Time Domain Techniques (3B)
for Noisy Signals

-
-

Young Won Lim
2/5/13

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Signal Detrending / Spike Removal

Remove low frequency noise

a constant
a line
a long period sinusoid

least square fitting a low frequency function $tr[i]$

subtracting the trend from the measurements $y[i] = x[i] - tr[i]$

example)

removing the dc component

$$y[i] = x[i] - tr[i] = x[i] - \frac{1}{N}\sum_{i=0}^{N-1} x[i-1]$$

removing spikes

$$y[i] = x[i] \qquad if \ \ |x[i] - y[i-1]| < threshold$$

$$y[i] = \frac{x[i-1] + x[i+1]}{2} \qquad else$$

Stacking

Remove background noise

Assume noise has a zero mean

Measuring multiple times (ensemble)

Averaging across the ensemble

Reduces the noise level in the averaged signal
Increases the SNR of the correlated component (signal)

$$\boldsymbol{E}[X[i]] \rightarrow x[i]$$

$$\boldsymbol{E}[N[i]] \rightarrow 0$$

$$\boldsymbol{E}[N^2[i]] \rightarrow \frac{\sigma_n}{\sqrt{M}} \qquad SNR \propto \sqrt{M}$$

Moving Average Filter

L-point Running Averager

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k] \qquad = \frac{1}{L}\{x[n] + x[n-1] + \cdots + x[n-L+1]\}$$

$$Y(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}\, X(e^{j\hat{\omega}}) \qquad = \frac{1}{L}\{X(e^{j\hat{\omega}}) + X(e^{j\hat{\omega}})e^{-j\hat{\omega}1} + \cdots + X(e^{j\hat{\omega}})e^{-j\hat{\omega}(L-1)}\}$$

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{j\hat{\omega}k} \qquad = \frac{1}{L}\{1 + e^{-j\hat{\omega}1} + \cdots + e^{-j\hat{\omega}(L-1)}\} \qquad = \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right)$$

$$= \frac{1}{L}\left(\frac{e^{-j\hat{\omega}L/2}}{e^{-j\hat{\omega}/2}}\right)\left(\frac{e^{+j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}}{e^{+j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}\right) \qquad = \frac{1}{L}\left(\frac{e^{+j\hat{\omega}L/2} - e^{-j\hat{\omega}L/2}}{e^{+j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}\right) e^{-j\hat{\omega}(L-1)/2}$$

$$= \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right) e^{-j\hat{\omega}(L-1)/2}$$

Dirichlet Function

$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}})\, e^{-j\hat{\omega}(L-1)/2}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

Young Won Lim
2/5/13

Dirichlet Function (1)

Frequency Response

$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}})\, e^{-j\hat{\omega}(L-1)/2}$$

Dirichlet Function

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

$$D_L(e^{j(\hat{\omega}+2\pi)}) = \frac{\sin((\hat{\omega}+2\pi)L/2)}{L\sin((\hat{\omega}+2\pi)/2)} = \frac{\sin(\hat{\omega}L/2 + L\pi)}{L\sin(\hat{\omega}/2+\pi)}$$

$$\begin{cases} +D_L(e^{j\hat{\omega}}) & \textit{for an odd } L \\ -D_L(e^{j\hat{\omega}}) & \textit{for an even } L \end{cases}$$

(period: 2π)

$$0 \le \hat{\omega} \le +\pi$$

$$0 \le \hat{\omega}/2 \le +\frac{\pi}{2} \qquad 0 \le \hat{\omega}L/2 \le +L\frac{\pi}{2}$$

$$0 \le \sin(\hat{\omega}/2) \le +1 \qquad -1 \le \sin(\hat{\omega}L/2) \le +1$$

a quarter period — L quarter periods

Envelope: $\dfrac{1}{\sin(\hat{\omega}/2)}$

Zeros: $\hat{\omega} = \dfrac{2\pi}{L}k$

$$\sin(\hat{\omega}L/2) = 0$$

$$D_L(e^{-j\hat{\omega}}) = \frac{\sin(-\hat{\omega}L/2)}{L\sin(-\hat{\omega}/2)} = D_L(e^{j\hat{\omega}})$$

an even function

$$\lim_{\hat{\omega}\to 0} D_L(e^{j\hat{\omega}}) = \lim_{\hat{\omega}\to 0} \frac{L/2\cos(\hat{\omega}L/2)}{L/2\cos(\hat{\omega}/2)} = 1$$

Maximum: $D_L(e^{j\hat{\omega}}) = 1 \quad \textit{when } \hat{\omega} = 0$

Dirichlet Function (2)

Dirichlet Function

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega} L/2)}{L \sin(\hat{\omega}/2)}$$

Zeros: $\hat{\omega} = \frac{2\pi}{L} k$

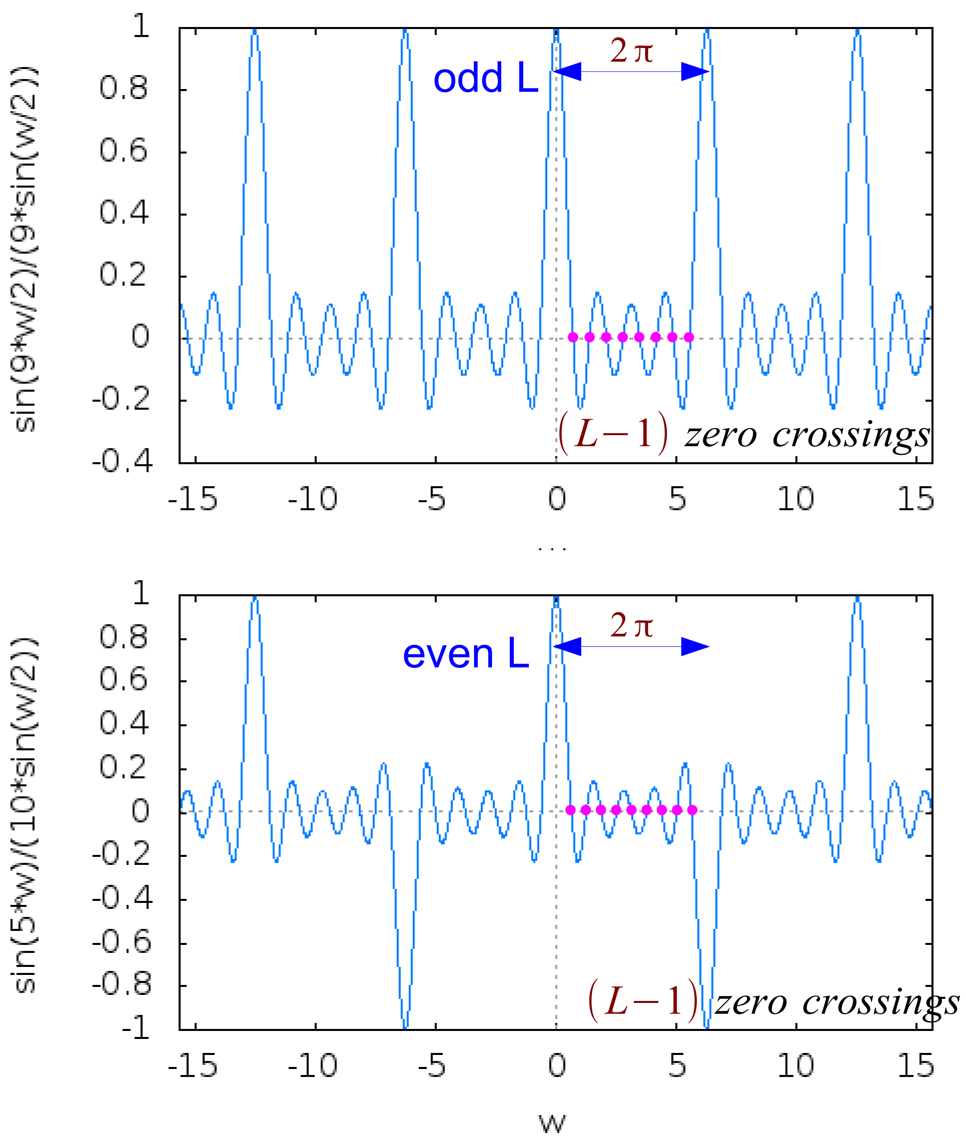
Maximum, Minimum:

odd L

$$\lim_{\hat{\omega} \to 2\pi n} D_L(e^{j\hat{\omega}}) = +1$$

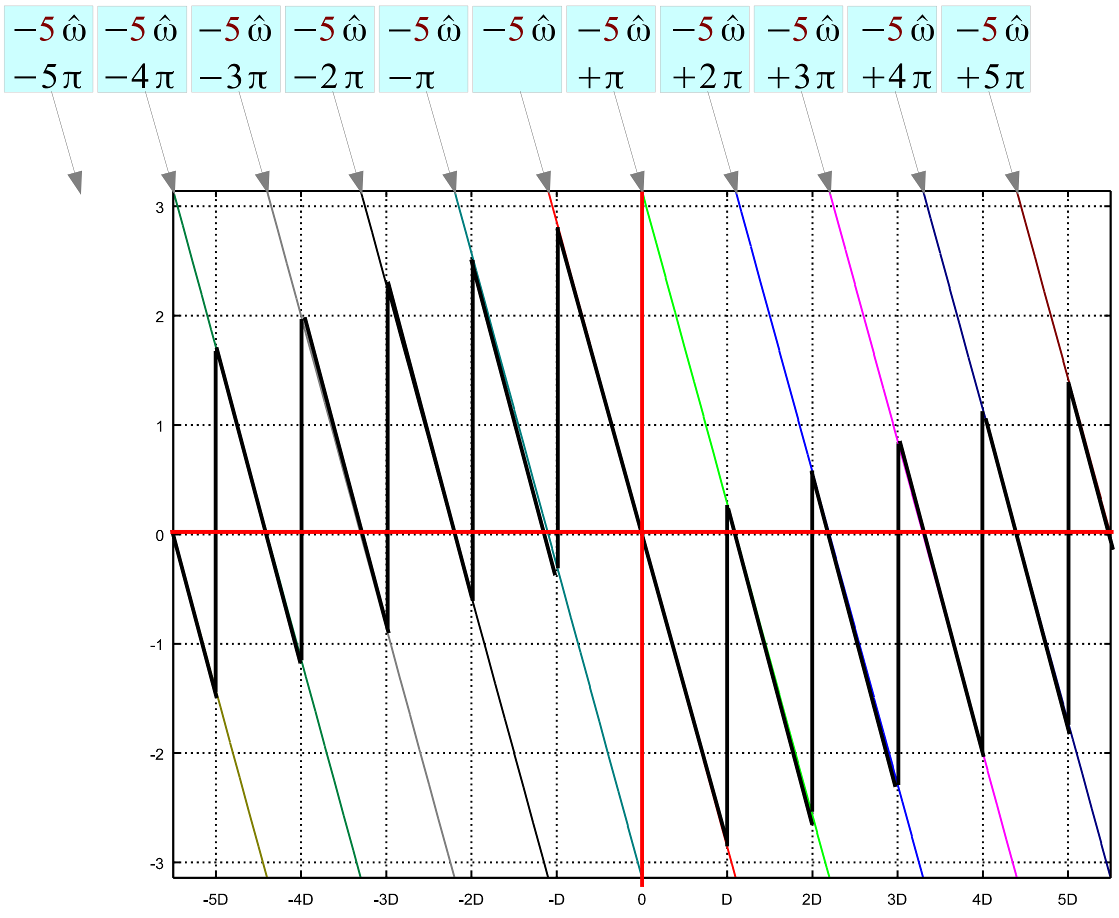
even L

$$\lim_{\hat{\omega} \to 2\pi n} D_L(e^{j\hat{\omega}}) = (-1)^n$$



Young Won Lim
2/5/13

Phase Response (1)



$$\Delta = \frac{2\pi}{11}$$

$$H(e^{j\hat{\omega}}) = D_L(e^{j\hat{\omega}})\, e^{-j\hat{\omega}(L-1)/2}$$

$$H(e^{j\hat{\omega}}) = D_{11}(e^{j\hat{\omega}})\, e^{-j5\hat{\omega}}$$

$$D_L(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

$$D_{11}(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)}$$

Zeros: $\hat{\omega} = \frac{2\pi}{L}k = k\Delta$

$$\Delta = \frac{2\pi}{11}$$

Phase Response (2)

$$H(e^{j\hat\omega}) = D_L(e^{j\hat\omega})\, e^{-j\hat\omega(L-1)/2}$$

$$D_L(e^{j\hat\omega}) = \frac{\sin(\hat\omega L/2)}{L\sin(\hat\omega/2)}$$

Zeros: $\hat\omega = \frac{2\pi}{L}k = k\Delta$

$$H(e^{j\hat\omega}) = D_{11}(e^{j\hat\omega})\, e^{-j5\hat\omega}$$

$$= \left[\frac{\sin(\hat\omega 11/2)}{11\sin(\hat\omega/2)}\right] e^{-j5\hat\omega}$$

$+ \quad - \quad$ *real number*

$$D_{11}(e^{j\hat\omega}) = \frac{\sin(\hat\omega 11/2)}{11\sin(\hat\omega/2)}$$

$$\Delta = \frac{2\pi}{11}$$

$$\arg\{H(e^{j\hat\omega})\} = 2\pi n \pm\pi - 5\hat\omega$$

$$\begin{cases} -5\hat\omega & \left(0\cdot\frac{2\pi}{11} \le \hat\omega < 1\cdot\frac{2\pi}{11}\right) \\ -5\hat\omega + \pi & \left(1\cdot\frac{2\pi}{11} \le \hat\omega < 2\cdot\frac{2\pi}{11}\right) \\ -5\hat\omega + 2\pi & \left(2\cdot\frac{2\pi}{11} \le \hat\omega < 3\cdot\frac{2\pi}{11}\right) \\ -5\hat\omega + 2\pi + \pi & \left(3\cdot\frac{2\pi}{11} \le \hat\omega < 4\cdot\frac{2\pi}{11}\right) \\ -5\hat\omega + 2\pi + 2\pi & \left(4\cdot\frac{2\pi}{11} \le \hat\omega < 5\cdot\frac{2\pi}{11}\right) \\ -5\hat\omega + 2\pi + 2\pi + \pi & \left(5\cdot\frac{2\pi}{11} \le \hat\omega < 11\cdot\frac{2\pi}{11}\right) \end{cases}$$

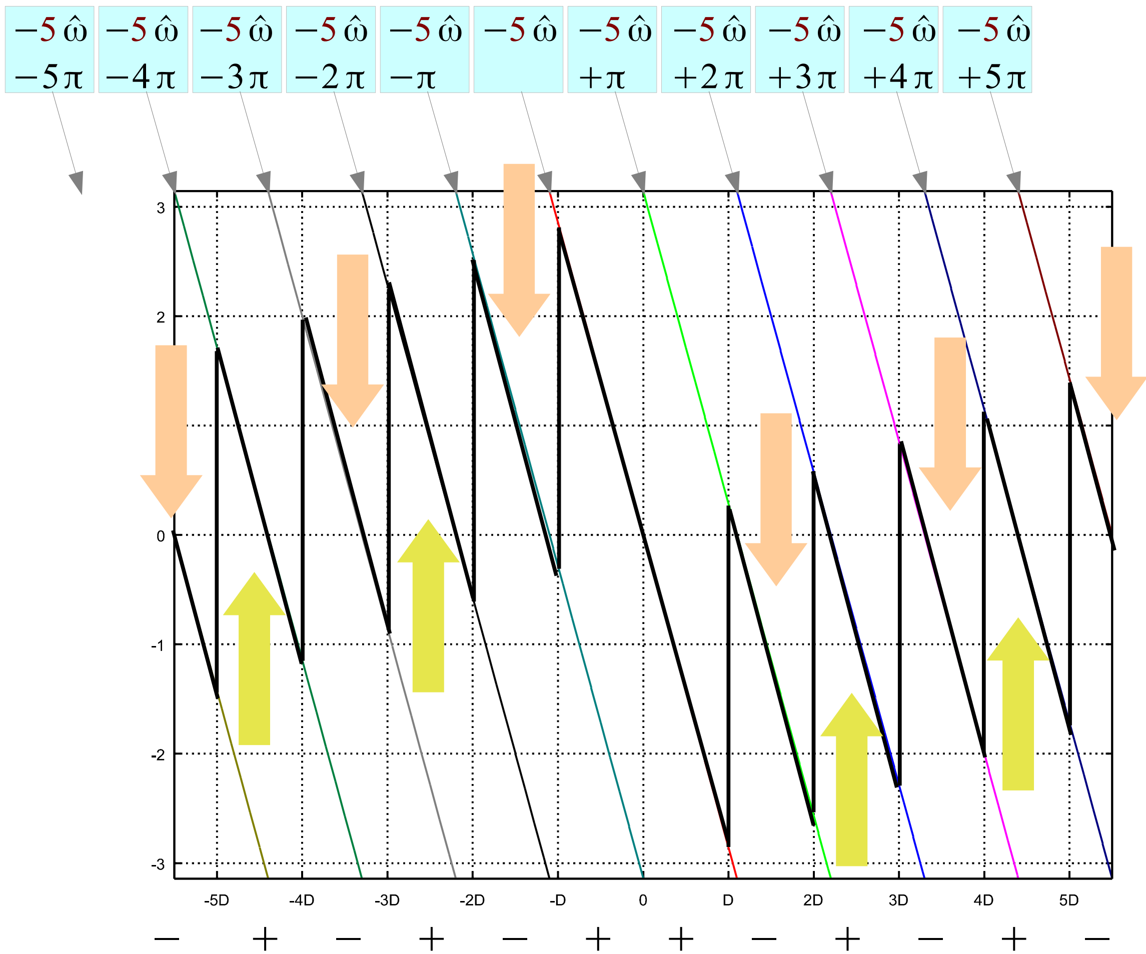
At each zero crossing of $D_{11}(e^{j\hat\omega})$

the signs of $D_{11}(e^{j\hat\omega})$ change

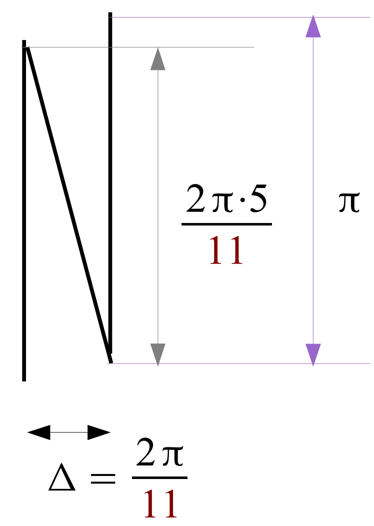
$\pm\pi$ *radian change*

Young Won Lim
2/5/13

Phase Response (3)



$slope = -5$



At each zero crossing of $D_{11}(e^{j\hat{\omega}})$

the signs of $D_{11}(e^{j\hat{\omega}})$ change

$\pm\pi$ *radian change*

Young Won Lim
2/5/13

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] J.C. Santamarina, D Fratta, "Discrete Signals and Inverse Problems: An Introduction for Engineers and Scientists", 2005