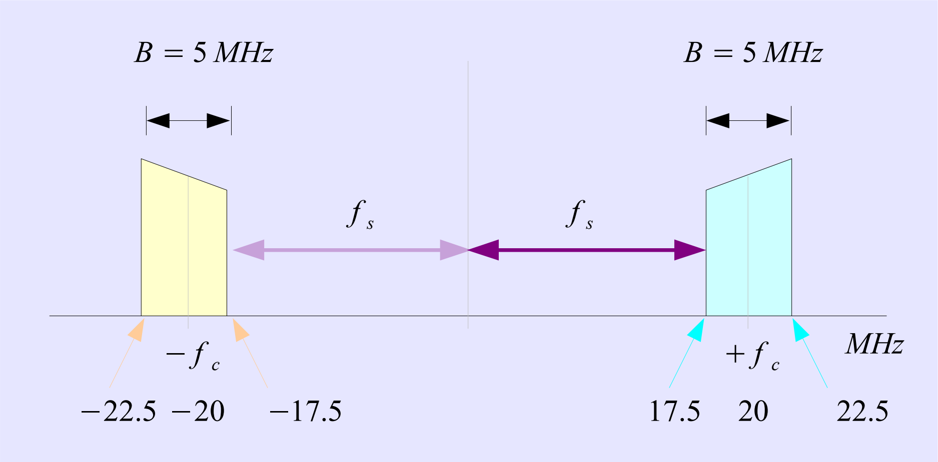
Band-limited Signal



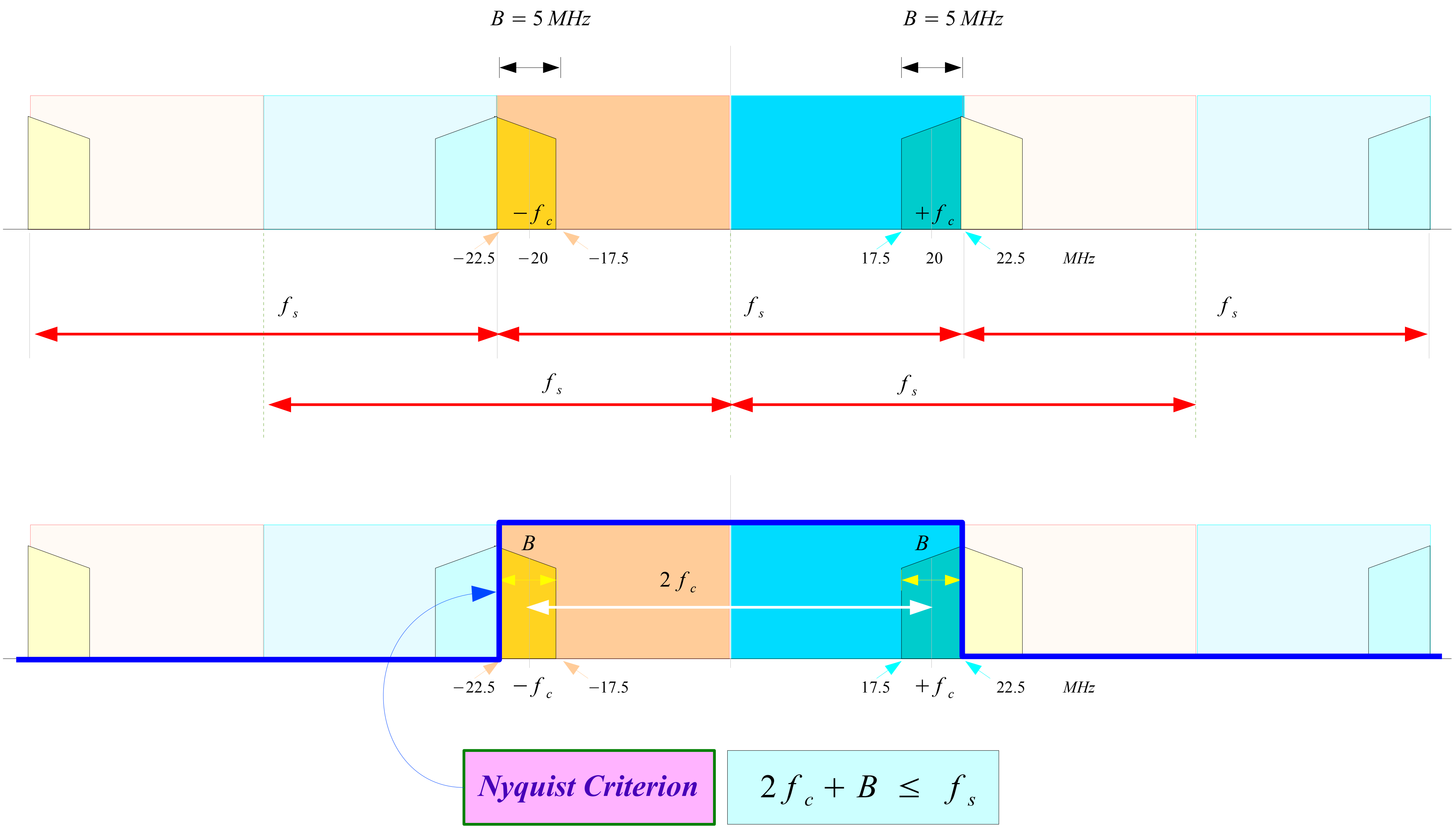


- Lowpass Sampling

- Bandpass Sampling
- IF filtering
- Harmonic Sampling
- Sub-Nyquist Sampling

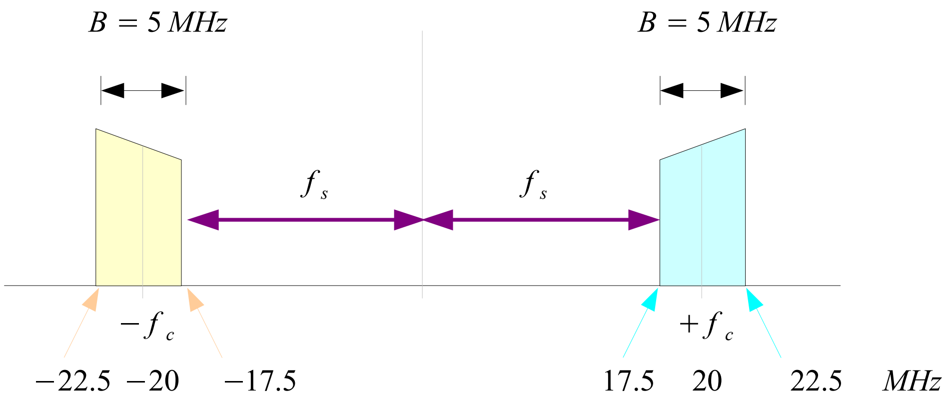


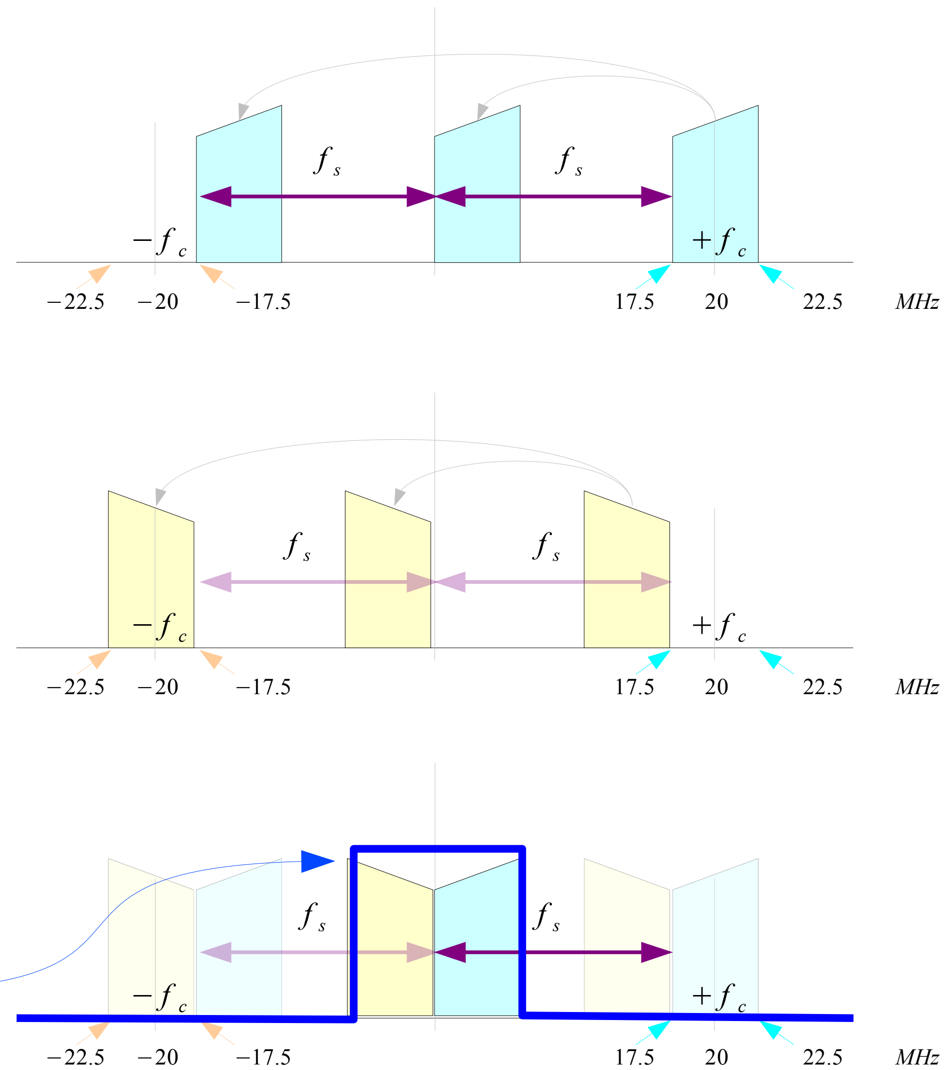
Low-pass Signal Sampling



$$2 f_c + B \leq f_s$$

Band-pass Signal Sampling





- **Bandpass Sampling**
- **IF filtering**
- **Harmonic Sampling**
- **Sub-Nyquist Sampling**

Nyquist Criterion	$2B \leq f_s$

Sampling Frequency f_s (1)

Assume there are m multiples of f_s

$$2f_c - B = m \cdot f_s$$

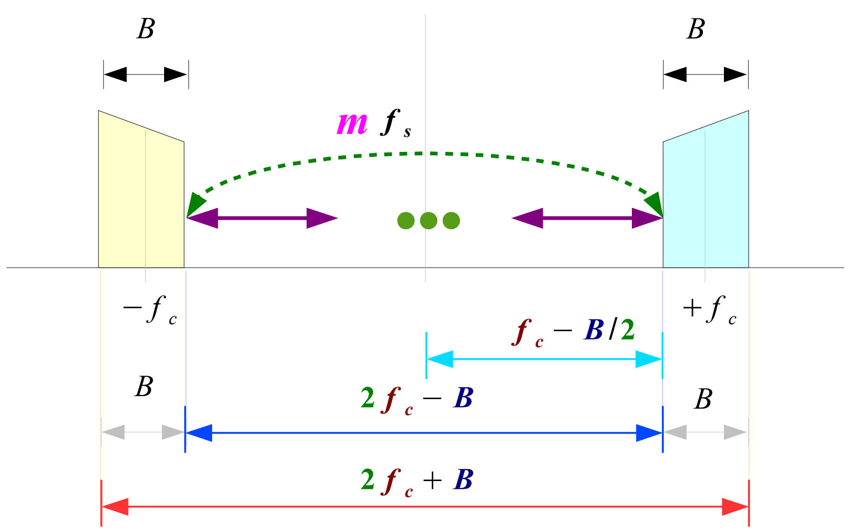
Given an integer m

Max f_s condition

f_s *can be decreased according to the following condition without introducing aliasing problems*

$$2f_c + B = (m+1) \cdot f_s$$

Min f_s condition



Given Band-pass Signal is characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Sampling Frequency f_s (2)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

Given Band-pass Signal is characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

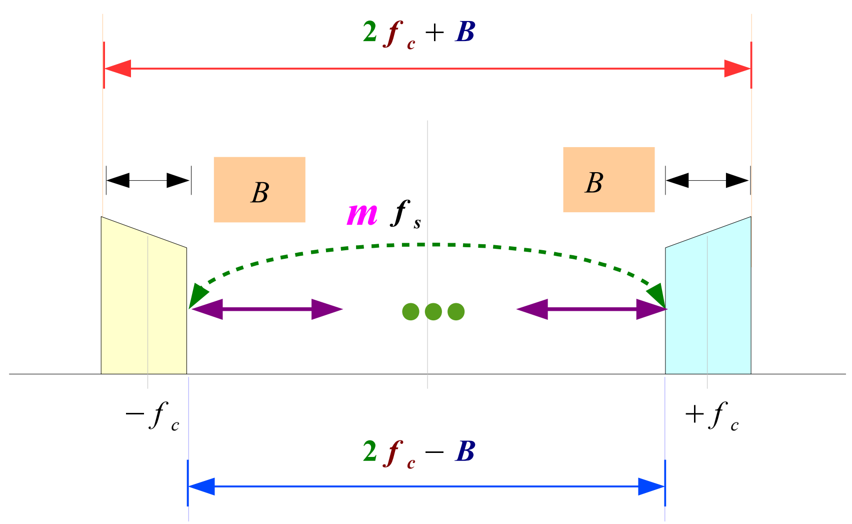
➡ ***Normalization by B***

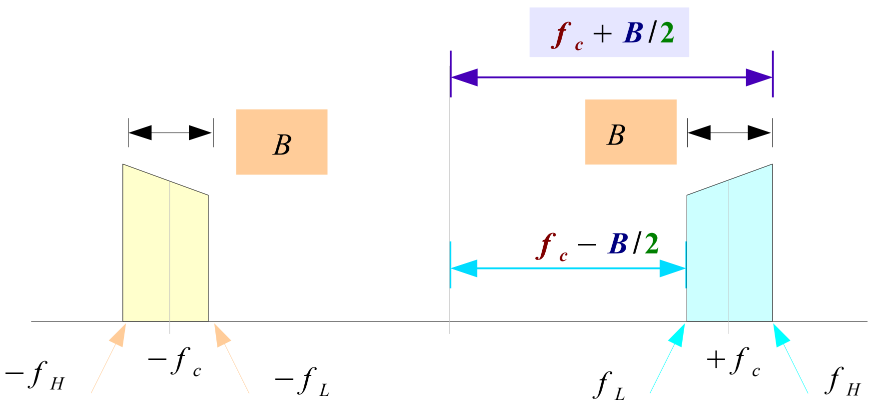
$$\frac{2f_c + B}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_c - B}{mB}$$

$$\frac{2f_H}{(m+1)B} \le \frac{f_s}{B} \le \frac{2f_L}{mB}$$

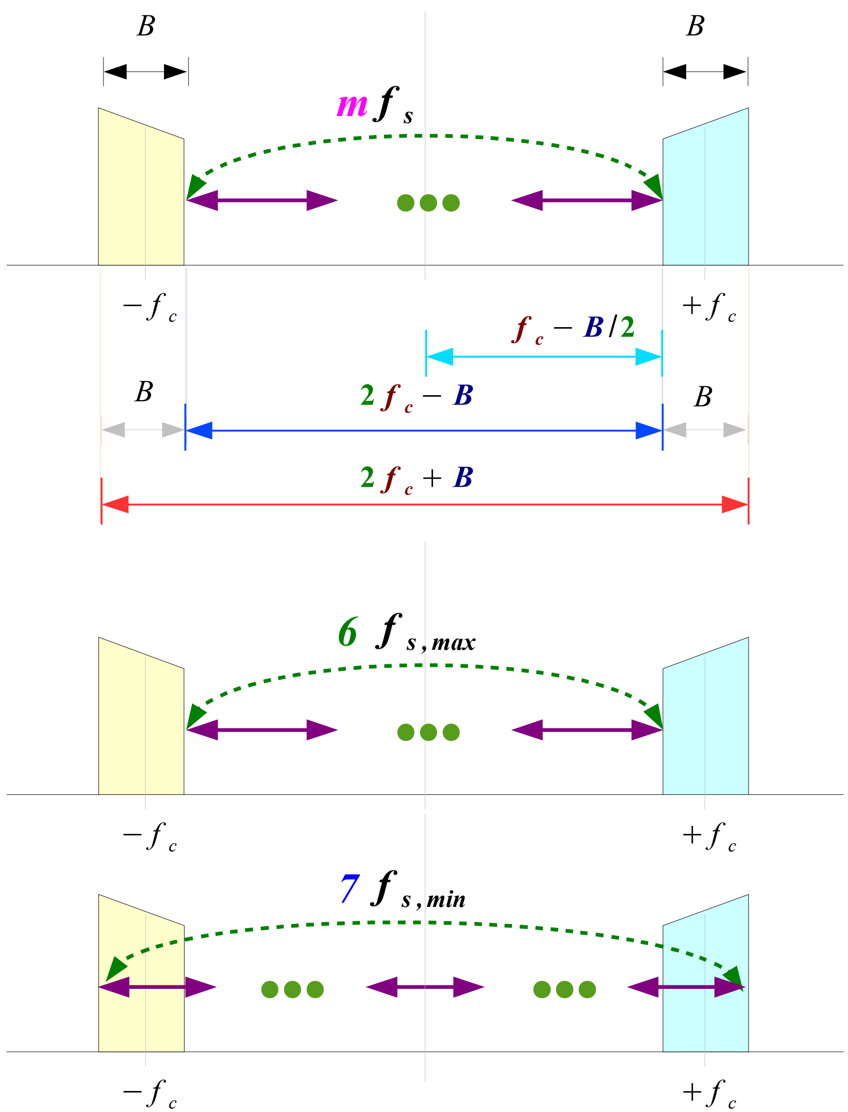
$f_H = f_c + B/2$ ***Highest frequency***

$f_L = f_c - B/2$ ***Lowest frequency***





Example m=6 (1)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

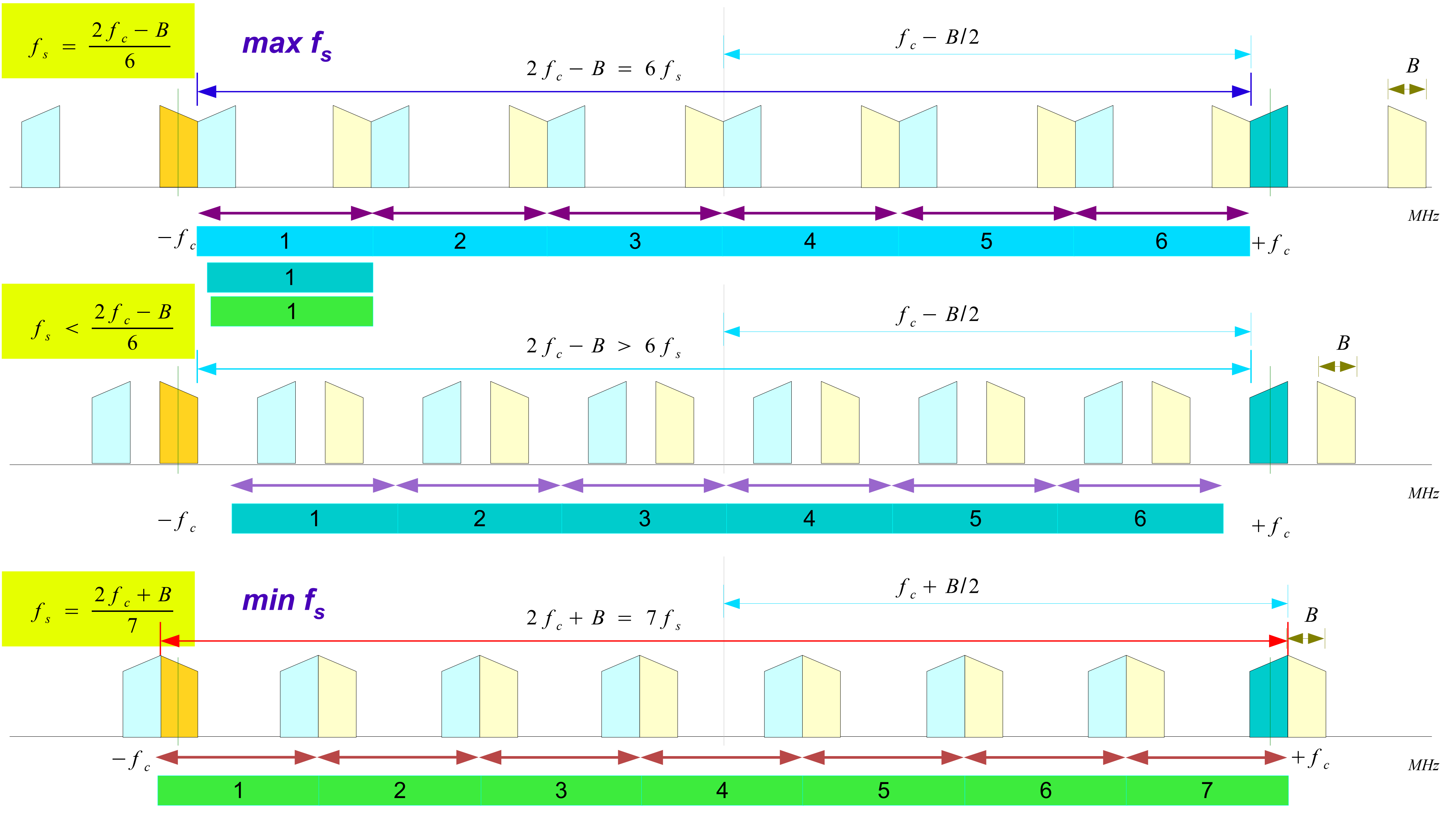
When m = 6

min f_s $\quad \frac{2f_c + B}{7} \le f_s \le \frac{2f_c - B}{6}$ $\quad$ max f_s

$$\text{max } f_s = \frac{2f_c - B}{6}$$

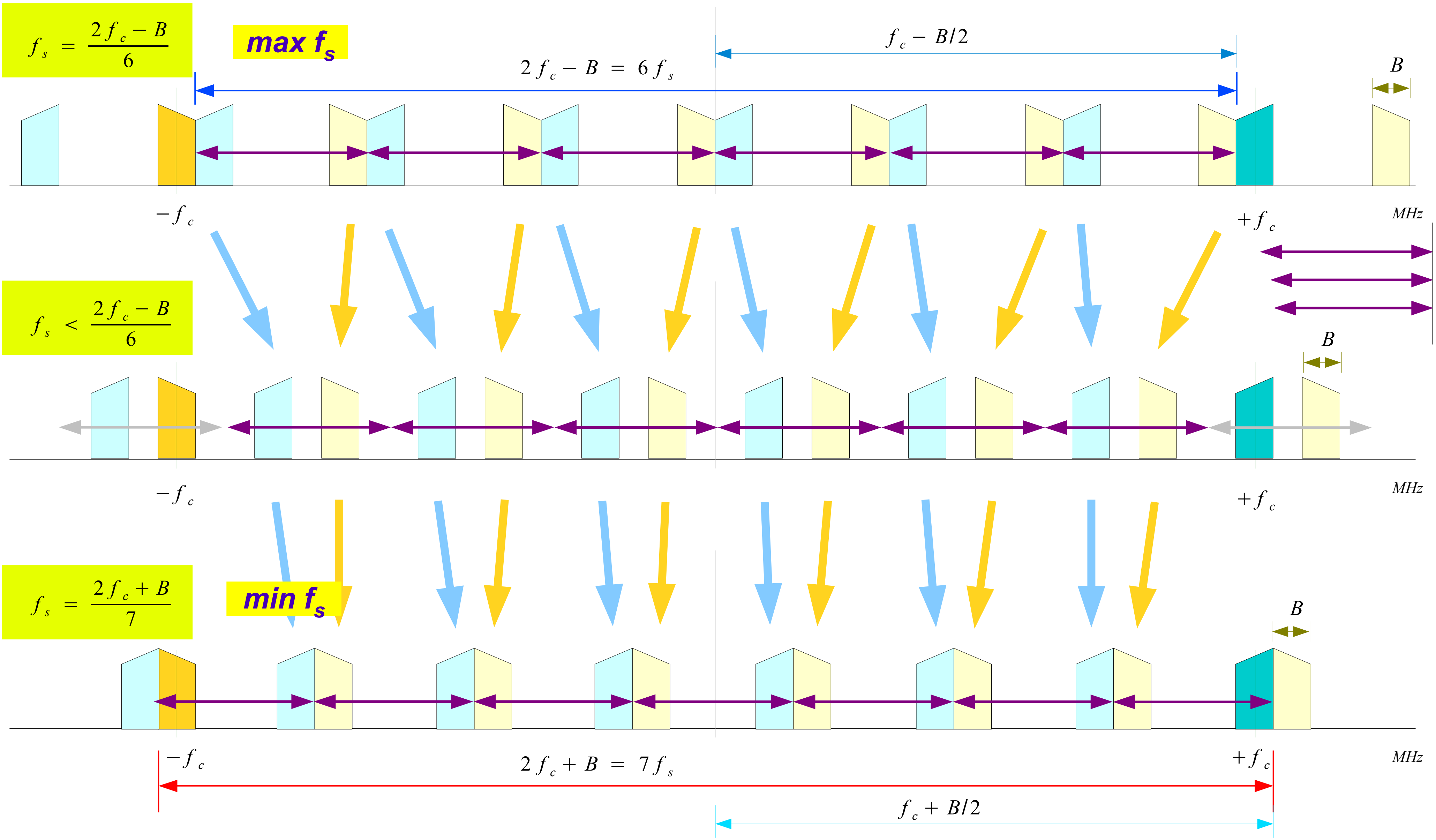
$$\text{min } f_s = \frac{2f_c + B}{7}$$

Example m=6 (2)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
MHz
$-f_c$
1
2
3
4
5
6
$+f_c$
1
1
$f_s < \frac{2f_c - B}{6}$
$f_c - B/2$
$2f_c - B > 6f_s$
B
MHz
$-f_c$
1
2
3
4
5
6
$+f_c$
$f_s = \frac{2f_c + B}{7}$
min f_s
$f_c + B/2$
$2f_c + B = 7f_s$
B
$-f_c$
$+f_c$
MHz
1
2
3
4
5
6
7

Example m=6 (3)



$f_s = \frac{2f_c - B}{6}$
max f_s
$f_c - B/2$
$2f_c - B = 6f_s$
B
$-f_c$
$+f_c$
MHz
$f_s < \frac{2f_c - B}{6}$
B
$-f_c$
$+f_c$
MHz
$f_s = \frac{2f_c + B}{7}$
min f_s
B
$-f_c$
$2f_c + B = 7f_s$
$+f_c$
MHz
$f_c + B/2$

Minimum f_s Plot (1)

$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

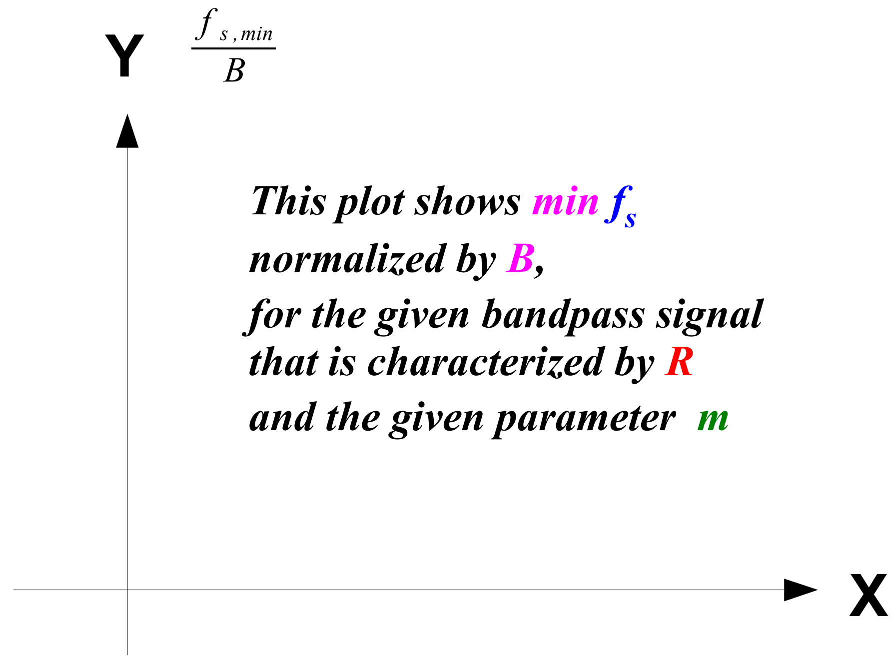
$$\frac{f_c + B/2}{B} = R \rightarrow \mathbf{X}$$

$\Rightarrow$ $\dfrac{\text{highest signal frequency}}{\text{bandwidth B}}$

$$\frac{2f_c + B}{(m+1)B} = \frac{f_{s,min}}{B} \rightarrow \mathbf{Y}$$

$\Rightarrow$ $\dfrac{\text{minimum sampling rate}}{\text{bandwidth B}}$

X-Y Plot



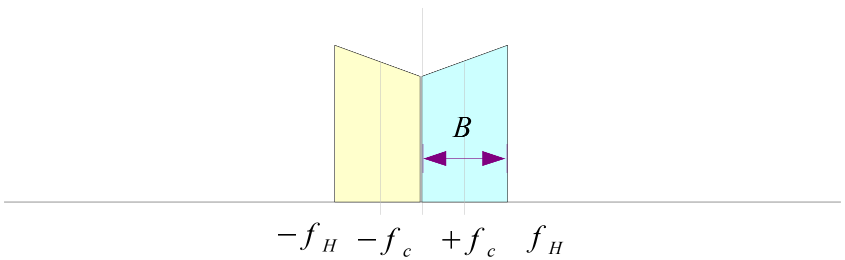
This plot shows min f_s normalized by B, for the given bandpass signal that is characterized by R and the given parameter m

Characterized by
- *Bandwidth B*
- *Carrier Frequency f_c*

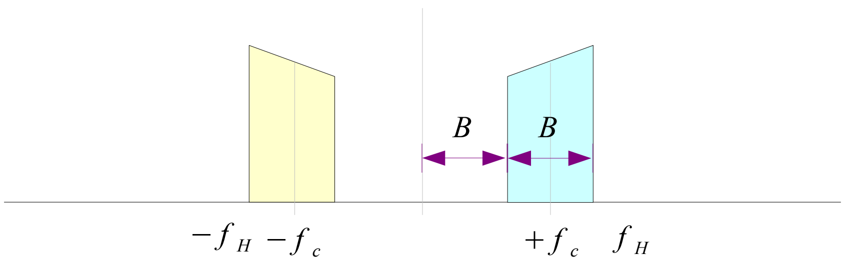
$$R = \frac{f_H}{B} = \frac{f_c + B/2}{B}$$

Minimum f_s Plot (2)

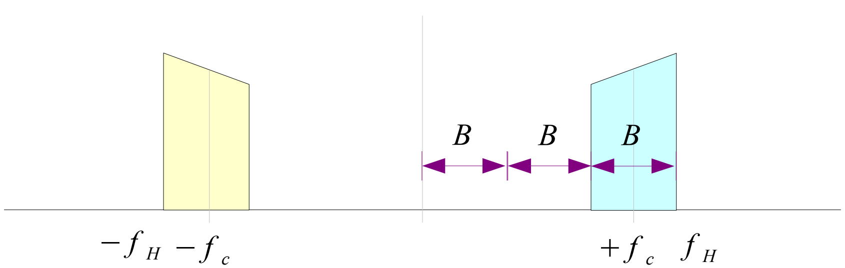
$f_H = f_c + B/2 = 1B$ | $R = f_H / B = 1$



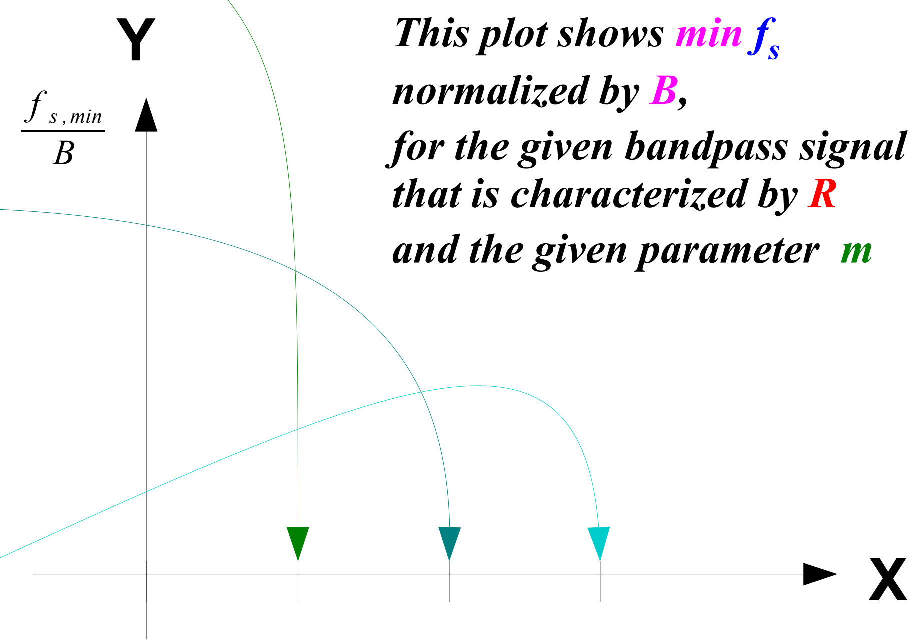
$f_H = f_c + B/2 = 2B$ | $R = f_H / B = 2$



$f_H = f_c + B/2 = 3B$ | $R = f_H / B = 3$



X-Y Plot



This plot shows min f_s normalized by B, for the given bandpass signal that is characterized by R and the given parameter m

Characterized by

- ***Bandwidth B***
- ***Carrier Frequency f_c***

$$R = \frac{f_H}{B} = \frac{f_c + B/2}{B}$$

Minimum f_s Plot (3)

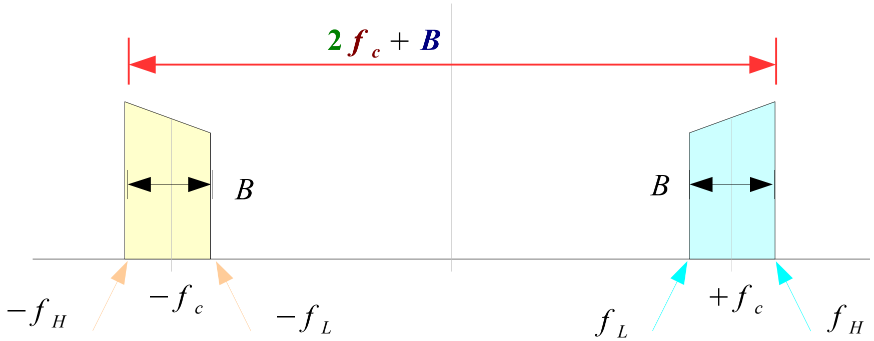
$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$g(m,R) = \frac{2}{(m+1)}\frac{f_H}{B} = \frac{2}{(m+1)}R$$

$\frac{f_H}{B}$ = X $\Rightarrow$ $\frac{f_c + B/2}{B} = R$

$\frac{f_{s,min}}{B}$ = Y $\Rightarrow$ $\frac{2f_c + B}{(m+1)B} = \frac{2f_H}{(m+1)B}$

$\Rightarrow$ $g(m,R)$



$m = 0$	$g(0,R) = 2R$	$slope = 2$
$m = 1$	$g(1,R) = R$	$slope = 1$
$m = 2$	$g(2,R) = \frac{2}{3}R$	$slope = 2/3$
$m = 3$	$g(3,R) = \frac{1}{2}R$	$slope = 1/2$
$m = 4$	$g(4,R) = \frac{2}{5}R$	$slope = 2/5$
$m = 5$	$g(5,R) = \frac{1}{3}R$	$slope = 1/3$
$m = 6$	$g(6,R) = \frac{2}{7}R$	$slope = 2/7$
$m = 7$	$g(7,R) = \frac{1}{4}R$	$slope = 1/4$
$m = 8$	$g(8,R) = \frac{2}{9}R$	$slope = 2/9$

Minimum f_s Plot (4)

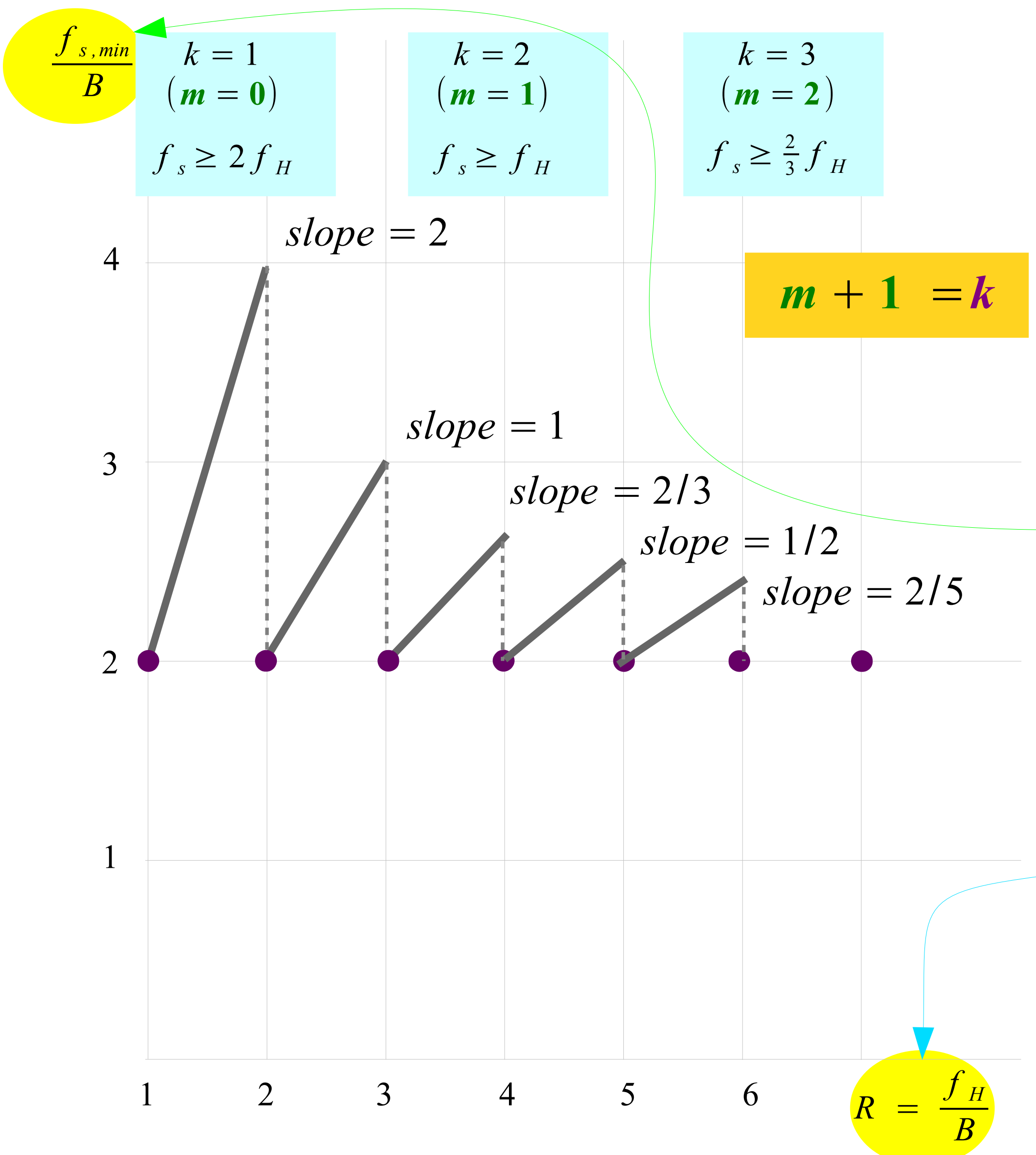
$$g(m,R) = \frac{2}{(m+1)}\frac{f_H}{B} = \frac{2}{(m+1)}R$$

$$R = m+1 \Rightarrow g(m, m+1) = 2$$

$m = 0$	$g(0,R) = 2R$	$slope = 2$
$m = 1$	$g(1,R) = R$	$slope = 1$
$m = 2$	$g(2,R) = \frac{2}{3}R$	$slope = 2/3$
$m = 3$	$g(3,R) = \frac{1}{2}R$	$slope = 1/2$
$m = 4$	$g(4,R) = \frac{2}{5}R$	$slope = 2/5$
$m = 5$	$g(5,R) = \frac{1}{3}R$	$slope = 1/3$
$m = 6$	$g(6,R) = \frac{2}{7}R$	$slope = 2/7$
$m = 7$	$g(7,R) = \frac{1}{4}R$	$slope = 1/4$
$m = 8$	$g(8,R) = \frac{2}{9}R$	$slope = 2/9$

$m = 0$	$R = 1$	$\Rightarrow g(0,1) = 2$
$m = 1$	$R = 2$	$\Rightarrow g(1,2) = 2$
$m = 2$	$R = 3$	$\Rightarrow g(2,3) = 2$
$m = 3$	$R = 4$	$\Rightarrow g(3,4) = 2$
$m = 4$	$R = 5$	$\Rightarrow g(4,5) = 2$
$m = 5$	$R = 6$	$\Rightarrow g(5,6) = 2$
$m = 6$	$R = 7$	$\Rightarrow g(6,7) = 2$
$m = 7$	$R = 8$	$\Rightarrow g(7,8) = 2$
$m = 8$	$R = 9$	$\Rightarrow g(8,9) = 2$

Minimum f_s Plot (5)



$$\frac{2f_c + B}{m+1} \leq f_s \leq \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \leq f_s \leq \frac{2f_c - B}{k-1}$$

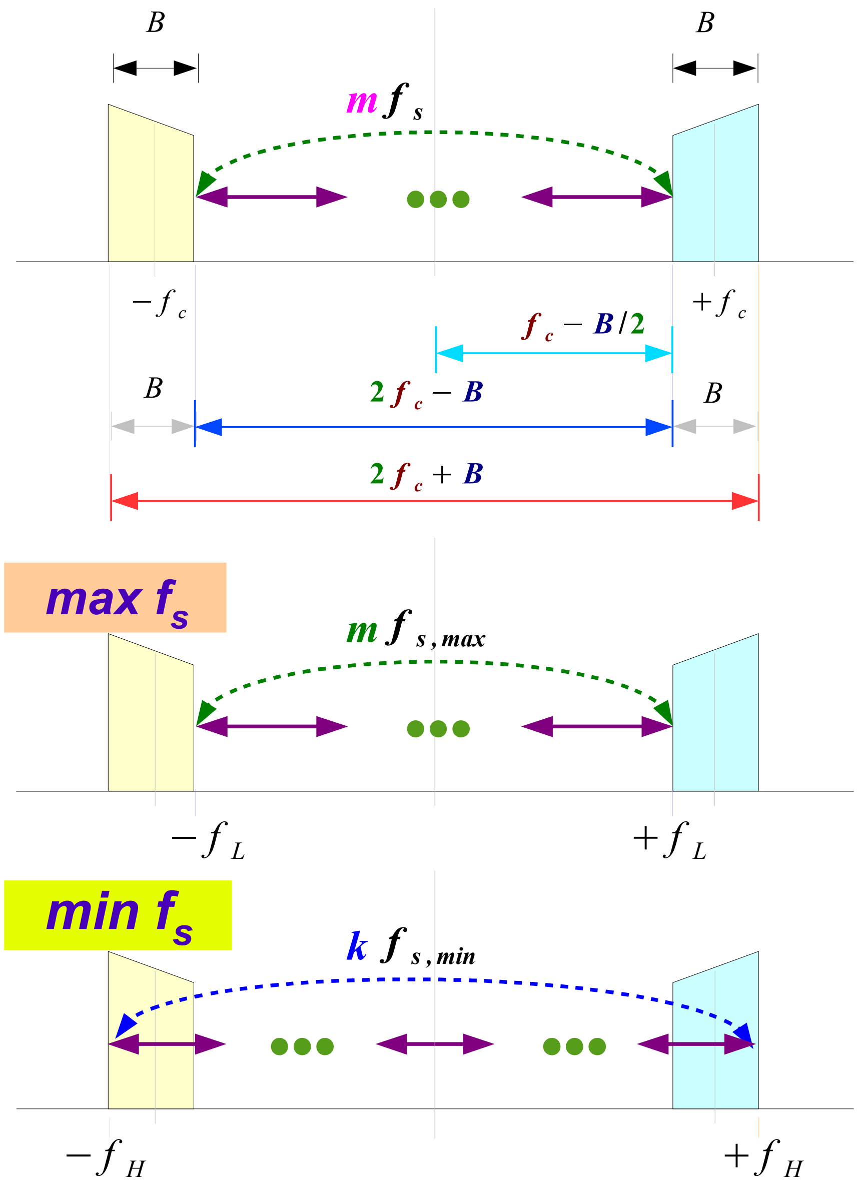
$$\frac{2f_c + B}{(m+1)B} = \frac{f_{s,min}}{B} = g(m, R)$$

→ **minimum sampling rate** / **bandwidth B**

$$\frac{f_c + B/2}{B} = R$$

→ **highest signal frequency** / **bandwidth B**

Min, Max Condition on f_s (1)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \le f_s \le \frac{2f_c - B}{k-1}$$

$$m + 1 = k$$

min f_s **max f_s**

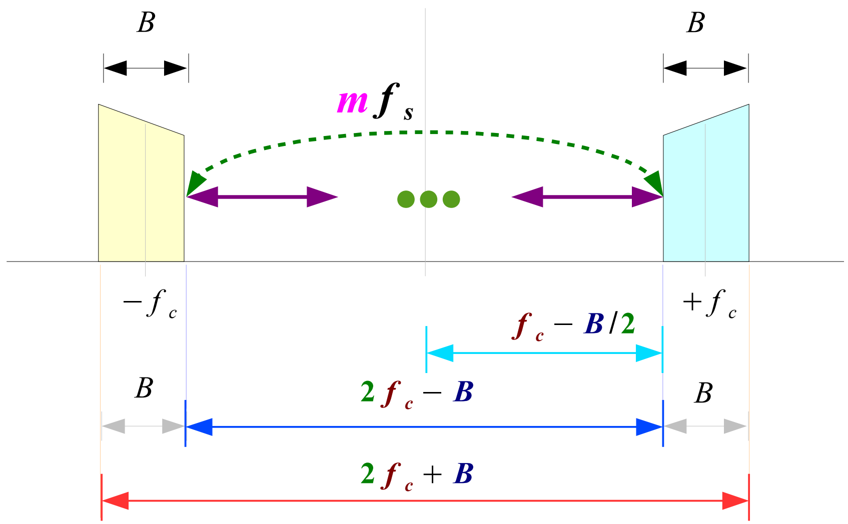
$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

$$k = 2 \qquad f_H \le f_s \le 2f_L \qquad m = 1$$

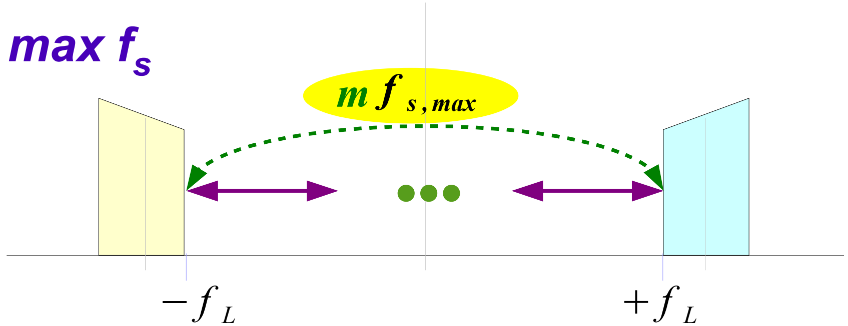
$$k = 3 \qquad \frac{2}{3}f_H \le f_s \le f_L \qquad m = 2$$

$$k = 4 \qquad \frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L \qquad m = 3$$

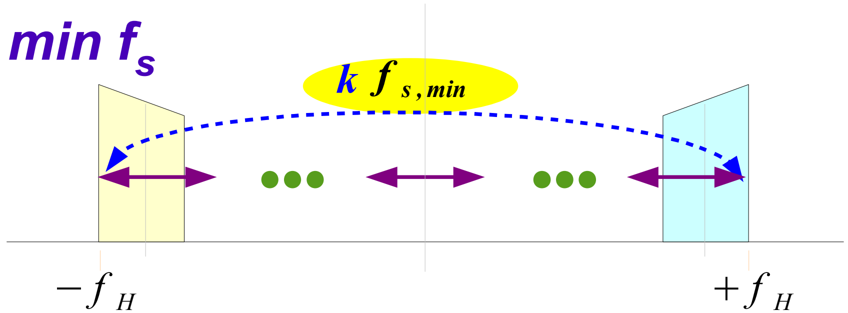
Min, Max Condition on f_s (2)



max f_s



min f_s



min f_s		max f_s
$\dfrac{2f_H}{k}$	$\le f_s \le$	$\dfrac{2f_L}{m}$

$$k = m + 1$$

m represents how many f_s are in $2f_c - B$ in **max f_s**

$$\text{max } f_s = \frac{2f_c - B}{m} = \frac{2f_L}{m}$$

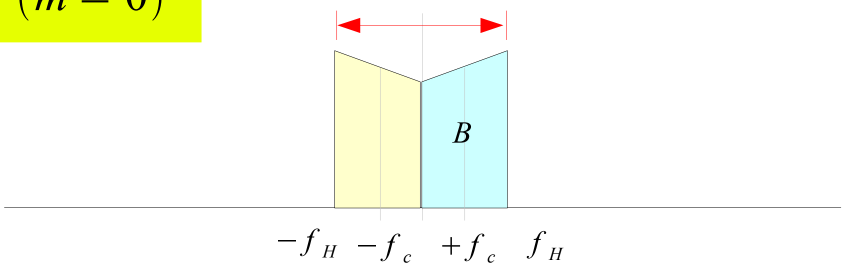
k represents how many f_s are in $2f_c + B$ in **min f_s**

$$\text{min } f_s = \frac{2f_c + B}{k} = \frac{2f_H}{k}$$

Example k=1 (m=0)

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1\,B$

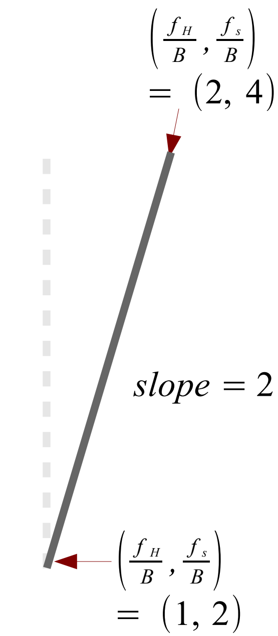


$R = f_H / B = 1$

$R \in [1,\ 2]$

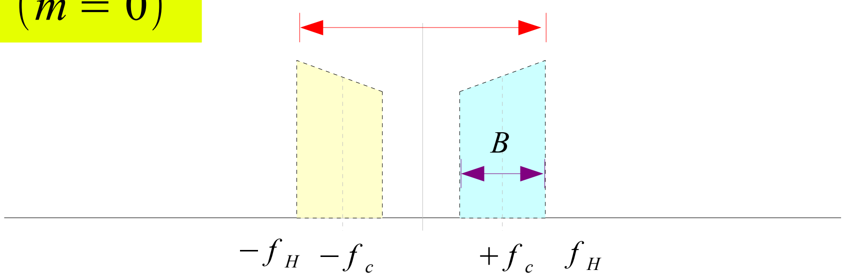
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$



$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 1.5\,B$



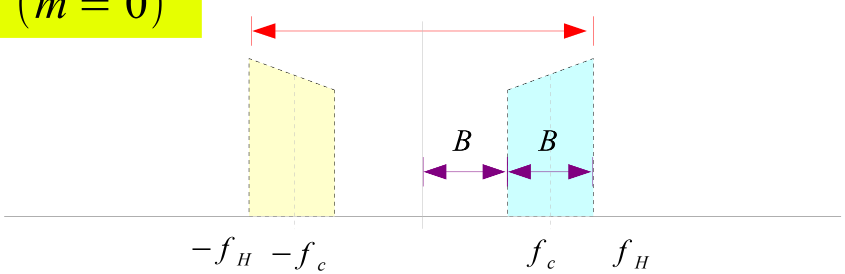
$R = f_H / B = 1.5$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$

$k = 1$
$(m = 0)$

$f_H = f_c + B/2 = 2\,B$



$R = f_H / B = 2$

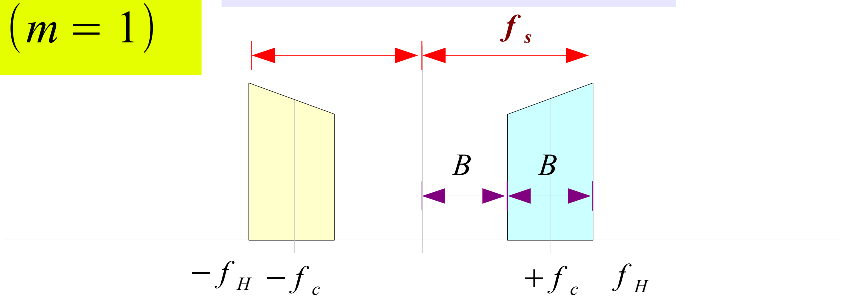
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{4}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = +\infty$$

Example k=2 (m=1)

$k = 2$ $(m = 1)$

$f_H = f_c + B/2 = 2B$

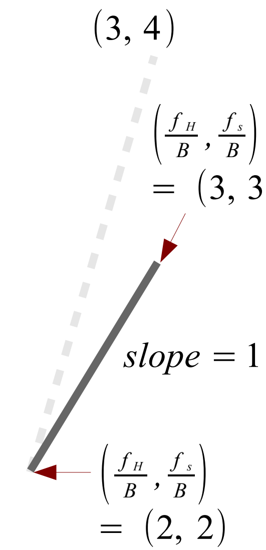


$R = f_H / B = 2$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2}$$

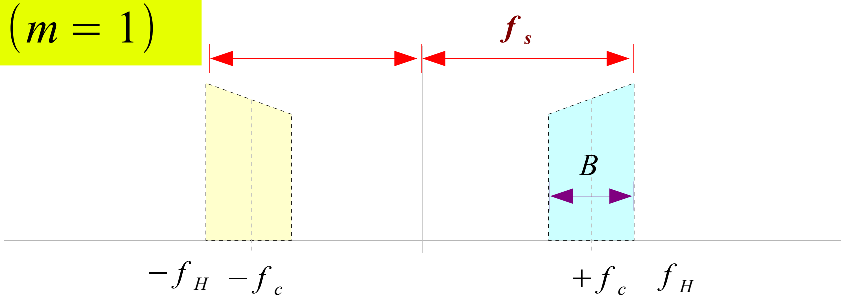
$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{2}$$

$R \in [2, 3]$



$k = 2$ $(m = 1)$

$f_H = f_c + B/2 = 2.5B$



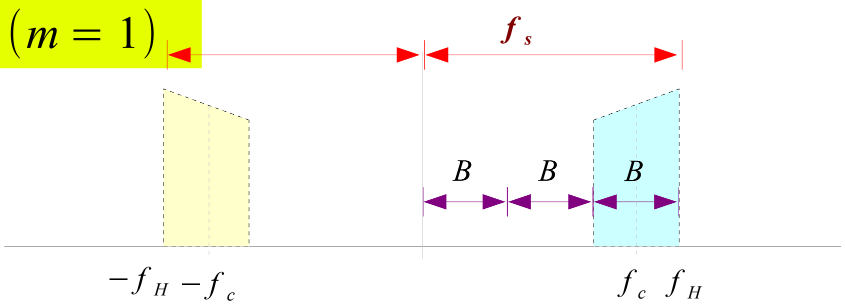
$R = f_H / B = \mathbf{2.5}$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2.5}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{3}$$

$k = 2$ $(m = 1)$

$f_H = f_c + B/2 = 3B$



$R = f_H / B = 3$

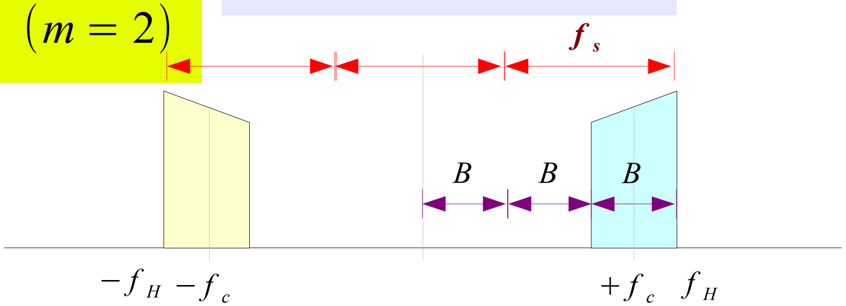
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{3}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1)B} = \mathbf{4}$$

Example k=3 (m=2)

$k = 3$ $(m = 2)$

$f_H = f_c + B/2 = 3B$



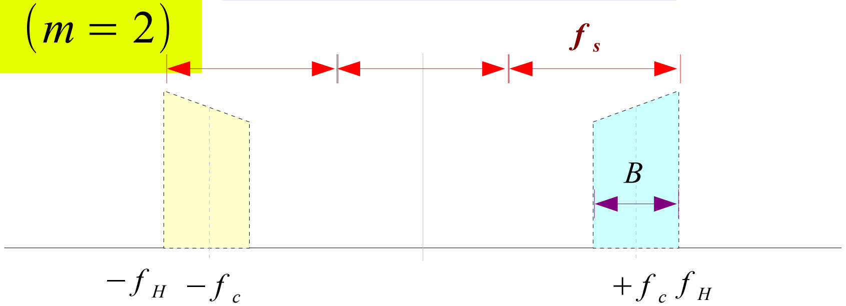
$R = f_H / B = 3$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{2}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = \mathbf{2}$$

$k = 3$ $(m = 2)$

$f_H = f_c + B/2 = 3.5B$



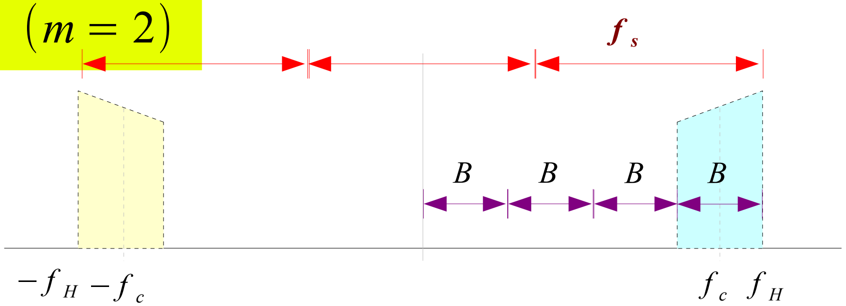
$R = f_H / B = 3.5$

$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{\frac{7}{3}}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = \mathbf{3}$$

$k = 3$ $(m = 2)$

$f_H = f_c + B/2 = 4B$

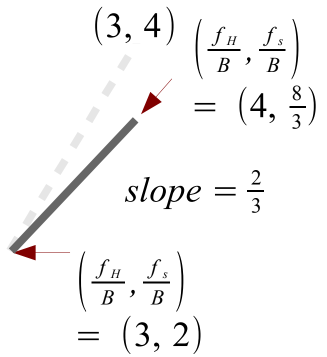


$R = f_H / B = 4$

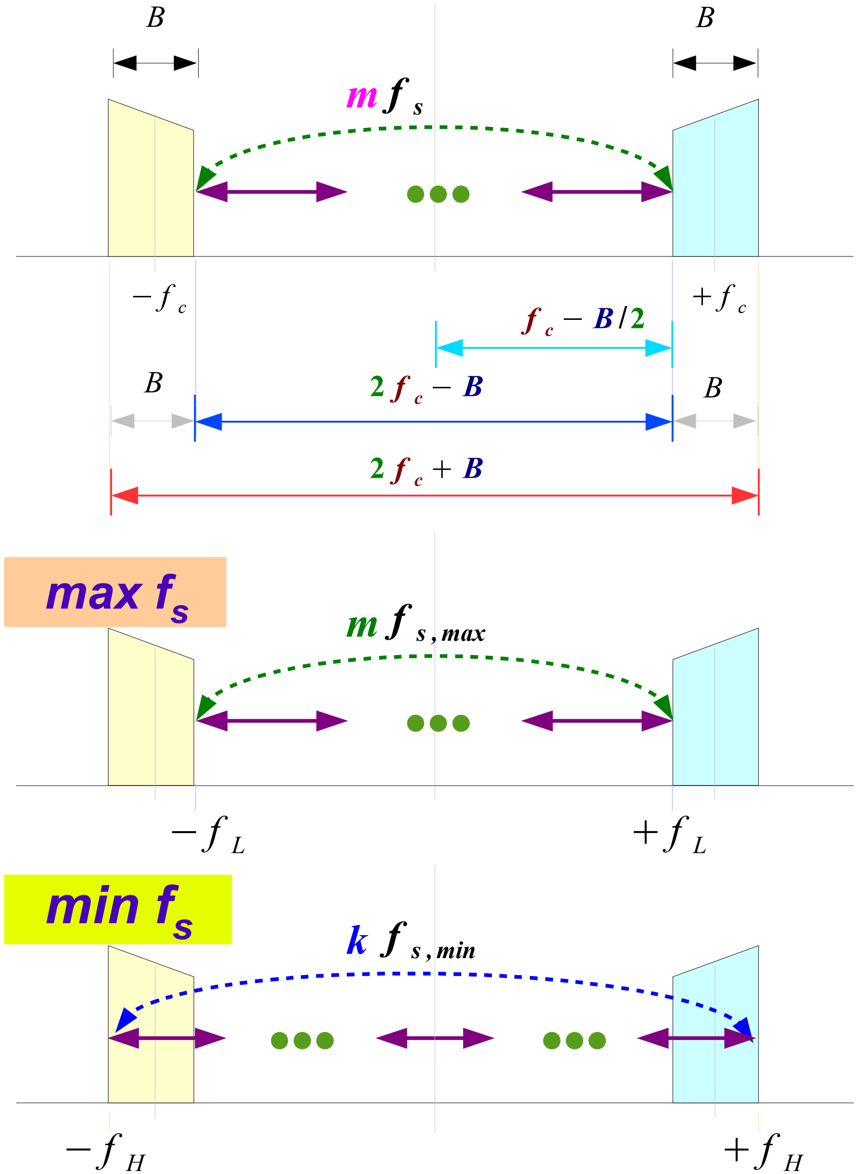
$$\frac{f_{s,min}}{B} = \frac{2 f_H}{k B} = \mathbf{\frac{8}{3}}$$

$$\frac{f_{s,max}}{B} = \frac{2(f_H - B)}{(k-1) B} = \mathbf{4}$$

$R \in [3,\ 4]$



Min, Max Condition on f_s (2)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \le f_s \le \frac{2f_c - B}{k-1}$$

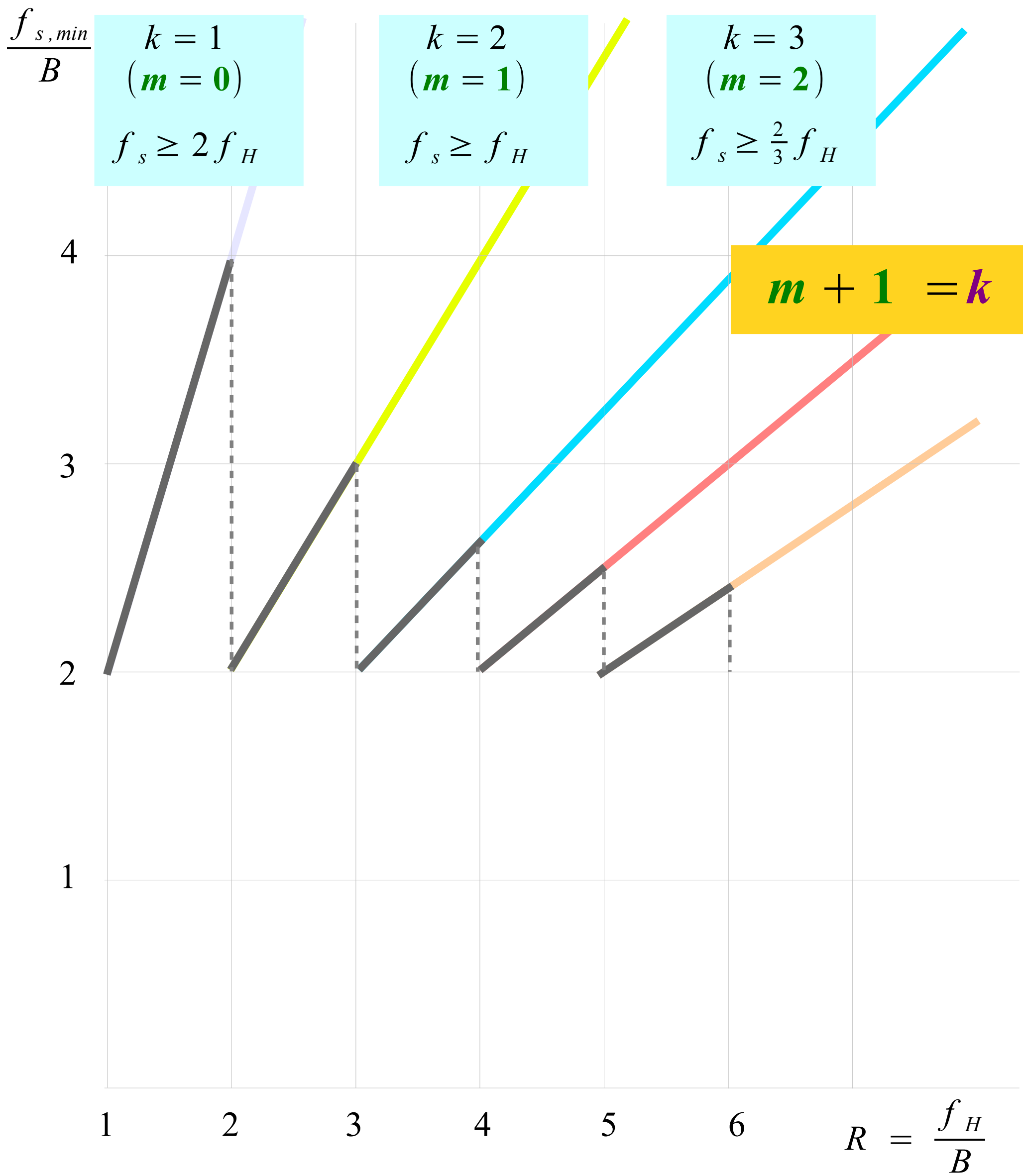
$$m + 1 = k$$

min f_s **max f_s**

$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

$k = 2$	$f_H \le f_s \le 2f_L$	$m = 1$
$k = 3$	$\frac{2}{3}f_H \le f_s \le f_L$	$m = 2$
$k = 4$	$\frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L$	$m = 3$

Min Max f_s Plot (1)

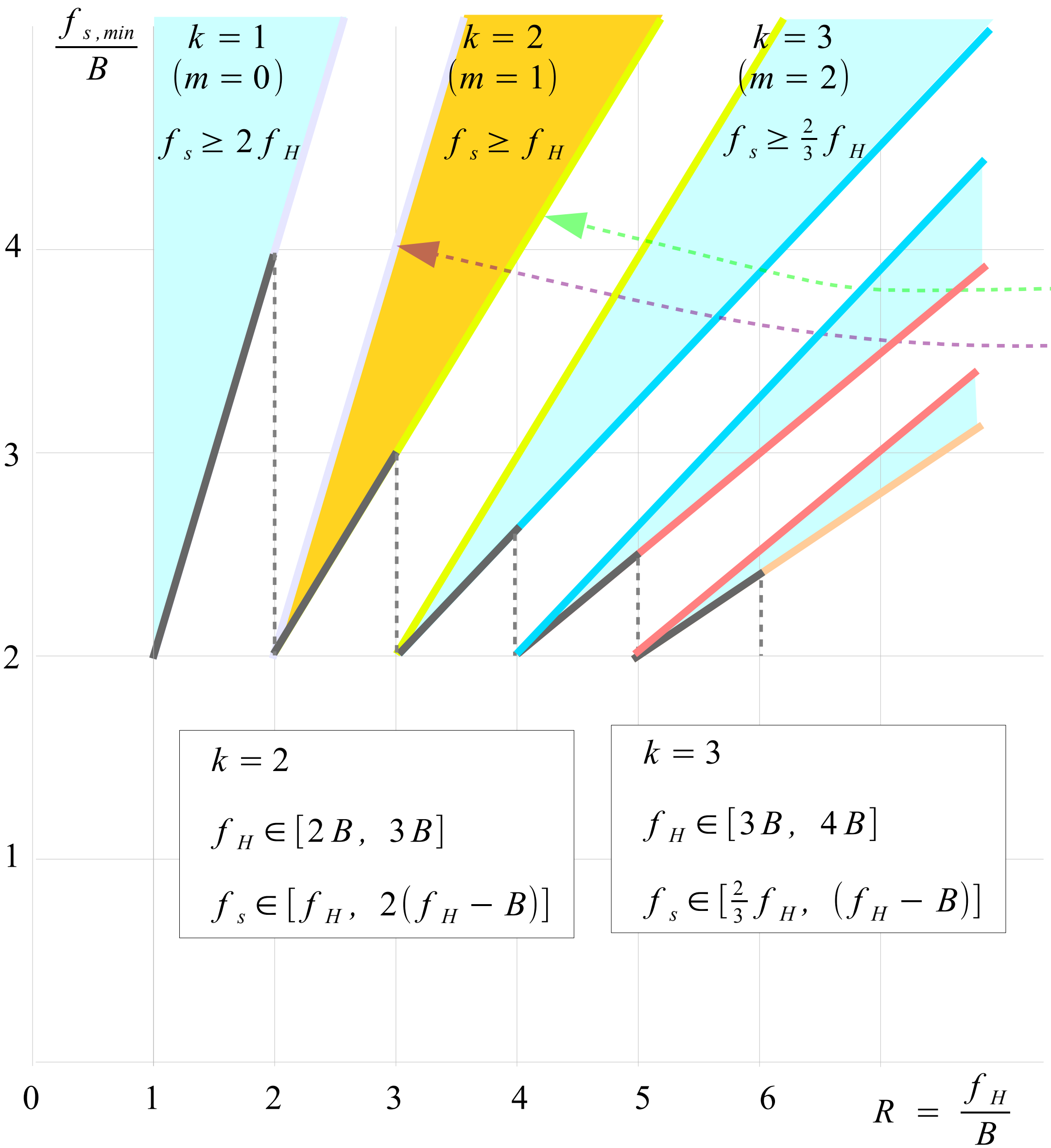


$$\frac{2f_c + B}{m+1} \leq f_s \leq \frac{2f_c - B}{m}$$

$$\frac{2f_c + B}{k} \leq f_s \leq \frac{2f_c - B}{k-1}$$

$$\frac{2f_H}{k} \leq f_s \leq \frac{2f_L}{m}$$

Min Max f_s Plot (2)



$$\frac{2f_H}{k} \leq f_s \leq \frac{2f_L}{m}$$

Min f_s **Max f_s**

$k = 2$	$f_H \leq f_s \leq 2f_L$
$k = 3$	$\frac{2}{3} f_H \leq f_s \leq f_L$

Min f_s		Max f_s
$y = 1(x-2)+2$ $y = x$	$k = 2$	$y = 2(x-2)+2$ $y = 2x-2$
$y = \frac{2}{3}(x-3)+2$ $y = \frac{2}{3}x$	$k = 3$	$y = 1(x-3)+2$ $y = x-1$

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Range of f_s when R=4.5, B=5 (1)

For a given m: $\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$

Nyquist Criterion: $2B \le f_s$

$f_c = 20\ MHz$

$B = 5\ MHz$

	min f_s		max f_s	Optimum Sampling Frequency
$m = 1$	$\frac{2\cdot 20 + 5}{1+1} = 22.5$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{1} = 35$	$f_s = 22.5\ MHz \quad (10 \le f_s)$
$m = 2$	$\frac{2\cdot 20 + 5}{2+1} = 15$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{2} = 17.5$	$f_s = 17.5\ MHz \quad (10 \le f_s)$
$m = 3$	$\frac{2\cdot 20 + 5}{3+1} = 11.25$	$\le f_s \le$	$\frac{2\cdot 20 - 5}{3} = 11.67$	$f_s = 11.25\ MHz \quad (10 \le f_s)$
$m = 4$	$\frac{2\cdot 20 + 5}{4+1} = 9$	$\ge$	$\frac{2\cdot 20 - 5}{4} = 8.75$	X
$m = 5$	$\frac{2\cdot 20 + 5}{5+1} = 7.5$	$\ge$	$\frac{2\cdot 20 - 5}{5} = 7.0$	X

Range of f_s when R=4.5, B=5 (2)

For a given m	$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$	Nyquist Criterion	$2B \le f_s$

$f_c = 20\ MHz$

$B = 5\ MHz$

$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

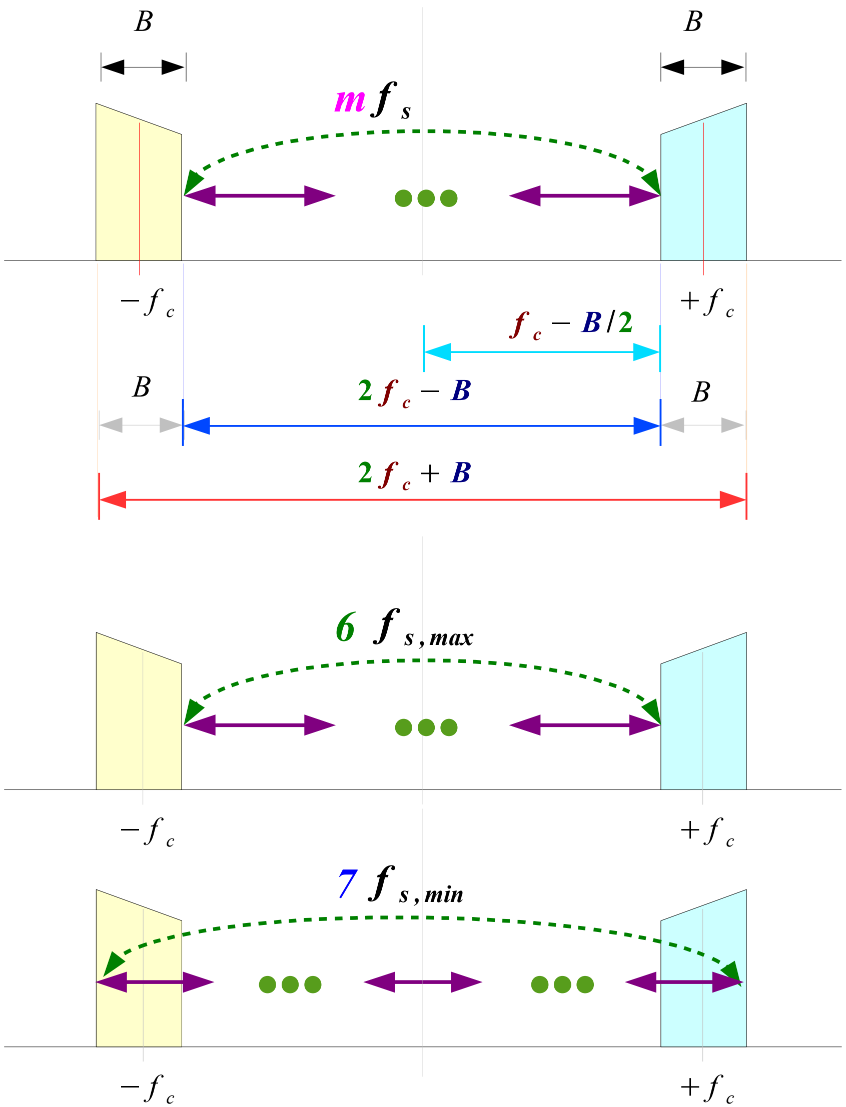
min f_s — max f_s

$f_H = f_c + B/2 = 22.5\ MHz$

$f_L = f_c - B/2 = 17.5\ MHz$

$k=2$	$m=1$	$f_H \le f_s \le 2f_L$	$22.5 \le f_s \le 35$
$k=3$	$m=2$	$\frac{2}{3}f_H \le f_s \le f_L$	$15.0 \le f_s \le 17.5$
$k=4$	$m=3$	$\frac{1}{2}f_H \le f_s \le \frac{2}{3}f_L$	$11.2 \le f_s \le 11.67$
$k=5$	$m=4$	$\frac{2}{5}f_H \le f_s \le \frac{1}{2}f_L$	9.0 X 8.75
$k=6$	$m=5$	$\frac{1}{3}f_H \le f_s \le \frac{2}{5}f_L$	7.5 X 7.0

Range of f_s when R=4.5, B=5 (3)



$$\frac{2f_c + B}{m+1} \le f_s \le \frac{2f_c - B}{m}$$

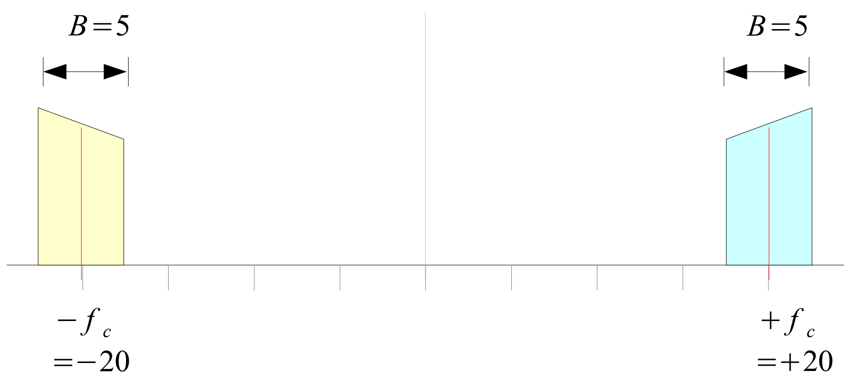
When m = 6

$$\text{min } f_s \quad \frac{2f_c + B}{7} \le f_s \le \frac{2f_c - B}{6} \quad \text{max } f_s$$

$$\text{max } f_s = \frac{2f_c - B}{6}$$

$$\text{min } f_s = \frac{2f_c + B}{7}$$

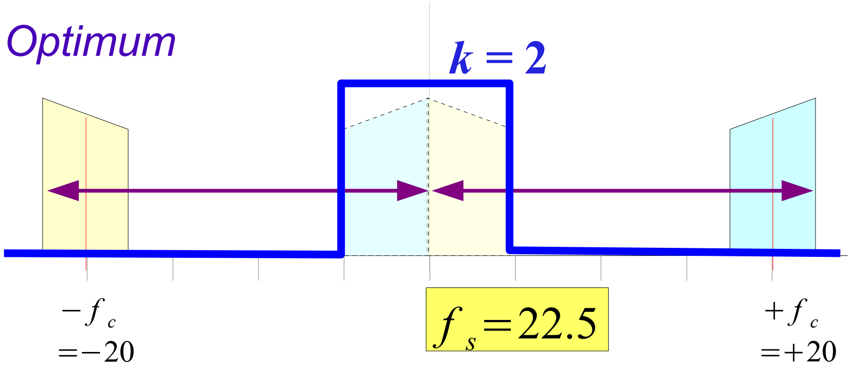
Range of f_s when R=4.5, B=5 (4)

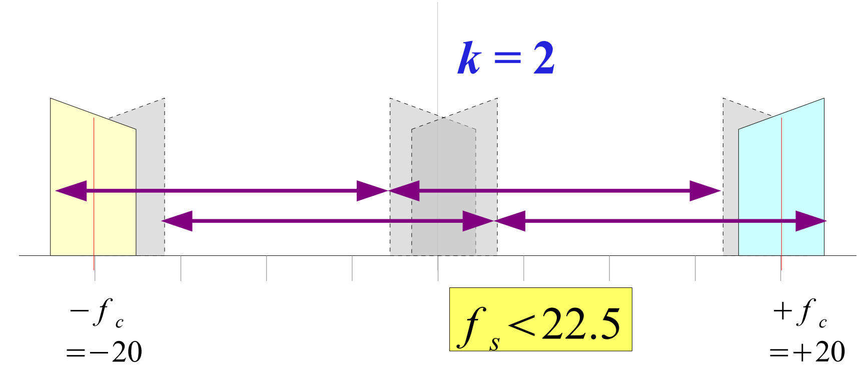


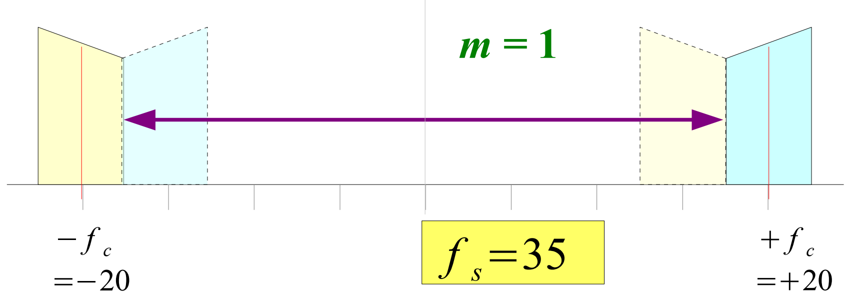
$$\frac{2f_H}{k} \le f_s \le \frac{2f_L}{m}$$

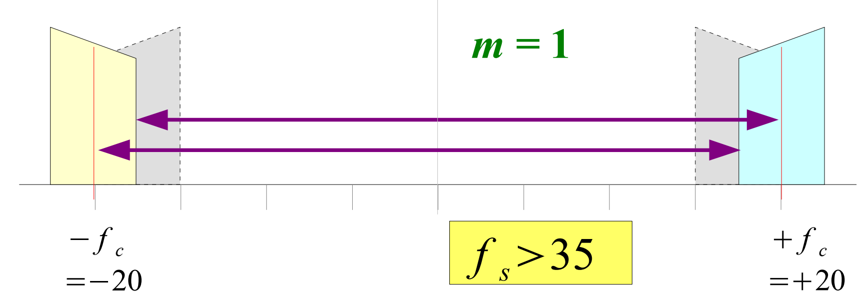
min f_s $k=2$ | $m=1$ **max f_s**

$$22.5 = f_H \le f_s \le 2f_L = 35$$

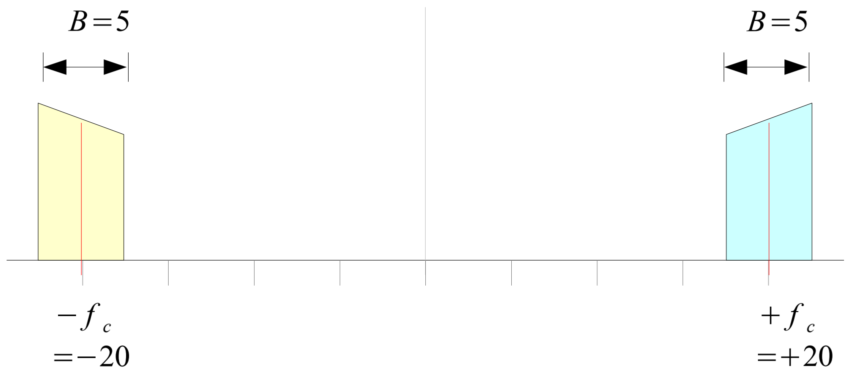








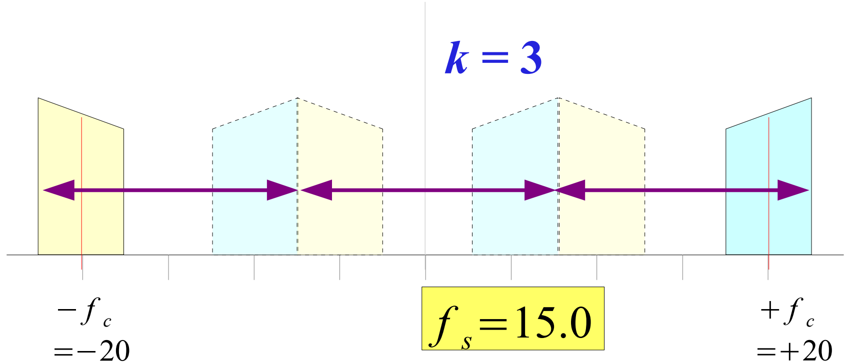
Range of f_s when R=4.5, B=5 (5)

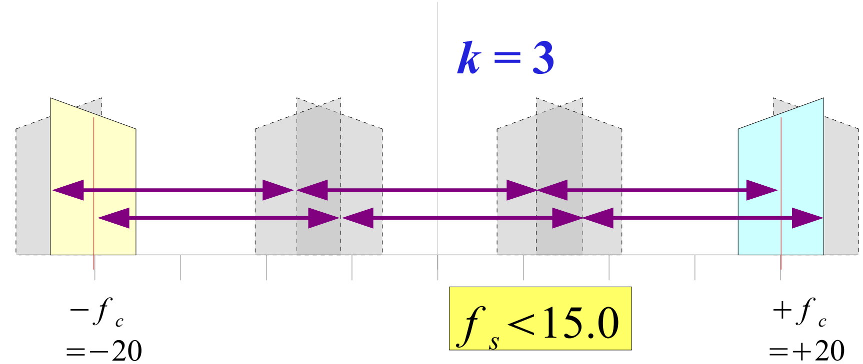


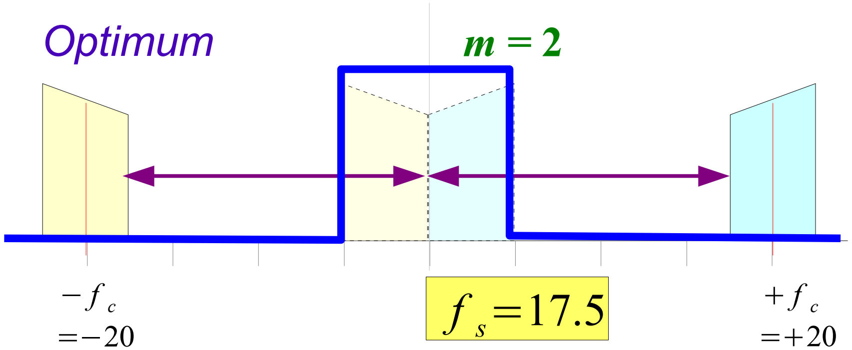
$$\frac{2 f_H}{k} \le f_s \le \frac{2 f_L}{m}$$

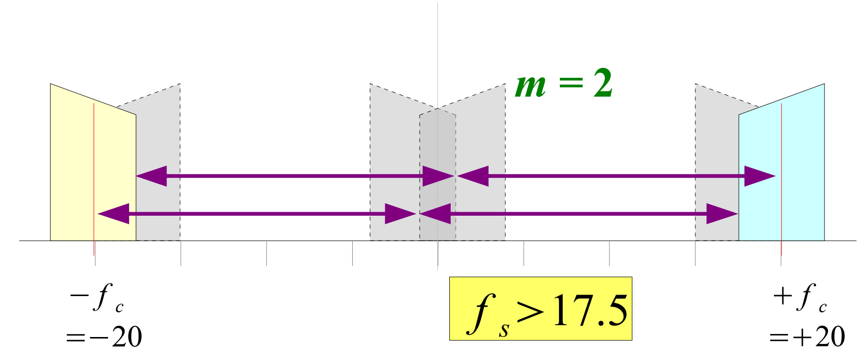
min f_s $k=3$ | $m=2$ **max f_s**

$$15.0 = \frac{2}{3} f_H \le f_s \le f_L = 17.5$$

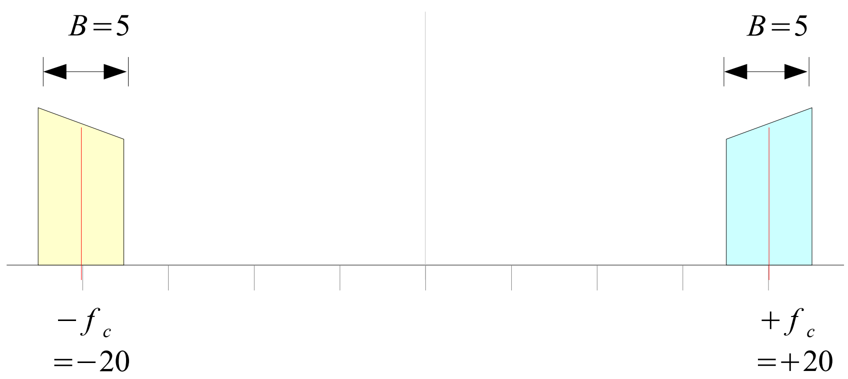








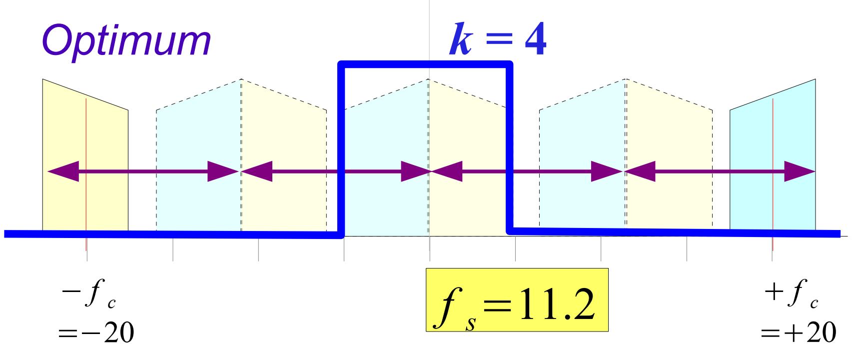
Range of f_s when R=4.5, B=5 (6)

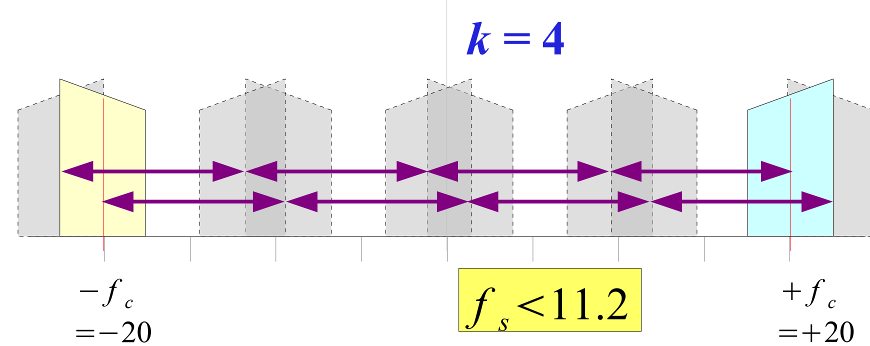


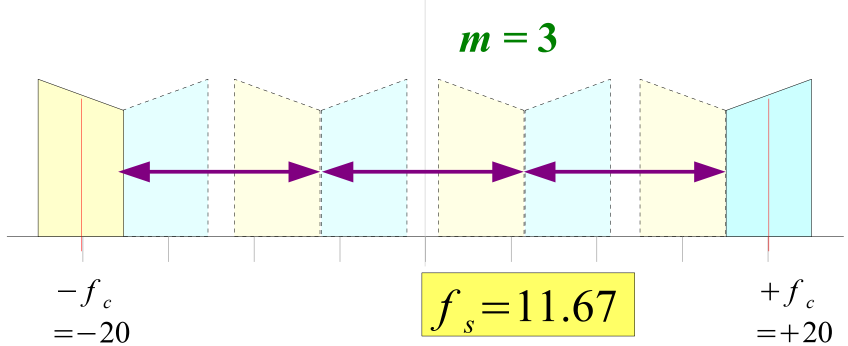
$$\frac{2 f_H}{k} \le f_s \le \frac{2 f_L}{m}$$

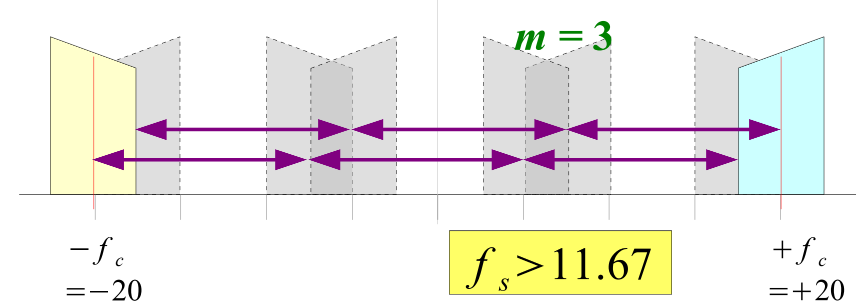
min f_s $k = 4$ | $m = 3$ **max f_s**

$$11.2 = \frac{1}{2} f_H \le f_s \le \frac{2}{3} f_L = 11.67$$

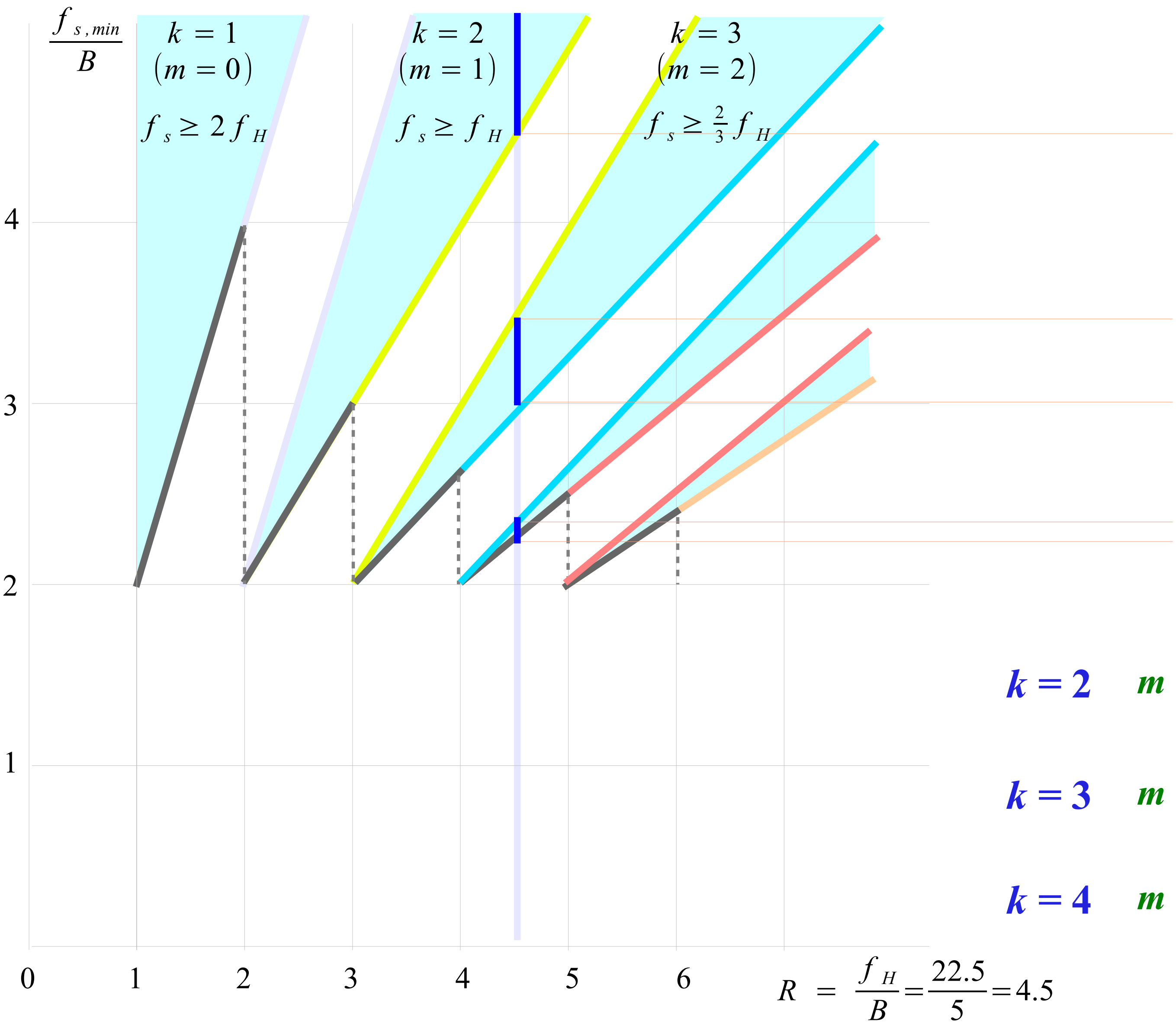








Range of f_s when R=4.5, B=5 (7)



$f_s \geq 22.5$ $\quad \frac{f_s}{B} = 4.5$

$f_s \leq 17.5$ $\quad \frac{f_s}{B} = 3.5$

$f_s \geq 15.0$ $\quad \frac{f_s}{B} = 3.0$

$f_s \leq 11.67$ $\quad \frac{f_s}{B} = 2.33$

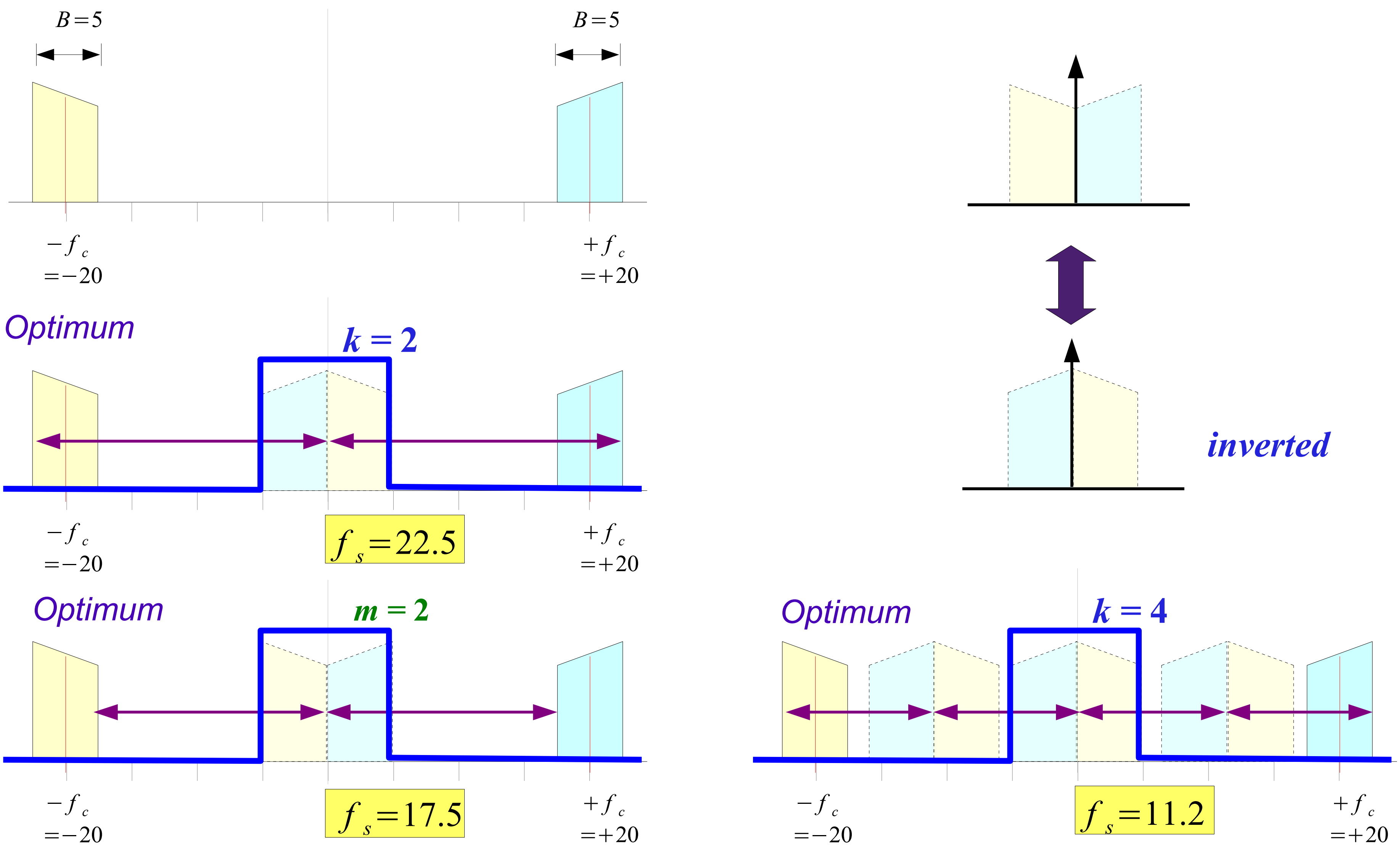
$f_s \geq 11.2$ $\quad \frac{f_s}{B} = 2.24$

$k = 2$ $\quad m = 1$ ➡ $22.5 \leq f_s \leq 35$

$k = 3$ $\quad m = 2$ ➡ $15.0 \leq f_s \leq 17.5$

$k = 4$ $\quad m = 3$ ➡ $11.2 \leq f_s \leq 11.67$

Spectral Inversion



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[4] R. G. Lyons, Understanding Digital Signal Processing, 1997