Gaussian Random Variable Gaussian Function Background

## Gaussian Random Variables

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Based on Probability, Random Variables and Random Signal Principles, P.Z. Peebles,Jr. and B. Shi







## Gaussian Density Function

### Definition

A Gaussian random variable X if its density function has the form

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-a_X)^2/2\sigma_X^2}$$

where  $\sigma_X$  and  $a_X$  are real constants.

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## Gaussian Distribution Function

#### Definition

A Gaussian distribution function has the form

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2} d\xi$$

where  $\sigma_X$  and  $a_X$  are real constants.

## Normalized Gaussian Distribution Function

### Definition

A normalized Gaussian distribution function has the form

$$F_X(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\xi^2/2} d\xi$$

where  $\sigma_X = 1$  and  $a_X = 0$ Let the function  $F(x) = F_X(x)$ 

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$
 (=  $\Phi(x)$ )

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## Normalized Gaussian Distribution Function Property

#### Theorem

Assume F(x) is a normalized (standard) Gaussian distribution function, then F(-x) = 1 - F(x)

$$F(-x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-x} e^{-\xi^2/2} d\xi$$
$$= 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$
$$= 1 - F(x)$$

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## Computing General Gaussian Distribution Function

#### Definition

A general Gaussian distribution function via a variable change  $u = (x - a_X)/\sigma_X$ 

$$x \sim \mathcal{N}(a_X, \sigma_X) \implies u \sim \mathcal{N}(0, 1)$$

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^x e^{-(\xi - a_X)^2/2\sigma_X^2} d\xi$$

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$$

$$F_X(x) = F\left(\frac{x-a_x}{\sigma_X}\right) = F(u)$$

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# Change of variables $u = (x - a_X)/\sigma_X$

Definition

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x} e^{-(\xi - a_X)^2/2\sigma_X^2} d\xi$$

et 
$$y = \xi - a_x$$
  
 $\frac{1}{\sqrt{2\pi\sigma_X^2}} \int_{-\infty}^{x-a_X} e^{-y^2/2\sigma_X^2} dy$ 

let  $z = y/\sigma_X$ 

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-a_X)/\sigma_X} e^{-z^2/2} dz = F\left(\frac{x-a_X}{\sigma_X}\right)$$

therefore  $u = (x - a_X)/\sigma_X \implies F_X(x) = F\left(\frac{x - a_X}{\sigma_X}\right)$ 

# Q Function

## Definition

 ${\sf Q}$  function is defined as

$$F(x) = 1 - Q(x)$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\xi^2/2} d\xi$$

$$Q(x) \approx \left[\frac{1}{(1-a)x + a\sqrt{x^2 + b}}\right] \frac{e^{-\xi^2/2}}{\sqrt{2\pi}}$$
$$x \ge 0$$

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## Gaussian Function

#### Definition

A Gaussian functon has the form

$$f(x) = ae^{-(x-b)^2/2c^2}$$

- *a* is the height of the curve's peak,
- *b* is the position of the center of the peak
- c (the standard deviation) controls the width of the "bell".

https://en.wikipedia.org/wiki/Gaussian\_function

## Gaussian PDF

#### Definition

- Gaussian functions
- used to represent the pdf of a normally distributed r.v.
- with expected value  $\mu=b$  and variance  $\sigma^2=c^2$

$$g(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$$

https://en.wikipedia.org/wiki/Gaussian\_function

## Gaussian Integral

## Definition

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

$$\int_{-\infty}^{\infty} a e^{-(x-b)^2/2c^2} dx = ac \cdot \sqrt{2\pi}$$

- This integral is 1 if and only if  $a = \frac{1}{c\sqrt{2\pi}}$
- in this case, the Gaussian is the p.d.f
  - of a normally distributed r.v.
  - with expected value  $\mu = b$  and variance  $\sigma^2 = c^2$ :

https://en.wikipedia.org/wiki/Gaussian\_integral

## Standard Gaussian Integral

## Definition

$$\int_{-\infty}^{\infty} a e^{-(x-b)^2/2c^2} dx$$

let y = x - b

$$a\int_{-\infty}^{\infty}e^{-y^2/2c^2}dy$$

let 
$$z = y/\sqrt{2c^2}$$

$$a\sqrt{2c^2}\int_{-\infty}^{\infty}e^{-z^2}dz$$

using 
$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$
  
$$\int_{-\infty}^{\infty} a e^{-(x-b)^2/2c^2} dx = a \cdot \sqrt{2\pi c^2}$$

 $https://en.wikipedia.org/wiki/Gaussian\_integral$ 

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## Standard Gaussian Density Function

### Definition

• the standard normal distribution.

• when 
$$\mu = 0$$
 and  $\sigma = 1$ 

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

https://en.wikipedia.org/wiki/Gaussian\_integral

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## Standard Gaussian Distribution Function

#### Definition

the standard Gaussian density function: when  $\mu = 0$  and  $\sigma = 1$ 

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

the standard Gaussian distribution function: when  $\mu = 0$  and  $\sigma = 1$ 

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-x^2/2} dx$$

https://en.wikipedia.org/wiki/Gaussian\_integral

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# General Gaussian Density Function

### Definition

- Every normal distribution is
  - a version of the standard normal distribution
  - whose domain has been stretched
  - by a factor  $\sigma$  (the standard deviation)
  - then translated by  $\mu$  (the mean value):

$$f(x|\mu,\sigma^2) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$$
$$= \frac{1}{\sqrt{2\pi\sigma}}e^{-(x-\mu)^2/2\sigma^2}$$

https://en.wikipedia.org/wiki/Gaussian\_integral

## General Gaussian Distribution Function

#### Definition

The general Gaussian density function

$$f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}$$

The general Gaussian distribution function

$$F(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx$$

https://en.wikipedia.org/wiki/Gaussian\_integral

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# Error Function (1)

#### Definition

### the Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

#### the error function

$$erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-x^2} dx$$
$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-x^2} dx$$

https://en.wikipedia.org/wiki/Error\_function

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# Error Function (2)

#### Fact

- erf(x) is a sigmoid function
- for nonnegative values of x
- for a r.v. Y that is normally distributed
- with mean 0 and variance 1/2
- erf(x) is the probability that Y falls in the range [-x,x].
- the complementary error function

$$erfc(x) = 1 - erf(x)$$

 $https://en.wikipedia.org/wiki/Error\_function$ 

# Q Function (1)

#### Definition

- the Q-function is
  - the tail distribution function
  - of the standard normal distribution.
- Q(x) is the probability
  - that a normal r.v. will obtain a value
  - larger than x standard deviations.

https://en.wikipedia.org/wiki/Q-function

# Q Function (2)

#### Theorem

If Y is a Gaussian random variable
with mean μ and variance σ<sup>2</sup>,
then X = Y - μ/σ is standard normal
and x = y - μ/σ

https://en.wikipedia.org/wiki/Q-function

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# Q Function (3)

#### Definition

$$P(Y > y) = P(X > x) = Q(x)$$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-u^2/2} du$$

$$Q(x) = 1 - Q(-x) = 1 - \Phi(x)$$

https://en.wikipedia.org/wiki/Q-function

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