Redundant CORDIC Timmermann (B)

20161231

MSD Inspection - Sign Estimation from p MSD's

Parallel Prediction of Angle Directions

Low Latency Time CORDIC Algorithms - Timmermann (1992)
Redundant and on-line CORDIC - Ercegovac & Lang (1990)

MSD Inspection - Sign Estimation from p MSD's

Parallel Prediction of Angle Directions

Constant Scaling Factor
Redundant CORDIC

$$\boxed{\begin{aligned} x_{i+1} &= x_i - m\,\sigma_i\, 2^{-s(m,i)}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-s(m,i)}\, x_i \\ z_{i+1} &= z_i - \sigma_i\, \alpha_{m,i} \end{aligned}}$$

$$\alpha_{m,i} = \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\; 2^{-s(m,i)}\right)$$

$$k_m = \prod_i \sqrt{1 + m\,\sigma_i^2\, 2^{-s(m,i)\cdot 2}}$$

$$\sigma_i = \begin{cases} \text{sign}(z_i) & \text{for } z_0 = \theta,\ z_n \to 0 \ (\text{rotation} : \sin\theta ?) \\ -\text{sign}(x_i) \cdot \text{sign}(y_i) & \text{for } y_0 = y,\ y_n \to 0 \ (\text{vector} : \sin^{-1} y ?) \end{cases}$$

$$\boxed{m = 1, \quad s(m,i) = i}$$

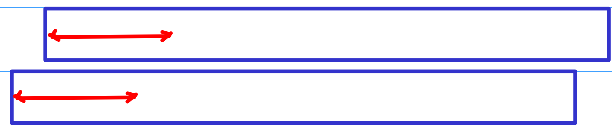
$$\boxed{\begin{aligned} x_{i+1} &= x_i - \sigma_i\, 2^{-i}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-i}\, x_i \\ z_{i+1} &= z_i - \sigma_i\, \alpha_{m,i} \end{aligned}}$$

$$\alpha_{1,i} = \tan^{-1}\left(2^{-i}\right)$$

Redundant Number in CORDIC arithmetic (SD)

σ_i has to be estimated from the inspection of some of the most significant digits MSD's

Inspecting p MSD (Most Significant Digit)



intermediate sum

intermediate carry

$\underbrace{00\cdots00}$xx ......

very close to zero : too small ← after some iterations

if all the inspected digits are all zero,
for the precise value of σ_i
need more bits from the remaining digits

the easy choice : $\sigma_i \leftarrow 0$ **freezing iteration**
but this affects k_m (data dependent)

may increase latency and chip area
by at least 80 %
still requires n sequential decisions to generate all σ_i
→ preventing a parallelization

$i \in \left(\frac{1}{4}(n-3), \frac{1}{2}(n+1)\right]$ $\boxed{\overbrace{00\cdots 0}^{p}\ \square\square\cdots\square\qquad\qquad}$

if p MSD's are all 0

for the accurate sign bit, need more lower bits

cannot use $\sigma_i = 0$ directly

without scaling factor compensation

→ whenever $\sigma_i = 0$ from inspecting a few MSD's

$m=+1$ increase the length of the vector (circular)

$m=-1$ decrease 〃 〃 〃 〃 (hyperbolic)

	$m=0$	linear
increase	$m=+1$	circular
decrease	$m=-1$	hyperbolic

→ the scale factor can be maintained while no rotation is performed.

MSD Inspection (Most Significant Digit)

determining $\sigma_i \in \{-1, 0, +1\}$ (the rotation direction)

from the inspection of the p most significant digits
$(p << n)$

the choice of $\sigma_i = 0$ is avoided
to keep a constant scaling factor

in order to allow $\sigma_i = 0$ as a valid choice

- do not rotate the vector
- but just increase the length of the vector according to the scale factor

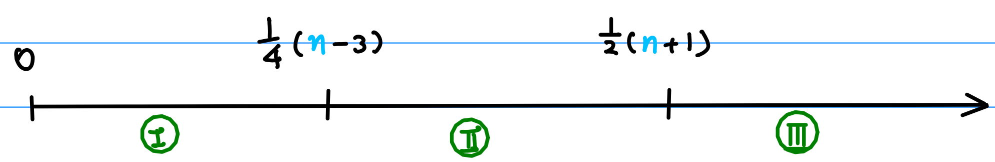
$$\boxed{\sigma_i = \begin{cases} \text{sign}(z_i) & : \text{rotation } z_n \to 0 \\ -\text{sign}(x_i)\,\text{sign}(y_i) & : \text{vectoring } y_n \to 0 \end{cases}}$$

$\sigma_i = \text{sign}(z_{i-1})$ Conventional index

(6) problem when the remaining angle is too small to allow exact detection of the sign only from inspection of a few MSD's $\boxed{\frac{1}{4}(n-3) < i}$

$z_i \approx 0$

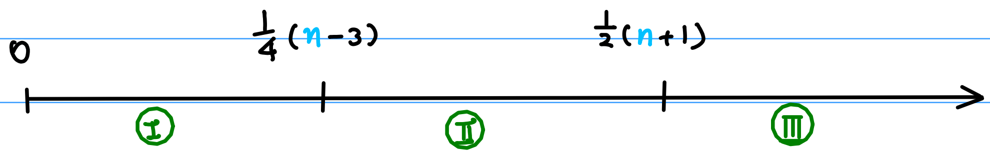
(6) No problem when $\boxed{i < \frac{1}{4}(n-3)}$ $z_i > 0$



i : iteration

n : word length

$$\begin{cases} i \leq \frac{1}{4}(n-3) & \sigma_i \neq 0 \text{ never } (z_i \text{ is sufficiently large}) \\ i > \frac{1}{4}(n-3) & \boxed{\sigma_i = 0} \text{ possible } (z_i \text{ is very small } \not\approx 0) \\ i > \frac{1}{2}(n+1) & \text{no Change in Scale Factors} \end{cases}$$



$n = 16$ $\qquad \frac{16-3}{4} = 3,25$

$\frac{32-3}{4} = 7,25$

		$\sigma_i \neq 0$	$\sigma_i = 0$
Cond I	$0 \leq i \leq (n-3)/4$	regular rotation	× z_i sufficiently large
Cond II	$(n-3)/4 < i \leq (n+1)/2$	regular rotation	no rotation, inc/dec
Cond III	$(n+1)/2 < i$	regular rotation	no rotation, no compensation

Approximation of a scale factor

i :	iteration
n :	word length

$$k_m = \prod_i \sqrt{1 + m\sigma_i^2\, 2^{-2\boxed{s(m,i)}}}$$

assume $\boxed{s(m,i) \Rightarrow i}$

$$k_m = \prod_i \sqrt{1 + m\cdot 2^{-2i}}$$

assume $\boxed{\frac{1}{4}(n-3) < i}$ *

$$k_m = \prod_i \left(1 + m\cdot 2^{-2i-1}\right)$$

0 $\quad\frac{1}{4}(n-3)\quad$ $\frac{1}{2}(n+1)$

I	II	III
$[0, \frac{1}{4}(n-3)]$	$(\frac{1}{4}(n-3), \frac{1}{2}(n+1)]$	$(\frac{1}{2}(n+1), \infty)$

scale factor does not affect

inspecting a few MSD's does not matter when $\boxed{i < \frac{1}{4}(n-3)}$

$$\sqrt{1 + m\cdot 2^{-2i}} \approx 1 + m\cdot 2^{-2i-1}$$

$$k_m = \prod_i \sqrt{1 + m\cdot 2^{-2i}} \approx \prod_i \left(1 + m\cdot 2^{-2i-1}\right)$$

i :	iteration
n :	word length

$z_i \to 0$ as $i \to \infty$

n-bit

$i \in [0, \frac{1}{4}(n-3)]$ $\square\square \cdots \square$

regular rotation

checking a few MSD : no problem

$i \in (\frac{1}{4}(n-3), \frac{1}{2}(n+1)]$ $00 \cdots 0 \; \square\square \cdots \square$

$\sigma_i = 0$ allowed — no rotation
only with the compesation of
the scale factor $\Rightarrow$ inc/dec

$i \in (\frac{1}{2}(n+1), \infty)$ $00 \cdots 0 \; 00 \cdots 0 \; \square\square \cdots \cdots \cdots \square$

$\sigma_i = 0$ allowed — rotation with
no compensation of the scale factor
no inc/dec a rotating vector

Low Latency Time CORDIC Algorithms - Timmerman (1992)

$\overleftrightarrow{\quad p \quad}$

$\boxed{00 \cdots 0 \, \square\square \cdots \square \qquad\qquad\qquad}$

if p MSD's are all 0 and $i \in \left[\frac{1}{4}(n-3), \frac{1}{2}(n+1)\right]$

- no rotation
- only inc/dec the length of a vector

$m=+1$ / $m=-1$

$$\boxed{\begin{aligned} x_{i+1} &= x_i + m \cdot 2^{-2i-1} x_i && \text{inc/dec} \\ y_{i+1} &= y_i + m \cdot 2^{-2i-1} y_i && \text{inc/dec} \\ z_{i+1} &= z_i \end{aligned}}$$

$$\begin{aligned} x_{i+1} &= \left(\boxed{1 + m \cdot 2^{-2i-1}}\right) x_i && \text{inc/dec} \\ y_{i+1} &= \left(\boxed{1 + m \cdot 2^{-2i-1}}\right) y_i && \text{inc/dec} \end{aligned}$$

$$\sqrt{1 + 2^{-2i}} \approx 1 + 2^{-2i-1}$$

$m=0$	linear	
$m=+1$	circular	increase
$m=-1$	hyperbolic	decrease

$$m=1, \quad s(m,i)=i$$

Cond (I) $\quad 0 \leq i \leq \frac{1}{4}(n-3)$

$$\boxed{\begin{aligned} x_{i+1} &= x_i - \sigma_i\, 2^{-i}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-i}\, x_i \\ z_{i+1} &= z_i - \sigma_i \tan^{-1}(2^{-i}) \end{aligned}}$$

regular rotation

Cond (II) $\quad \frac{1}{4}(n-3) < i \leq \frac{1}{2}(n+1)$

$\sigma_i \neq 0$

$$\boxed{\begin{aligned} x_{i+1} &= x_i - \sigma_i\, 2^{-i}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-i}\, x_i \\ z_{i+1} &= z_i - \sigma_i \tan^{-1}(2^{-i}) \end{aligned}}$$

regular rotation

$\sigma_i = 0$

$$\boxed{\begin{aligned} x_{i+1} &= x_i + m \cdot 2^{-2i-1}\, x_i \\ y_{i+1} &= y_i + m \cdot 2^{-2i-1}\, y_i \\ z_{i+1} &= z_i \end{aligned}}$$

inc/ dec the length

Cond (III) $\quad \frac{1}{2}(n+1) < i$

$\sigma_i \neq 0$

$$\boxed{\begin{aligned} x_{i+1} &= x_i - \sigma_i\, 2^{-i}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-i}\, x_i \\ z_{i+1} &= z_i - \sigma_i \tan^{-1}(2^{-i}) \end{aligned}}$$

regular rotation

$\sigma_i = 0$

$$\boxed{\begin{aligned} x_{i+1} &= x_i \\ y_{i+1} &= y_i \\ z_{i+1} &= z_i \end{aligned}}$$

no change

MSD Inspection - Sign Estimation from p MSD's

Parallel Prediction of Angle Directions

Constant Scaling Factor
Redundant CORDIC

Parallel Generation of σ_i

Simple Conversion Rule

Z_{i-1}	Z_i	σ_i
0	□	-1
1	□	+1

$z_0 = \boxed{z_0 \; z_1 \; \cdots \; z_{n-2} \; z_{n-1}}$

n-bit

$z_0 = \boxed{\sigma_0 \; \sigma_1 \; \cdots \; \sigma_{n-2} \; \sigma_{n-1}}$

$Z_i \in \{0, 1\}$

conversion

$\sigma_i \in \{-1, 1\}$

Conversion Criteria

$\boxed{\text{the absolute value of the remaining rotation angle}} \leq \boxed{\text{two times of the actual rotation angle}}$

$$|z_i| \leq 2 \cdot \alpha_{m,i}$$

rotation angle
residue angle

j k

binary angle $\boxed{0 \text{ — } 0 \; Z_j \; Z_{j+1} \; \cdots \; Z_k \quad\quad}$

$\Downarrow$

Signed Digit angle (SD) $\boxed{\sigma_j \; \sigma_{j+1} \; \cdots \; \sigma_k}$

$\boxed{\sigma_j \; \sigma_{j+1} \; \cdots \; \sigma_k}\boxed{\sigma_k}$

repeat one more time → error correction

$$\boxed{k \lesssim 3j + 1.5}$$

$j=1$ $\quad 1 \sim 3\cdot 1+1 = 4$

$j=4$ $\quad 4 \sim 3\cdot 4+1 = 13$

$j=13$ $\quad 13 \sim 3\cdot 13+1 = 40$

σ_i prediction $\quad \sigma_1 \cdots \sigma_4 \quad \sigma_4 \cdots \sigma_{13} \quad \sigma_{13} \cdots \sigma_{40}$

Notation z_i & Z_i

iteration

$z_0 \leftarrow \theta$ initial angle

z_1

z_2

$\vdots$

$z_n \to 0$ (rotation mode)

residual angle (remaining rotation angle)

each z_i has a binary representation $\{Z_i\}$

i-th bit $Z_i \in \{0, 1\}$ ordinary binary digit

n- bits

z_i : $\boxed{Z_0 \, Z_1 \quad \cdots \quad Z_i \quad \cdots \quad Z_{n-2} \, Z_{n-1}}$

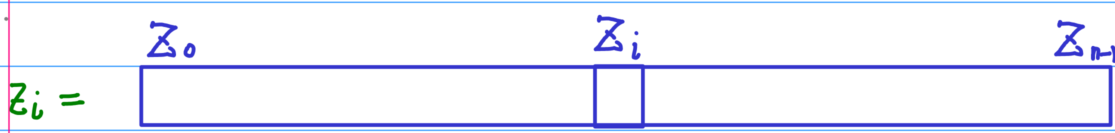
msb

lsb

Convergence Criteria

after a sufficiently large number of iteration,
at the i-th iteration

$$\alpha_{m,i} = \tan^{-1}(2^{-i}) \approx 2^{-i}$$



z_i: 0 —— 1 0 0 —— 0 $\quad \boxed{z_i = 2\alpha_i}$

α_i: 0 —— 0 $\boxed{1}$ 0 —— 0

$z_i - \alpha_i$: 0 —— 0 1 0 —— 0

can be eliminated at the $(i+1)^{st}$ iteration. (next)

z_i: 0 —— 1 1 X —— X $\quad \boxed{z_i > 2\alpha_i}$

α_i: 0 —— 0 $\boxed{1}$ 0 —— 0

$z_i - \alpha_i$: 0 —— 1 0 X —— X

more than one iteration required to eliminate this 1

the absolute value of the remaining rotation angle	$\leqslant$	two times of the actual rotation angle

$$|z_i| \leqslant 2 \cdot \alpha_{m,i}$$

Conversion

(I) Ordinary Binary $\boxed{z_0\, z_1 \;\cdots\; z_{n-2}\, z_{n-1}}$ $\longrightarrow$ Redundant Binary $\boxed{\sigma_0\, \sigma_1 \;\cdots\; \sigma_{n-2}\, \sigma_{n-1}}$

msb

z_0	σ_i
0	1
1	$\bar{1}$

z_{i-1}	$\boxed{z_i}$	$\boxed{\sigma_i}$
0	$\square$	$\bar{1}$
1	$\square$	1

Simple Conversion Rule

(II) Ordinary Binary $\boxed{z_0\, z_1 \;\cdots\; z_{n-2}\, z_{n-1}}$ $\longleftarrow$ Redundant Binary $\boxed{\sigma_0\, \sigma_1 \;\cdots\; \sigma_{n-2}\, \sigma_{n-1}}$

	1	0	1	0	1	0	0	0
$\ominus$	0	1	0	1	0	1	1	1
	0	1	0	1	0	0	0	1

1	$\bar{1}$	1	$\bar{1}$	1	$\bar{1}$	$\bar{1}$	$\bar{1}$

conventional binary adder CLA

Ordinary Binary $\longrightarrow$ Redundant Binary

$\boxed{z_0 \; z_1 \; \cdots \; z_{n-2} \; z_{n-1}} \longrightarrow \boxed{\sigma_0 \; \sigma_1 \; \cdots \; \sigma_{n-2} \; \sigma_{n-1}}$

Based on Baker's Prediction Scheme

for explanation,
assume multiply-and-add

$m=0$	linear
$m=+1$	circular
$m=-1$	hyperbolic

$\boxed{m=0, \quad \text{rotation mode}}$

$$
\begin{aligned}
x_{i+1} &= x_i & &\leftarrow x_0 \\
y_{i+1} &= y_i + \sigma_i 2^{-i} x_i & &\rightarrow y_0 + z_0 x_0 \\
z_{i+1} &= z_i - \sigma_i 2^{-i} & &\rightarrow 0 \ \text{(rotation)}
\end{aligned}
$$

$\leftarrow x_0$ $(i=0,$ n-bit num$)$

$\leftarrow y_0$ $(i=0,$ n-bit num$)$

σ_0	σ_1	σ_2	σ_3	$\cdots$	σ_{n-1}
2^0	2^{-1}	2^{-2}	2^{-3}		2^{-n+1}

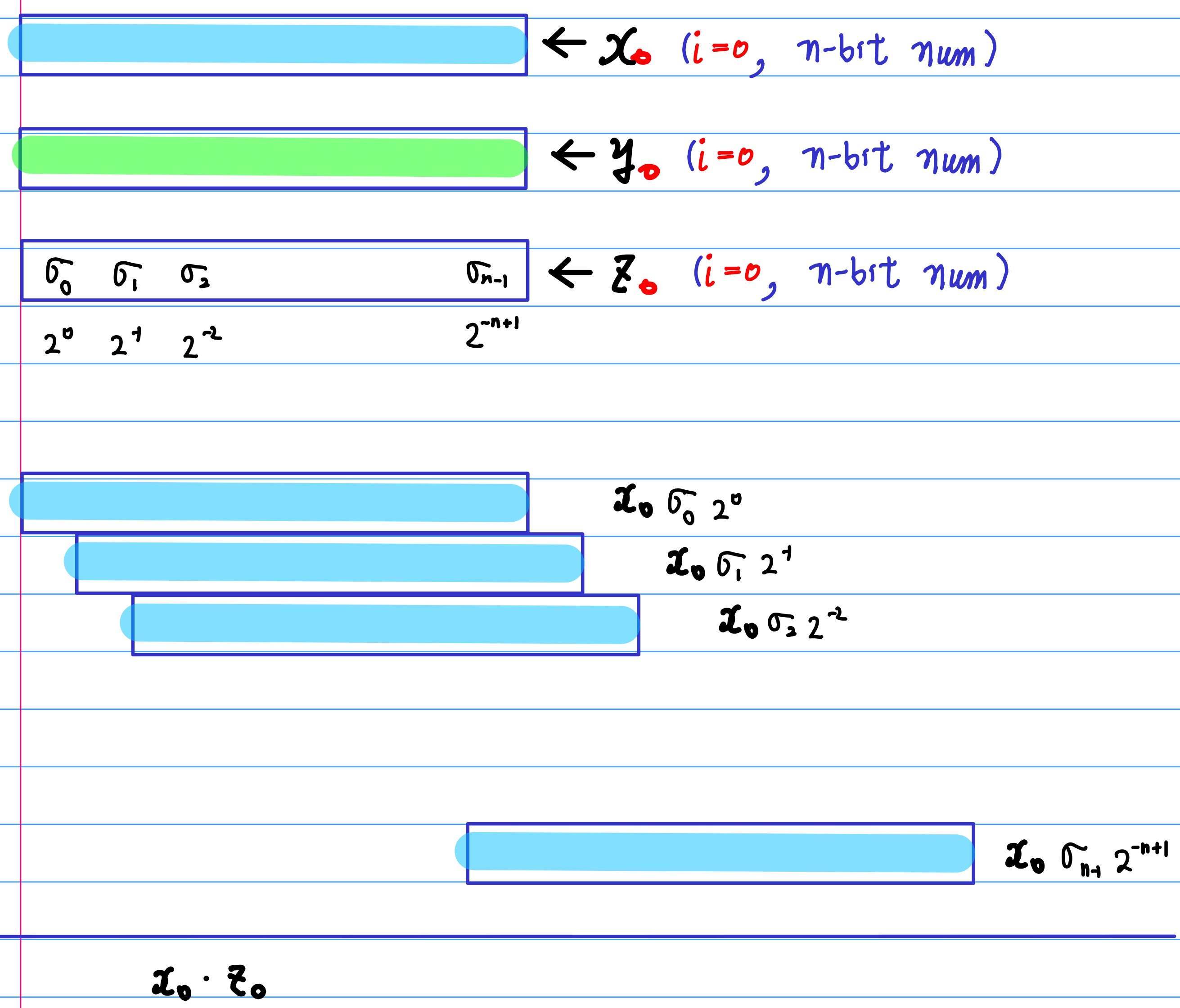
$\leftarrow z_0$ $(i=0,$ n-bit num$)$

$$\boxed{y_0 + z_0 \cdot x_0 \leftarrow y_n} = \boxed{y_0} + \boxed{\begin{matrix} x_0 \\ \sigma_0 2^0 \end{matrix}} + \boxed{\begin{matrix} x_1 \\ \sigma_1 2^{-1} \end{matrix}} + \boxed{\begin{matrix} x_2 \\ \sigma_2 2^{-2} \end{matrix}} + \boxed{\begin{matrix} x_3 \\ \sigma_3 2^{-3} \end{matrix}} + \cdots + \boxed{\begin{matrix} x_{n-1} \\ \sigma_{n-1} 2^{-n+1} \end{matrix}}$$

$$= \boxed{y_0} + \boxed{\begin{matrix} x_0 \\ \sigma_0 2^0 \end{matrix}} + \boxed{\begin{matrix} x_0 \\ \sigma_1 2^{-1} \end{matrix}} + \boxed{\begin{matrix} x_0 \\ \sigma_2 2^{-2} \end{matrix}} + \boxed{\begin{matrix} x_0 \\ \sigma_3 2^{-3} \end{matrix}} + \cdots + \boxed{\begin{matrix} x_0 \\ \sigma_{n-1} 2^{-n+1} \end{matrix}}$$

$\boxed{x_{i+1} = x_i = x_0}$

$$
\begin{aligned}
x_{i+1} &= x_i & \leftarrow x_0 \\
y_{i+1} &= y_i + \sigma_i 2^{-i} x_i & \rightarrow y_0 + z_0 x_0 \\
z_{i+1} &= z_i - \sigma_i 2^{-i} & \rightarrow 0 \ \text{(rotation)}
\end{aligned}
$$



$$\boxed{\begin{aligned} x_{i+1} &= x_i \\ y_{i+1} &= y_i + \sigma_i 2^{-i} x_i \\ z_{i+1} &= z_i - \sigma_i 2^{-i} \end{aligned}} \quad \begin{aligned} &\leftarrow x_0 \\ &\rightarrow y_0 + z_0 x_0 \\ &\rightarrow 0 \text{ (rotation)} \end{aligned}$$

$$\boxed{0 \leftarrow z_n} = \boxed{z_0} \boxed{- \sigma_0 2^0 - \sigma_1 2^{-1} - \sigma_2 2^{-2} - \sigma_3 2^{-3} - \cdots - \sigma_{n-1} 2^{-n+1}}$$

$$\boxed{z_0} = \boxed{+ \sigma_0 2^0 + \sigma_1 2^{-1} + \sigma_2 2^{-2} + \sigma_3 2^{-3} + \cdots + \sigma_{n-1} 2^{-n+1}}$$

signed digit binary $(\sigma_0, \quad \sigma_1, \quad \sigma_2, \quad \sigma_3, \quad \cdots, \quad \sigma_{n-1}) = z_0$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

ordinary binary $(z_0, \quad z_1, \quad z_2, \quad z_3, \quad \cdots \quad, \quad z_{n-1}) = z_0$

$$\boxed{\begin{aligned} x_{i+1} &= x_i - m\,\sigma_i\, 2^{-s(m,i)}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-s(m,i)}\, x_i \\ z_{i+1} &= z_i - \sigma_i\, \alpha_{m,i} \end{aligned}}$$

$$\alpha_{m,i} = \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\; 2^{-s(m,i)}\right)$$

$$k_m = \prod_i \sqrt{1 + m\,\sigma_i^2\, 2^{-2s(m,i)}}$$

$$\sigma_i = \begin{cases} \mathrm{sign}(z_i) & \text{for } z_n \to 0 \quad (\text{rotation}) \\ -\mathrm{sign}(x_i)\cdot \mathrm{sign}(y_i) & \text{for } y_n \to 0 \quad (\text{vectoring}) \end{cases}$$

$$\boxed{m=0, \quad s(m,i) = i}$$ linear, rotation

$$\boxed{\begin{aligned} x_{i+1} &= x_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-i}\, x_i \\ z_{i+1} &= z_i - \sigma_i\, 2^{-i} \end{aligned}}$$

$$\alpha_{0,i} = \lim_{m\to 0} \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\; 2^{-s(m,i)}\right) = 2^{-i}$$

$$k_0 = \prod_i \sqrt{1 + 0\cdot\sigma_i^2\, 2^{-2s(m,i)}} = 1$$

$$\sigma_i = \mathrm{sign}(z_i) \qquad \text{for } z_n \to 0 \; (\text{rotation})$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| \le 1, x \ne \pm i$$

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \cdots$$

$$\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,k)}\right)$$

$$= \frac{1}{\sqrt{m}} \left[\left(\sqrt{m}\, 2^{-s(m,k)}\right) - \frac{1}{3}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^3 + \frac{1}{5}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^5 - \cdots \right]$$

$$= 2^{-s(m,k)} - \frac{1}{3} m\, 2^{-3s(m,k)} + \frac{1}{5} m^2 2^{-5s(m,k)} - \cdots$$

$$\alpha_{0,i} = \lim_{m \to 0} \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right) = 2^{-i}$$

$$= \lim_{m \to 0} 2^{-s(m,k)} - \frac{1}{3} \cancelto{0}{m}\, 2^{-3s(m,k)} + \frac{1}{5} \cancelto{0}{m^2}\, 2^{-5s(m,k)} - \cdots$$

$$= 2^{-s(m,k)} = 2^{-i}$$

$$\boxed{m = 0, \quad s(0,i) = i}$$

linear, rotation

$$\boxed{m=0, \quad s(m,i) = i}$$ linear, rotation

$$y_n = y_0 + \left(\sigma_0 2^0 + \sigma_1 2^{-1} + \sigma_2 2^{-2} + \sigma_3 2^{-3} + \cdots + \sigma_{n-1} 2^{-n+1}\right) x_0 .$$

$$\boxed{0 \leftarrow z_n} = \boxed{z_0} \boxed{- \sigma_0 2^0 - \sigma_1 2^{-1} - \sigma_2 2^{-2} - \sigma_3 2^{-3} - \cdots - \sigma_{n-1} 2^{-n+1}}$$

$$\boxed{z_0} = \boxed{\sigma_0 2^0 + \sigma_1 2^{-1} + \sigma_2 2^{-2} + \sigma_3 2^{-3} + \cdots + \sigma_{n-1} 2^{-n+1}} , \quad \sigma_i \in \{-1, 1\}$$

conversion

$$= \boxed{Z_0 2^0 + Z_1 2^{-1} + Z_2 2^{-2} + Z_3 2^{-3} + \cdots + Z_{n-1} 2^{-n+1}} , \quad Z_i \in \{0, 1\}$$

redundant binary $\boxed{\overset{2^0}{\sigma_0} \; \overset{2^{-1}}{\sigma_1} \; \cdots \; \overset{2^{-n+2}}{\sigma_{n-2}} \; \overset{2^{-n+1}}{\sigma_{n-1}}}$ $\quad \sigma_i \in \{-1, 1\}$

conversion

ordinary binary $\boxed{\overset{2^0}{z_0} \; \overset{2^{-1}}{z_1} \; \cdots \; \overset{2^{-n+2}}{z_{n-2}} \; \overset{2^{-n+1}}{z_{n-1}}}$ $\quad Z_i \in \{0, 1\}$

$\boxed{m = \pm 1, \quad s(m,i) = i}$

$m = 0$	linear
$m = +1$	circular
$m = -1$	hyperbolic

$$\boxed{\begin{aligned} x_{i+1} &= x_i - m\,\sigma_i\, 2^{-s(m,i)}\, y_i \\ y_{i+1} &= y_i + \sigma_i\, 2^{-s(m,i)}\, x_i \\ z_{i+1} &= z_i - \sigma_i\, \alpha_{m,i} \end{aligned}}$$

$0 \leftarrow$

* complicated conversion
complex decomposition

$$z_0 = \sum_i \sigma_i \boxed{\alpha_{m,i}} = \sum_i \sigma_i \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\; 2^{-s(m,i)}\right)}$$

$$\boxed{\alpha_{m,i} = \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\; 2^{-s(m,i)}\right)} \neq 2^{-i}$$

Since $\alpha_{m,i} \neq 2^{-i}$, the simple conversion rule
can not be directly applied

$\lim_{i\to\infty} \alpha_{m,i} \to 2^{-i}$ for sufficiently large i

∴ the simple conversion rule can still be used
if the prediction error correction is also used

As a result of the rotation operation

ordinary binary $\xrightarrow{\text{recode}}$ redundant binary

$\boxed{z_0\, z_1\, \cdots\, z_{n-2}\, z_{n-1}}$ $\quad$ $\boxed{\sigma_0\, \sigma_1\, \cdots\, \sigma_{n-2}\, \sigma_{n-1}}$

$z_i \in \{0, 1\}$ $\quad$ $\sigma_i \in \{-1, 1\}$

$z_0 = \boxed{z_0\, z_1\, \cdots\, z_{n-2}\, z_{n-1}}$ $\quad$ $z_i \in \{0, 1\}$

conversion

$\boxed{0 \leftarrow z_n}$ $\quad$ $z_0 = \boxed{\sigma_0\, \sigma_1\, \cdots\, \sigma_{n-2}\, \sigma_{n-1}}$ $\quad$ $\sigma_i \in \{-1, 1\}$

- recoding the **binary** representation z_0 using $\boxed{z_i \in \{0, 1\}}$

$\Updownarrow$

- finding σ_i's : **SD** notation of z_0 using $\boxed{\sigma_i \in \{-1, +1\}}$

Simple Conversion Rule

directly obtaining all σ_i's from $Z_0 = Z_0 Z_1 Z_2 \cdots Z_{n-1}$

$$\boxed{Z_0\ Z_1\ \cdots\ Z_{n-2}\ Z_{n-1}} \longrightarrow \boxed{\sigma_0\ \sigma_1\ \cdots\ \sigma_{n-2}\ \sigma_{n-1}}$$

given

parallel processing

Z_{i-1}	Z_i	σ_i
0	0	-1
0	1	-1
1	0	$+1$
1	1	$+1$

1 0 1 0 $\quad 2+8=10$

$1\ 1\ \bar{1}\ 1 \quad 8+4-2$

$= 10$

$1\ 0 \rightarrow 1$

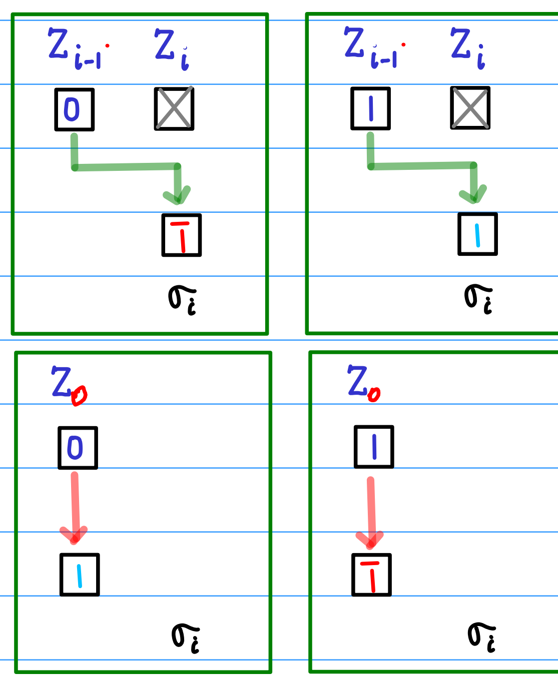
$0\ 1 \rightarrow \bar{1}$

$1\ 0 \rightarrow 1$

Z_{i-1}	Z_i	σ_i
0	$\square$	$\bar{1}$
1	$\square$	1

msb

Z_0	σ_i
0	1
1	$\bar{1}$



msb

Z_0	σ_i
0	1
1	$\bar{1}$

Z_{i-1}	Z_i	σ_i
0	$\square$	$\bar{1}$
1	$\square$	1

0	0 0 0 0	0 0 0 0	1 $\bar{1}$ $\bar{1}$ $\bar{1}$	$8-4-2-1 = 1$	0
1	0 0 0 1	0 0 0 1	1 $\bar{1}$ $\bar{1}$ $\bar{1}$	$8-4-2-1 = 1$	1
2	0 0 1 0	0 0 1 0	1 $\bar{1}$ $\bar{1}$ 1	$8-4-2+1 = 3$ -1	2
3	0 0 1 1	0 0 1 1	1 $\bar{1}$ $\bar{1}$ 1	$8-4-2+1 = 3$	3
4	0 1 0 0	0 1 0 0	1 $\bar{1}$ 1 $\bar{1}$	$8-4+2-1 = 5$ -1	4
5	0 1 0 1	0 1 0 1	1 $\bar{1}$ 1 $\bar{1}$	$8-4+2-1 = 5$	5
6	0 1 1 0	0 1 1 0	1 $\bar{1}$ 1 1	$8-4+2+1 = 7$ -1	6
7	0 1 1 1	0 1 1 1	1 $\bar{1}$ 1 1	$8-4+2+1 = 7$	7
-8	1 0 0 0	1 0 0 0	$\bar{1}$ 1 $\bar{1}$ $\bar{1}$	$-8+4-2-1 = -7$ -1	-8
-7	1 0 0 1	1 0 0 1	$\bar{1}$ 1 $\bar{1}$ $\bar{1}$	$-8+4-2-1 = -7$	-7
-6	1 0 1 0	1 0 1 0	$\bar{1}$ 1 $\bar{1}$ 1	$-8+4-2+1 = -5$ -1	-6
-5	1 0 1 1	1 0 1 1	$\bar{1}$ 1 $\bar{1}$ 1	$-8+4-2+1 = -5$	-5
-4	1 1 0 0	1 1 0 0	$\bar{1}$ 1 1 $\bar{1}$	$-8+4+2-1 = -3$ -1	-4
-3	1 1 0 1	1 1 0 1	$\bar{1}$ 1 1 $\bar{1}$	$-8+4+2-1 = -3$	-3
-2	1 1 1 0	1 1 1 0	$\bar{1}$ 1 1 1	$-8+4+2+1 = -1$ -1	-2
-1	1 1 1 1	1 1 1 1	$\bar{1}$ 1 1 1	$-8+4+2+1 = -1$	-1

msb

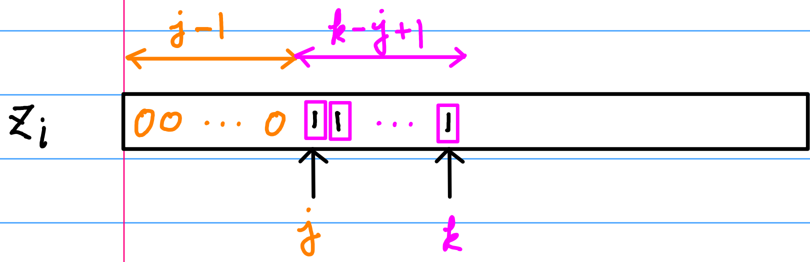
Increasing one more bit — no help

Como la aurora va en aumento de luz

0	0 0 0 0	0 0 0 0	1 T T T T	8-4-2-1 = 1-.5	0
1	0 0 0 1	0 0 0 1	1 T T T 1	8-4-2-1 = 1+.5	1
2	0 0 1 0	0 0 1 0	1 T T 1 T	8-4-2+1 = 3-.5	2
3	0 0 1 1	0 0 1 1	1 T T 1 1	8-4-2+1 = 3+.5	3
4	0 1 0 0	0 1 0 0	1 T 1 T T	8-4+2-1 = 5-.5	4
5	0 1 0 1	0 1 0 1	1 T 1 T 1	8-4+2-1 = 5+.5	5
6	0 1 1 0	0 1 1 0	1 T 1 1 T	8-4+2+1 = 7-.5	6
7	0 1 1 1	0 1 1 1	1 T 1 1 1	8-4+2+1 = 7+.5	7
-8	1 0 0 0	1 0 0 0	T 1 T T T	-8+4-2-1 = -7-.5	-8
-7	1 0 0 1	1 0 0 1	T 1 T T 1	-8+4-2-1 = -7+.5	-7
-6	1 0 1 0	1 0 1 0	T 1 T 1 T	-8+4-2+1 = -5-.5	-6
-5	1 0 1 1	1 0 1 1	T 1 T 1 1	-8+4-2+1 = -5+.5	-5
-4	1 1 0 0	1 1 0 0	T 1 1 T T	-8+4+2-1 = -3-.5	-4
-3	1 1 0 1	1 1 0 1	T 1 1 T 1	-8+4+2-1 = -3+.5	-3
-2	1 1 1 0	1 1 1 0	T 1 1 1 T	-8+4+2+1 = -1-.5	-2
-1	1 1 1 1	1 1 1 1	T 1 1 1 1	-8+4+2+1 = -1+.5	-1

msb

Prediction Error Upper Bound



$(j-1)$ leading zeros

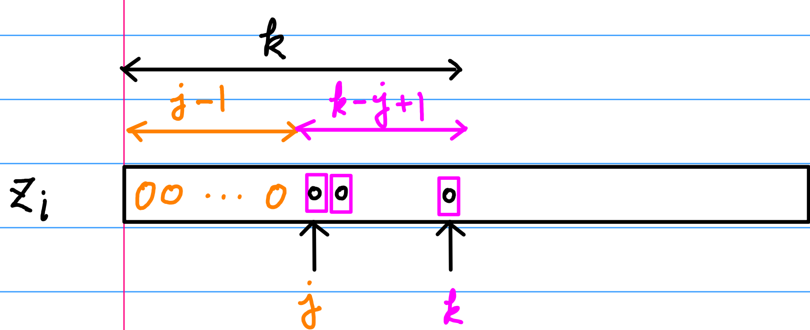
$(k-j+1)$ following ones

i-th iteration $(j \leqslant i \leqslant k)$

MSD inspection

→ prediction error

→ $\boxed{\text{the maximum accumulated error} \leqslant 2^{-s(m,k)}}$



+

$\boxed{\text{Repeating one more } k\text{-th iteration}}$

→ the maximum accumulated error can be eliminated

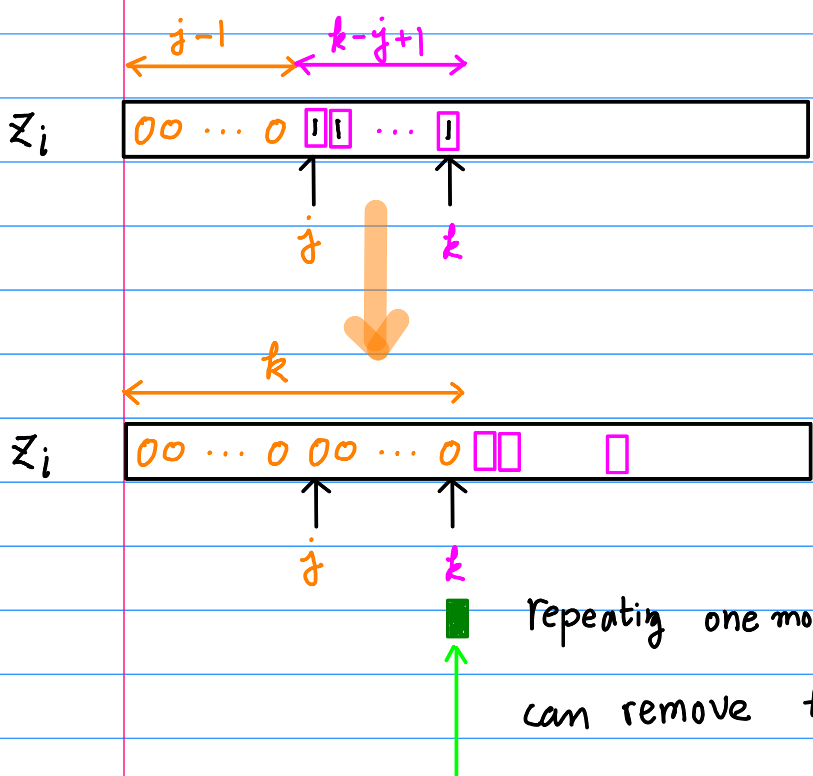
Permissible j, k combinations.

the maximum accumulated error from j-th to k-th iterations $\leqslant 2^{-s(m,k)}$

$$\sum_{i=j}^{k} \left| 2^{-i} - \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right) \right| \leqslant 2^{-s(m,k)}$$

→ $k < 3j + 1$

→ the repetition of the k-th iteration guarantees k leading zeros.



$(j-1)$ leading zeros

$(k-j+1)$ following ones

repeating one more k-th iteration

can remove the maximum accumulated error

① 1991, ISCAS, Timmermann

A Low Latency time CORDIC algorithm with increased parallelism

$$\boxed{\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| \leqslant \alpha_{m,k}}$$

② 1992, IEEE Trans. on Computers, Timmerman

Low Latency time CORDIC algorithms

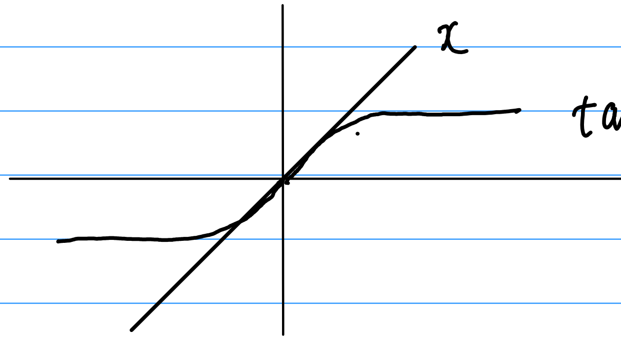
$$\boxed{\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| \leqslant 2^{-s(m,k)}}$$

$$\boxed{\alpha_{m,i} = \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)} \qquad \tan^{-1}(x) < x$$

$$\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| \leqslant \alpha_{m,k} < 2^{-s(m,k)}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| \le 1, x \ne \pm i$$

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots + \dots$$ Taylor Series



$\tan^{-1}(x) < x$

$\alpha_{m,k}$

$$= \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,k)}\right)$$

$$= \frac{1}{\sqrt{m}} \left[\left(\sqrt{m}\, 2^{-s(m,k)}\right) - \frac{1}{3}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^3 + \frac{1}{5}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^5 - \dots \right]$$

$$= 2^{-s(m,k)} - \frac{1}{3} m\, 2^{-3s(m,k)} + \frac{1}{5} m^2 2^{-5s(m,k)} - \dots$$

$< 2^{-s(m,k)}$

$$\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| < \alpha_{m,k} < 2^{-s(m,k)}$$

$$\boxed{s(m, i) = i}$$

$$z_0 = \sum_i \sigma_i \boxed{\alpha_{m,i}}$$

$$(m = 0) \quad = \sum_i \sigma_i \boxed{2^{-i}}$$

$$(m = \pm 1) \quad = \sum_i \sigma_i \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)}$$

$(j \leq i \leq k)$ $\qquad \sigma_i = 1$ for computing max error

$m = \pm 1$

j	k
0	1
1	4
4	13
13	40
40	121

→ only $m = +1$

z_i the remaining angle after the i-th iteration

$\alpha_{m,i}$ the rotation angle at the i-th iteration

$$|z_i| \leqslant 2 \cdot \alpha_{m,i}$$

convergence criterion:

$\boxed{\text{the absolute value of the remaining rotation angle}} \leqslant \boxed{\text{two times of the actual rotation angle}}$

$$|z_i| \leqslant 2 \cdot \alpha_{m,i}$$

	remaining angle	rotation angle	convergence criterion
$i=1$	z_1	$\pm \alpha_{m,1}$	$\lvert z_1 \rvert \leqslant 2 \cdot \alpha_{m,1}$
$i=2$	z_2	$\pm \alpha_{m,2}$	$\lvert z_2 \rvert \leqslant 2 \cdot \alpha_{m,2}$
$i=j$	z_j	$\pm \alpha_{m,j}$	$\lvert z_j \rvert \leqslant 2 \cdot \alpha_{m,j}$
$i=k$	z_k	$\pm \alpha_{m,k}$	$\lvert z_k \rvert \leqslant 2 \cdot \alpha_{m,k}$

$$\boxed{z_n \leftarrow 0}$$

	remaining angle	estimated angle		remaining angle	rotation angle	
$i=1$	$\hat{z}_1$	$\pm 2^{-1}$		z_1	$\pm \alpha_{m,1}$	$\lvert z_1 \rvert \leqslant 2 \cdot \alpha_{m,1}$
$i=2$	$\hat{z}_2$	$\pm 2^{-2}$		z_2	$\pm \alpha_{m,2}$	$\lvert z_2 \rvert \leqslant 2 \cdot \alpha_{m,2}$
			$\rightsquigarrow$			
$i=j$	$\hat{z}_j$	$\pm 2^{-j}$		z_j	$\pm \alpha_{m,j}$	$\lvert z_j \rvert \leqslant 2 \cdot \alpha_{m,j}$
$i=k$	$\hat{z}_k$	$\pm 2^{-k}$		z_k	$\pm \alpha_{m,k}$	$\lvert z_k \rvert \leqslant 2 \cdot \alpha_{m,k}$

prediction error!

converge?

$$
\begin{array}{rcccc}
 & 1 & 1 & \cdots & 1 \\
\hline
 & \bar{1} & 1 & & 1 \\
\hline
 & \bar{1} & 0 & \cdots & 0 \\
= & 0 & \bar{1} & \cdots & \bar{1} \\
+ & 0 & 0 & \cdots & \bar{1} \\
\hline
+ & 0 & 1 & \cdots & 1 \\
\hline
 & 0 & 0 & \cdots & \bar{1}
\end{array}
\qquad
\begin{array}{rccccc}
 & 0 & 0 & 0 & \cdots & 0 \\
\hline
 & 1 & \bar{1} & \bar{1} & \cdots & \bar{1} \\
\hline
 & 1 & 0 & 0 & \cdots & 0 \\
= & 0 & 1 & 1 & \cdots & 1 \\
+ & 0 & 0 & 0 & \cdots & 1 \\
\hline
+ & 0 & \bar{1} & \bar{1} & & \bar{1} \\
\hline
 & 0 & 0 & 0 & & 1
\end{array}
$$

$$
\boxed{\begin{array}{rcccc}
 & 1 & 1 & \cdots & 1 \\
= & \bar{1} & 1 & \cdots & 1 \\
= & 0 & 0 & \cdots & \bar{1}
\end{array}}
\qquad
\boxed{\begin{array}{rcccc}
 & 0 & 0 & \cdots & 0 \\
= & 1 & \bar{1} & \cdots & \bar{1} \\
= & 0 & 0 & \cdots & 1
\end{array}}
$$

Redundant Number system

many representation
not unique.

$$\boxed{\begin{array}{lcccc} & 1 & 1 & \cdots & 1 \\ = & \bar{1} & 1 & \cdots & 1 \\ = & 0 & 0 & \cdots & \bar{1} \end{array}}$$

$$\boxed{\begin{array}{lcccc} & 0 & 0 & \cdots & 0 \\ = & 1 & \bar{1} & \cdots & \bar{1} \\ = & 0 & 0 & \cdots & 1 \end{array}}$$

$i = 0 \quad 1 \quad 2 \quad \cdots \quad j \quad \cdots \quad k$

$$\begin{array}{cccccccc} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ 1 & \bar{1} & \bar{1} & \cdots & \bar{1} & 1 & 1 & \cdots & 1 \end{array}$$

$$\begin{array}{cccccccc} 0 & 0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ & & & & \bar{1} & 1 & 1 & \cdots & 1 \end{array}$$

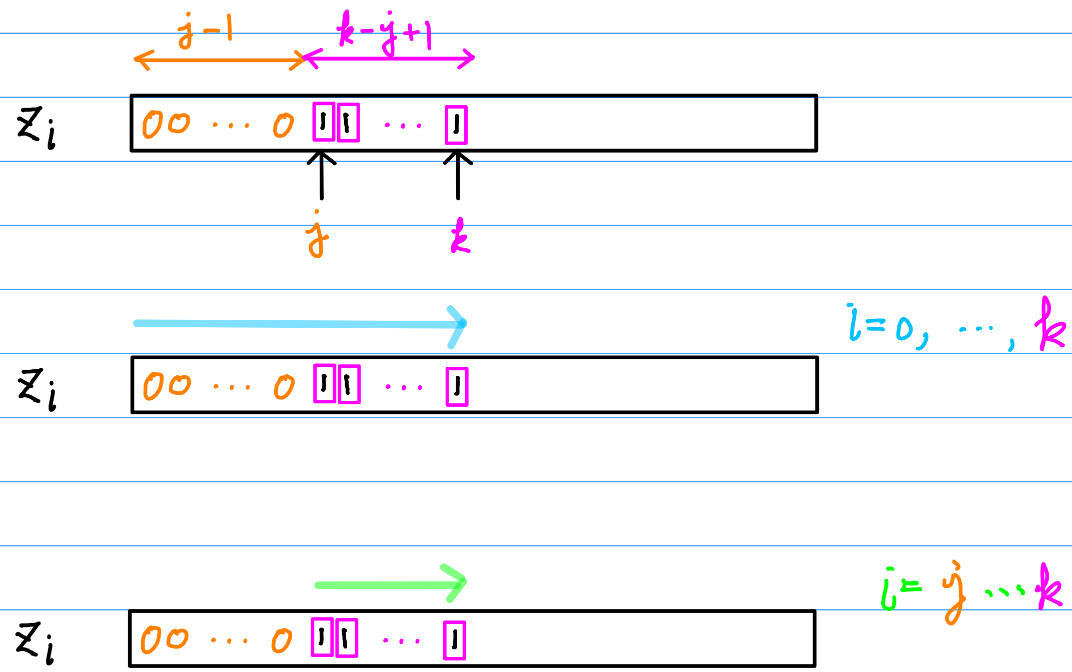
$$\boxed{0 \quad 0 \quad 0 \cdots 0 \quad 0 \quad 0 \cdots \bar{1}}$$

$=$

$i = 0 \quad 1 \quad 2 \quad \cdots \quad j \quad \cdots \quad k$

$$\begin{array}{cccccccc} 0 & 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\ & & & & & \bar{1} & 1 & \cdots & 1 \end{array}$$

$$\boxed{0 \quad 0 \quad 0 \cdots 0 \quad 0 \quad 0 \cdots \bar{1}}$$



	remaining angle	estimated angle		remaining angle	rotation angle	
$i=1$	$\hat{z}_1$	$\pm 2^{-1}$		z_1	$\pm\alpha_{m,1}$	$\lvert z_1 \rvert \leqslant 2\cdot\alpha_{m,1}$
$i=2$	$\hat{z}_2$	$\pm 2^{-2}$		z_2	$\pm\alpha_{m,2}$	$\lvert z_2 \rvert \leqslant 2\cdot\alpha_{m,2}$
$i=j$	$\hat{z}_j$	$\pm 2^{-j}$	$\rightsquigarrow$	z_j	$\pm\alpha_{m,j}$	$\lvert z_j \rvert \leqslant 2\cdot\alpha_{m,j}$
$i=k$	$\hat{z}_k$	$\pm 2^{-k}$		z_k	$\pm\alpha_{m,k}$	$\lvert z_k \rvert \leqslant 2\cdot\alpha_{m,k}$

prediction error!

converge?

	remaining angle	estimated angle		remaining angle	rotation angle	
$i=j$	$\hat{z}_j$	$\pm 2^{-j}$	$\rightsquigarrow$	z_j	$\pm \alpha_{m,j}$	$\lvert z_j \rvert \leqslant 2\cdot\alpha_{m,j}$
$i=k$	$\hat{z}_k$	$\pm 2^{-k}$		z_k	$\pm \alpha_{m,k}$	$\lvert z_k \rvert \leqslant 2\cdot\alpha_{m,k}$

prediction error!

			prediction error
$i=j$	$\hat{z}_j$	$\pm 2^{-j}$	$+\lvert 2^{-j}-\alpha_{m,j}\rvert$
$i=k$	$\hat{z}_k$	$\pm 2^{-k}$	$+\lvert 2^{-k}-\alpha_{m,k}\rvert$

prediction error!

$$\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| \leqslant \alpha_{m,k}$$

* sum of prediction error $\leqslant \alpha_{m,k}$

then one more repetition of the k-th iteration can eliminate errors

with the same rotation angle $\alpha_{m,k}$

* rotation from j to k-th iterations converge!

	remaining angle	estimated angle		remaining angle	rotation angle	
$i=j$	$\hat{z}_j$	$\pm 2^{-j}$		z_j	$\pm \alpha_{m,j}$	$\lvert z_j \rvert \leqslant 2 \cdot \alpha_{m,j}$
			$\approx$			
$i=k$	$\hat{z}_k$	$\pm 2^{-k}$		z_k	$\pm \alpha_{m,k}$	$\lvert z_k \rvert \leqslant 2 \cdot \alpha_{m,k}$
	$\hat{z}_k$	$\pm 2^{-k}$				

Correction step eliminates prediction error!

if z_k converges, then $\hat{z}_k$ converges too!

$$\boxed{\begin{array}{c} |z_j| \leqslant 2 \cdot \alpha_{m,j} \\ \vdots \\ |z_k| \leqslant 2 \cdot \alpha_{m,k} \end{array}} \Rightarrow \boxed{|\hat{z}_k| \leqslant 2 \cdot \alpha_{m,k}}$$

$m = 0$ linear
$m = +1$ circular
$m = -1$ hyperbolic

$$\boxed{m = \pm 1, \quad s(m,i) = i}$$

$$\boxed{\begin{aligned} x_{i+1} &= x_i - m\sigma_i 2^{-s(m,i)} y_i \\ y_{i+1} &= y_i + \sigma_i 2^{-s(m,i)} x_i \\ z_{i+1} &= z_i - \sigma_i \alpha_{m,i} \end{aligned}} \leftarrow 0$$

complicated conversion
complex decomposition

$$z_0 = \sum_i \sigma_i \boxed{\alpha_{m,i}} = \sum_i \sigma_i \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)}$$

$$\boxed{\alpha_{m,i} = \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)}$$

$$z_{i+1} = z_i - \sigma_i \alpha_{m,i}$$

$$\boxed{\alpha_{m,i}} = \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)}$$ rotation angle at the i-th iteration

$$\approx 2^{-i}$$ estimated, predicted rotation angle.

k-th iteration rotation angle

repeating

$$\boxed{\alpha_{m,k}} = \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,k)}\right)}$$

Maximum Prediction Error (Upper Bound)

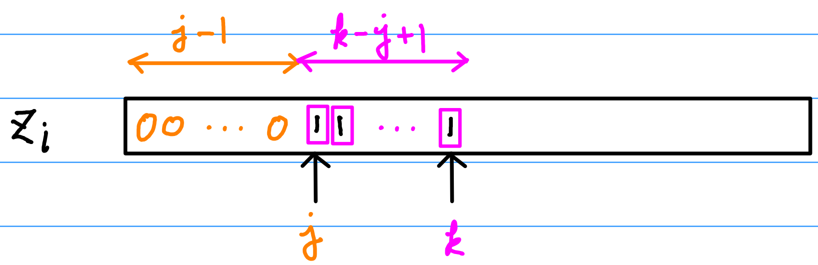
$$\sum_{i=j}^{k} \left| \boxed{2^{-i}} - \boxed{\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right)} \right| \leqslant \alpha_{m,k} \leqslant 2^{-s(m,k)}$$

i-th iteration true rotation angle

i-th iteration predicted rotation angle

$$\sum_{i=j}^{k} \left| \boxed{} - \boxed{\alpha} \right| \leqslant \boxed{\alpha}$$

($j \leqslant i \leqslant k$) accumulated (max) error



($j-1$) leading zeros

($k-j+1$) following ones

$\boxed{s(m, i) = i, \quad m = 1}$ assumed

$$\boxed{\text{the max accumulated error} \leqslant \alpha_{m,k} \leqslant 2^{-s(m,k)}}$$

if this condition is met,
by repeating the last k-th iteration
the max accumulated error
can be eliminated.

$$\sum_{i=j}^{k} \left| 2^{-i} - \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right) \right| \leqslant \alpha_{m,k} \leqslant 2^{-s(m,k)}$$

$$\sum_{i=j}^{k} \left| 2^{-i} - \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right) \right| \leqslant \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,k)}\right)$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \text{for } |x| \leq 1, x \neq \pm i$$

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \cdots + \cdots$$

$$\frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,k)}\right)$$

$$= \frac{1}{\sqrt{m}} \left[\left(\sqrt{m}\, 2^{-s(m,k)}\right) - \frac{1}{3}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^3 + \frac{1}{5}\left(\sqrt{m}\, 2^{-s(m,k)}\right)^5 - \cdots \right]$$

$$= 2^{-s(m,k)} - \frac{1}{3} m\, 2^{-3s(m,k)} + \frac{1}{5} m^2 2^{-5s(m,k)} - \cdots$$

$$\sum_{i=j}^{k} \left| 2^{-i} - 2^{-s(m,i)} + \frac{1}{3} m\, 2^{-3s(m,i)} - \frac{1}{5} m^2 2^{-5s(m,i)} + \cdots \right|$$

$$\leqslant 2^{-s(m,k)} - \frac{1}{3} m\, 2^{-3s(m,k)} + \frac{1}{5} m^2 2^{-5s(m,k)} - \cdots$$

$$\boxed{s(m,i) = i, \qquad m = 1}$$

$$\sum_{i=j}^{k} \left| \cancel{2^{-i}} - \cancel{2^{-i}} + \frac{1}{3} 2^{-3i} - \frac{1}{5} 2^{-5i} + \cdots \right| \leqslant 2^{-k} + \frac{1}{3} 2^{-3k} - \frac{1}{5} 2^{-5k} + \cdots$$

$$\sum_{i=j}^{k} \left| \frac{1}{3} 2^{-3i} - \frac{1}{5} 2^{-5i} + \cdots \right| \leqslant 2^{-k} + \frac{1}{3} 2^{-3k} - \frac{1}{5} 2^{-5k} + \cdots$$

$$\frac{1}{3} \sum_{i=j}^{k} 2^{-3i} \leqslant \boxed{2^{-k}} \quad \Leftarrow$$ ignoring higher order terms

$$2^{-3(j)} + 2^{-3(j+1)} + \cdots + 2^{-3(k)} = a\left(\frac{1-r^n}{1-r}\right) = \frac{a - ar^n}{1-r}$$

$$\frac{1}{3} \sum_{i=j}^{k} 2^{-3i} = \frac{1}{3} \frac{2^{-3j} - 2^{-3k}}{1 - 2^{-3}} = \frac{1}{3} \cdot \frac{8}{7} (2^{-3j} - 2^{-3k})$$

$$\frac{1}{3} \sum_{i=j}^{k} 2^{-3i} < \frac{1}{3} \cdot \frac{8}{7} (2^{-3j} - 2^{-3k}) \leqslant \boxed{2^{-k}}$$

$$\frac{1}{3} (2^{-3j} - 2^{-3k}) \leqslant \boxed{2^{-k}}$$

$$(2^{-3j} - 2^{-3k}) \leqslant 3 \cdot \boxed{2^{-k}}$$

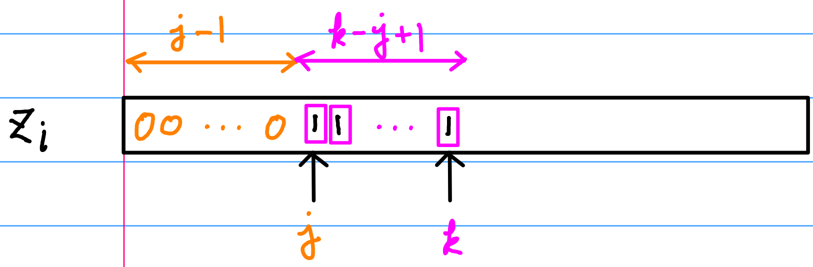
$$2^{-3j} \leqslant 3 \cdot 2^{-k} + 2^{-3k} = 2^{-k}(2^{-2k} + 3)$$

$$2^{-3j+k} \leqslant 2^{-2k} + 3$$

$$2^{-3j+k} \leqslant 3 \leqslant 2^{-2k} + 3$$

$$-3j + k \leqslant \log_2 3 = 1.58 \cdots$$

$$\boxed{k \leqslant 3j + 1.5} \qquad k \leqslant 3j + 1$$



the maximum accumulated error $\le \alpha_{m,k} \le 2^{-s(m,k)}$

$$\sum_{i=j}^{k} \left| 2^{-i} - \frac{1}{\sqrt{m}} \tan^{-1}\left(\sqrt{m}\, 2^{-s(m,i)}\right) \right| \le \alpha_{m,k} \le 2^{-s(m,k)}$$

Taylor Series Expansion

$$\sum_{i=j}^{k} \left| 2^{-i} - 2^{-s(m,i)} + \frac{1}{3} m\, 2^{-3s(m,i)} - \frac{1}{5} m^2 2^{-5s(m,i)} + \cdots \right|$$

$$\le 2^{-s(m,k)} - \frac{1}{3} m\, 2^{-3s(m,k)} + \frac{1}{5} m^2 2^{-5s(m,k)} - \cdots$$

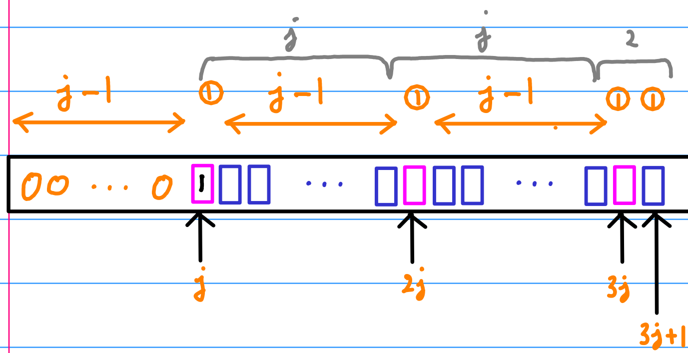
$$\frac{1}{3}\left(2^{-3j} - 2^{-3k}\right) < \frac{1}{3} \sum_{i=j}^{k} 2^{-3i} \lesssim 2^{-k}$$

$$2^{-3j+k} \lesssim 2^{-2k} + 3$$

$$\boxed{k \lesssim 3j + 1.5}$$

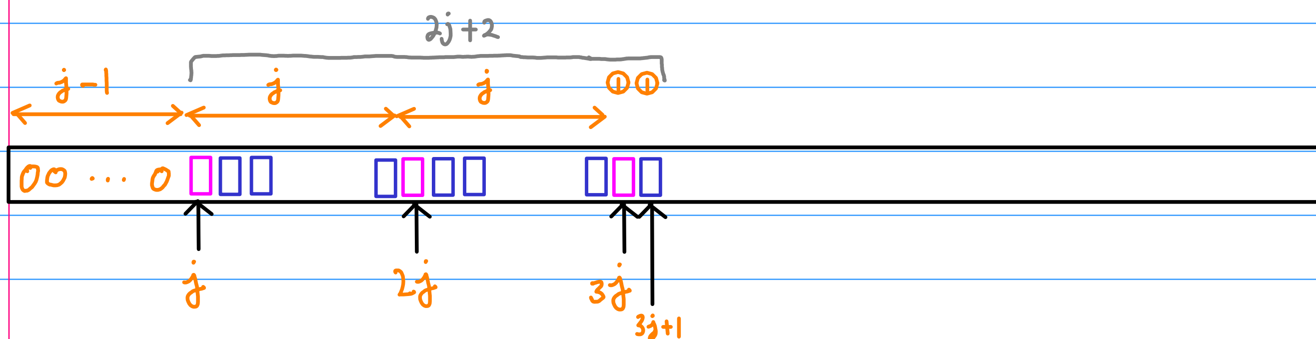
upper bound $k \leqslant 3j + 1$

$(j-1)$ leading zeros



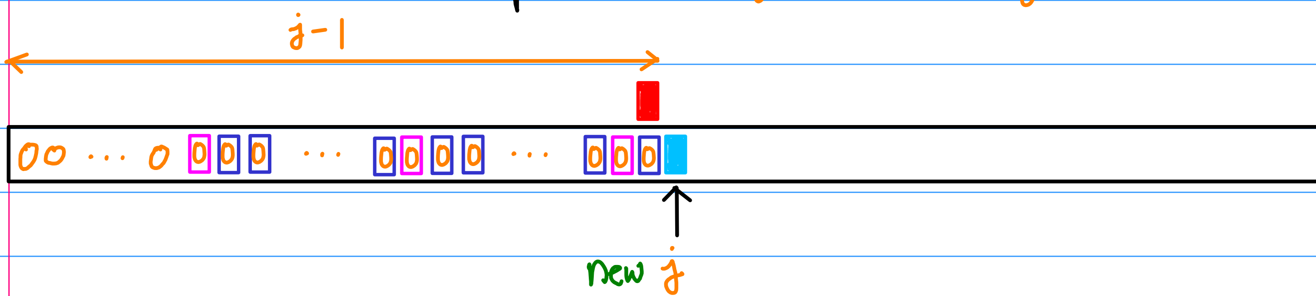
- the fully parallel bit conversion used for $2j+2$ iterations

$[j, 3j+1]$



repetition of the last iteration

After the correction step: $3j+2 \longrightarrow$ new j



$j=1$ $\sigma_{1..4}$ $1 \cdot 3 + 1 = 4 \rightarrow$ new j

$j=4$ $\sigma_{4..13}$ $4 \cdot 3 + 1 = 13 \rightarrow$ new j

$j=13$ $\sigma_{13..40}$ $13 \cdot 3 + 1 = 40 \rightarrow$ new j

		j		k	
binary angle	0 —— 0	Z_j	Z_{j+1}	Z_k	

Z_{i-1}	Z_i	σ_i
0	$\square$	-1
1	$\square$	$+1$

SD angle	σ_j	σ_{j+1}	$\cdots$	σ_k
rotation angle	α_{m_j}	$\alpha_{m_{j+1}}$		α_{m_k}
residue angle	z_j	z_{j+1}		z_k

$|z_k| \leqslant \alpha_{m_k}$

repeat one more time

σ_j	σ_{j+1}	$\cdots$	σ_k	σ_k
α_{m_j}	$\alpha_{m_{j+1}}$		α_{m_k}	α_{m_k}
z_j	z_{j+1}		z_k	z'_k

$|z'_k| \leqslant 2\alpha_{m_k}$

$$\boxed{\sum_{i=j}^{k} \left| 2^{-i} - \alpha_{m,i} \right| \leq \alpha_{m,k}}$$

$$\boxed{k \lesssim 3j + 1.5}$$

$j=1 \qquad 1 \sim 3\cdot 1 + 1 = 4$

$j=4 \qquad 4 \sim 3\cdot 4 + 1 = 13$

$j=13 \qquad 13 \sim 3\cdot 13 + 1 = 40$

σ_i prediction $\quad \sigma_1 \cdots \sigma_4 \quad \sigma_4 \cdots \sigma_{13} \quad \sigma_{13} \cdots \sigma_{40}$

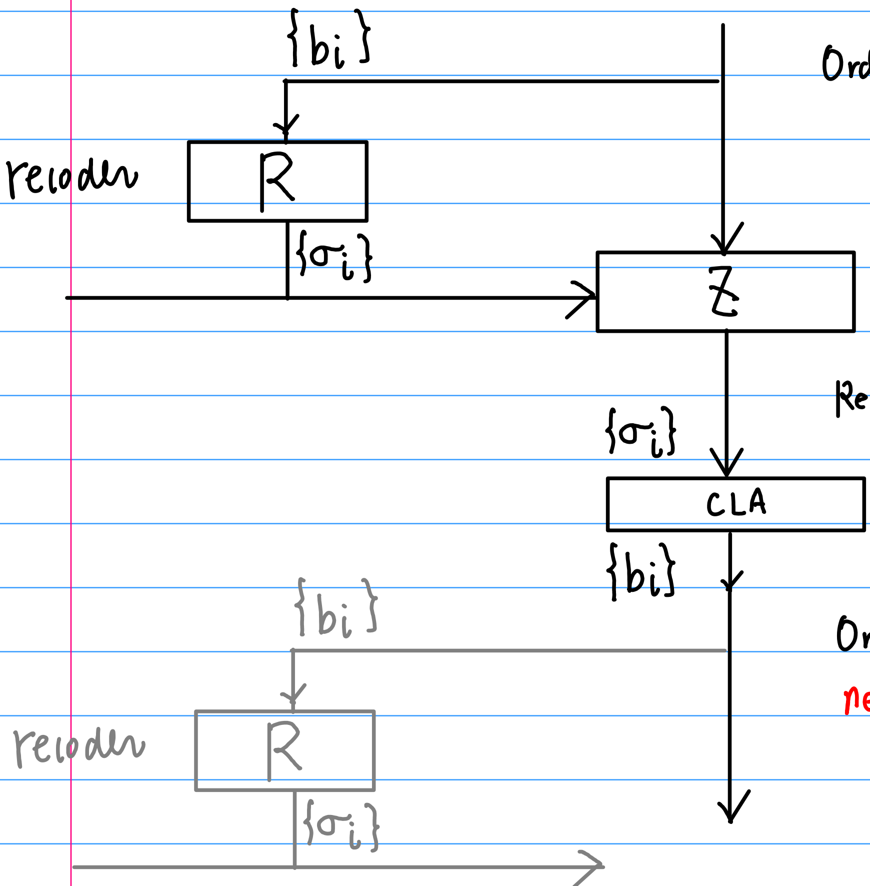
parallel recording $\quad \boxed{R}$

★

① **recoding** without $\sigma_i = 0$

② no scale factor problem

③ no iteration doubling



Ordinary Binary

$$z_{i+1} = z_i - \sigma_i \, \alpha_{m,i}$$

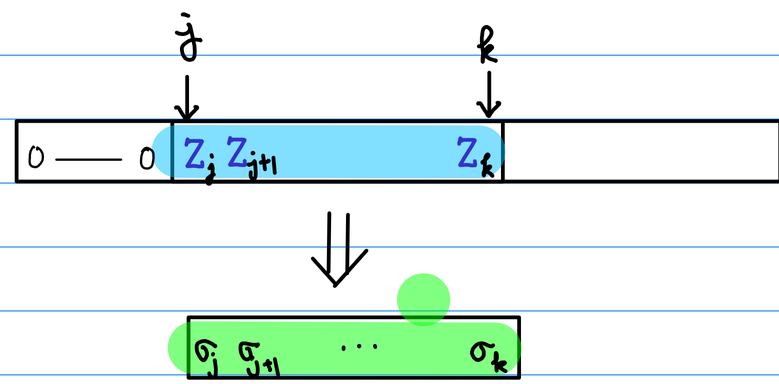
Redundant Binary

Ordinary Binary

necessary for the simple conversion

* Recoder R

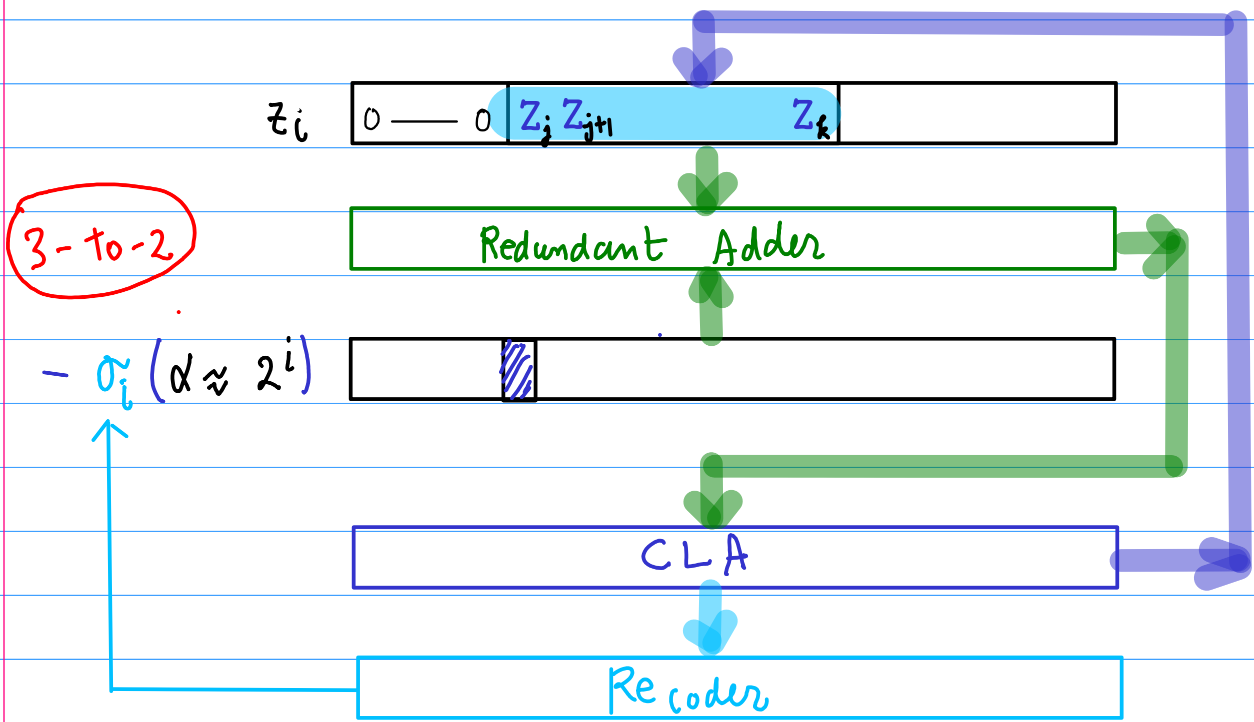
$(j \leq i \leq k)$ z_i

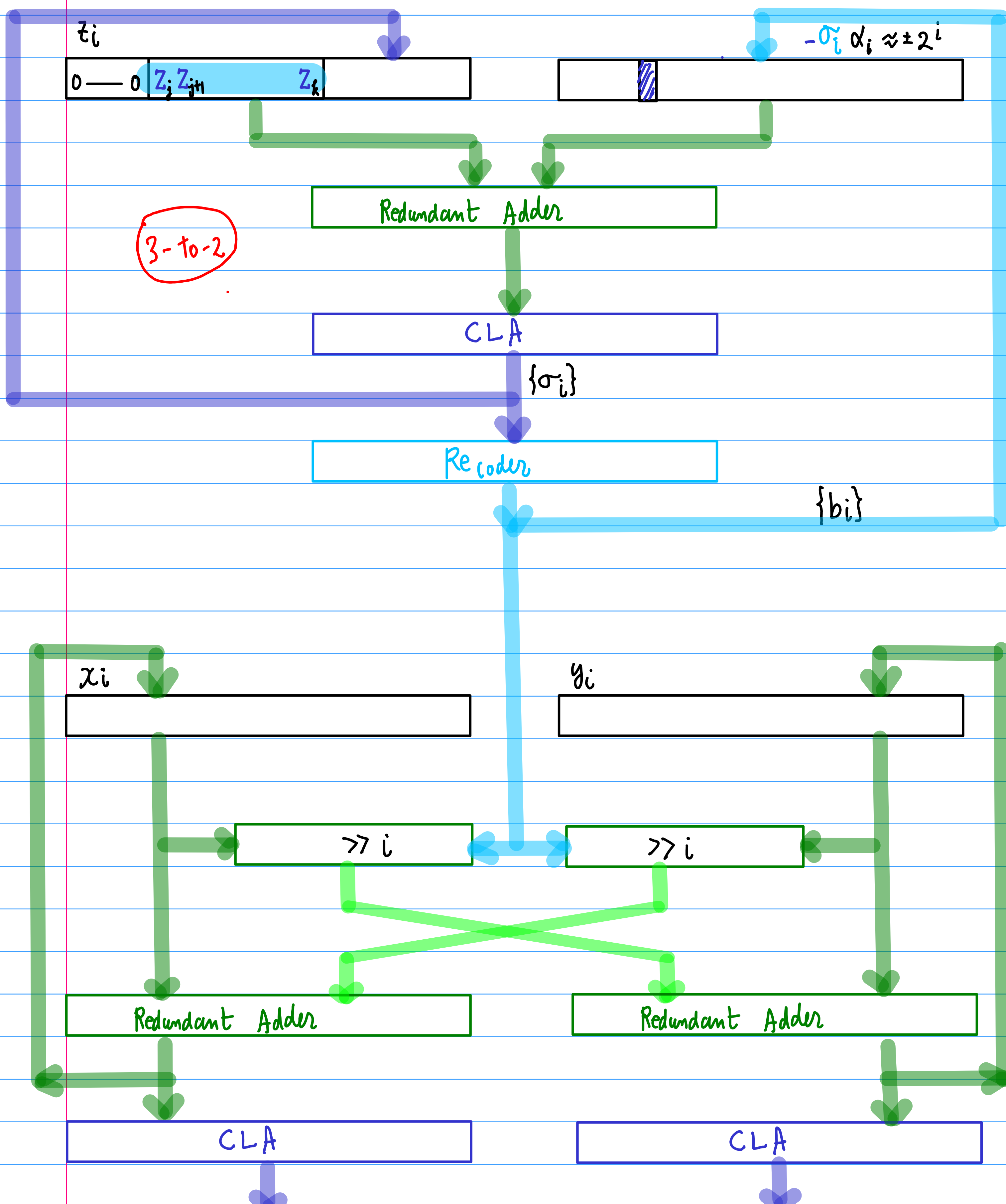


α_{m_j} $\alpha_{m_{j+1}}$ α_{m_k}

z_j z_{j+1} z_k

$|z_k| \leqslant \alpha_{m_k}$


z_i
0 — 0 Z_j Z_{j+1} Z_k
3-to-2
Redundant Adder
$-\sigma_i$ ($\alpha \approx 2^i$)
CLA
Recoder



t_i
0 — 0 $Z_j\ Z_{j+1}\ Z_k$
$-\sigma_i\ \alpha_i \approx \pm 2^i$
Redundant Adder
3-to-2
CLA
$\{\sigma_i\}$
Recoder
$\{b_i\}$
x_i
y_i
>> i
>> i
Redundant Adder
Redundant Adder
CLA
CLA