

# Green's Function (6A)

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# *Green's Function*

# Zero State IVP

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'(x_0) = y_1$$

$$y(x_0) = y_0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y'(x_0) = 0$$

$$y(x_0) = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x)y_1 + u_2(x)y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x)f(x)}{W}$$

$$\begin{vmatrix} f & y_2 \\ 0 & y_2' \end{vmatrix} = W$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W}$$

$$\begin{vmatrix} y_1 & f \\ y_1' & 0 \end{vmatrix} = W$$

$$y_p(x_0) = 0$$

$$y_p'(x_0) = 0$$

# Anti-derivatives of $u_1(x)$ & $u_2(x)$

$$u_1'(x) = \frac{W_1}{W} = -\frac{y_2(x)f(x)}{W}$$

$$u_2'(x) = \frac{W_2}{W} = \frac{y_1(x)f(x)}{W}$$

$$u_1(x) = \int u_1'(x) dx$$

$$u_2(x) = \int u_2'(x) dx$$

$$\text{anti-derivative} = \int -\frac{y_2(t)f(t)}{W(t)} dt + c_1$$

$$\text{anti-derivative} = \int \frac{y_1(t)f(t)}{W(t)} dt + c_2$$

$$= \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt$$

$$= \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$u_1(x_0) = 0$$

$$u_2(x_0) = 0$$

$$\rightarrow u_1(x_0)y_1(x_0) = 0$$

$$\rightarrow u_2(x_0)y_2(x_0) = 0$$

$$\rightarrow u_1(x_0)y_1'(x_0) = 0$$

$$\rightarrow u_2(x_0)y_2'(x_0) = 0$$

# Zero Initial Conditions

$$u_1(x_0) = 0 \quad \rightarrow \quad u_1(x_0)y_1(x_0) = 0 \\ \rightarrow \quad u_1(x_0)y_1'(x_0) = 0$$

$$u_2(x_0) = 0 \quad \rightarrow \quad u_2(x_0)y_2(x_0) = 0 \\ \rightarrow \quad u_2(x_0)y_2'(x_0) = 0$$

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \quad \rightarrow \quad y_p(x_0) = u_1(x_0)y_1(x_0) + u_2(x_0)y_2(x_0) = 0$$

$$\begin{aligned} y_p'(x) &= u_1'(x)y_1(x) + u_1(x)y_1'(x) \\ &\quad + u_2'(x)y_2(x) + u_2(x)y_2'(x) \end{aligned} \quad \rightarrow \quad \begin{aligned} y_p'(x_0) &= u_1'(x_0)y_1(x_0) + u_1(x_0)y_1'(x_0) \\ &= -\frac{y_1(x_0)y_2(x_0)f(x_0)}{W} + \frac{y_1(x_0)y_2(x_0)f(x_0)}{W} = 0 \end{aligned}$$

$$y_p(x) = u_1(x)y_1 + u_2(x)y_2 \quad \rightarrow \quad y_p(x_0) = 0 \quad y_p'(x_0) = 0$$

$$y_p(x) = y_1(x) \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt + y_2(x) \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt \quad \rightarrow \quad y_p(x_0) = 0 \quad y_p'(x_0) = 0$$

# Zero State Solution

$$y_p(x) = y_1(x) \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt + y_2(x) \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt \quad \Rightarrow \quad y_p(x_0) = 0 \quad y_p'(x_0) = 0$$

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= f(x) \\ y'(x_0) &= 0 \\ y(x_0) &= 0 \end{aligned}$$

$$\begin{aligned} y_p(x) &= y_1(x) \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt \\ &\quad + y_2(x) \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt \end{aligned}$$

# Green's Function and IVP's (1)

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$[x_0, x] \subset I$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W(x)$$

$$u_1'(x) = -\frac{y_2(x)f(x)}{W(x)}$$

$$u_2'(x) = \frac{y_1(x)f(x)}{W(x)}$$

$$u_1(x) = \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt$$

$$u_2(x) = \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt$$

$$\begin{aligned}
 y_p &= u_1(x)y_1 + u_2(x)y_2 \\
 &= \left[ \int_{x_0}^x -\frac{y_2(t)f(t)}{W(t)} dt \right] y_1(x) + \left[ \int_{x_0}^x \frac{y_1(t)f(t)}{W(t)} dt \right] y_2(x) \\
 &= \left[ \int_{x_0}^x -\frac{y_1(x)y_2(t)}{W(t)} f(t) dt \right] + \left[ \int_{x_0}^x \frac{y_1(t)y_2(x)}{W(t)} f(t) dt \right] \\
 &= \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t) dt \\
 &= \int_{x_0}^x G(x, t) f(t) dt
 \end{aligned}$$

# Green's Function and IVP's (2)

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= 0 \\ y'(x_0) &= y_1 \\ y(x_0) &= y_0 \end{aligned}$$

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= f(x) \\ y'(x_0) &= 0 \\ y(x_0) &= 0 \end{aligned}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2 = \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t) dt = \int_{x_0}^x G(x, t) f(t) dt$$

at the end, this  $x$  will replace the literal  $t$

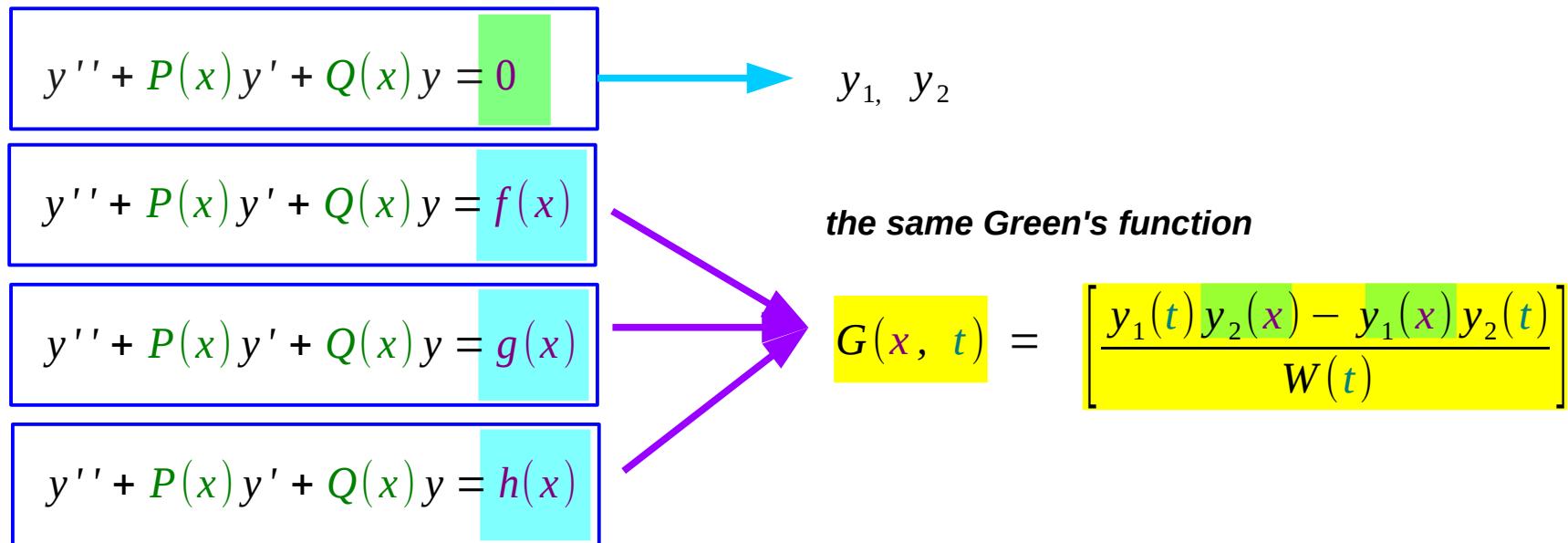
$$= \int_{x_0}^x G(x, t) f(t) dt$$

The diagram shows a mathematical expression  $\int_{x_0}^x G(x, t) f(t) dt$ . Two purple arrows point from the variable  $x$  in the upper limit of the integral to the variable  $x$  in the argument of the Green's function  $G(x, t)$ .

this  $x$  and  $t$  appear in the indefinite integral

# Green's Function

$$G(x, t) = \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] \quad W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$



# Three Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

**Homogeneous DEQ**

**Nonhomogeneous Initial Conditions**

**Nonzero Initial Conditions**

**Nonhomogeneous DEQ**

**Zero Initial Conditions**

**Initially at rest**

**Rest Solution**

# General Solutions of the Initial Value Problem

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y = y_h + y_p$$

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$

$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y(x_0) = y_0$$

$$y'(x_0) = y_1$$

$$y_h$$

**Nonhomogeneous Initial Conditions**  
**Nonzero Initial Conditions**

**Response due to the initial conditions**

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

$$y_p = \int_{x_0}^x G(x, t)f(t)dt$$

**Zero Initial Conditions**  
**Initially at rest**

**Response due to the forcing function  $f$**

**Rest Solution**

# Rest Solution

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

**Nonhomogeneous DEQ**

**Zero Initial Conditions**

**Initially at rest**

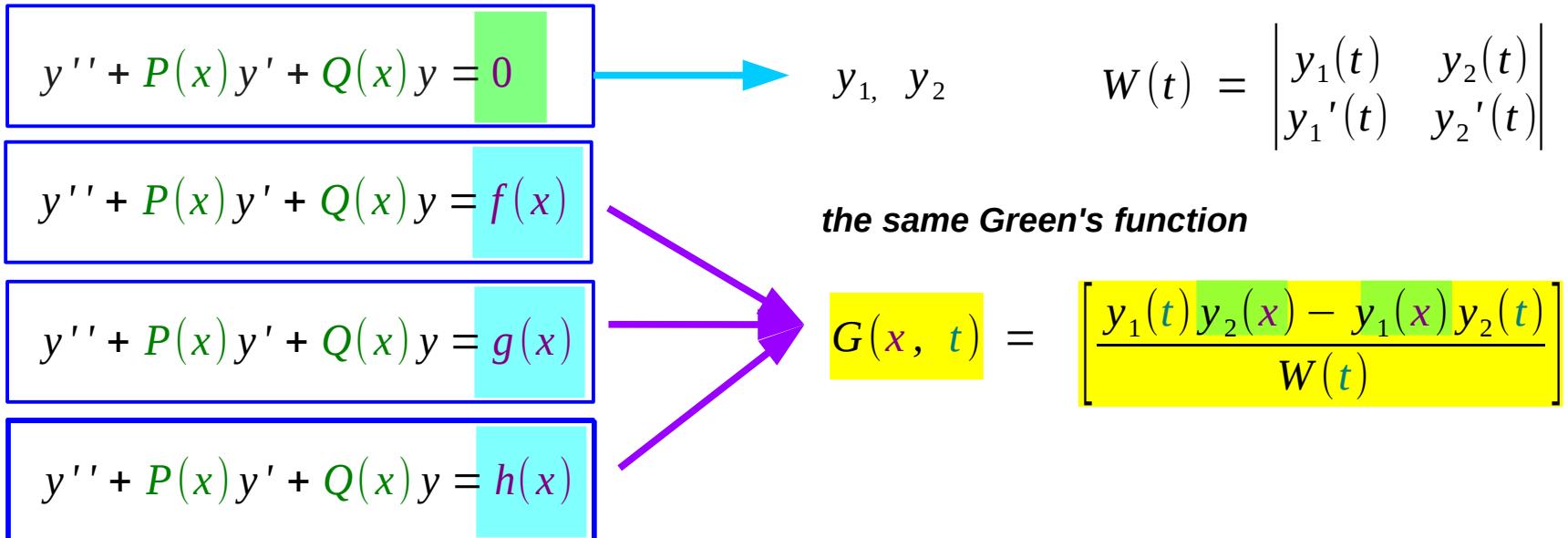
**Rest Solution**

$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t) dt = \int_{x_0}^x G(x, t)f(t)dt$$

$$\begin{cases} y_p(x) = \int_{x_0}^x G(x, t)f(t)dt \\ y_p'(x) = G(x, x)f(x) + \int_{x_0}^x \frac{\partial}{\partial x} [G(x, t)f(t)] dt = \int_{x_0}^x \left[ \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)} \right] f(t) dt \end{cases}$$

$$\begin{cases} y_p(x_0) = \int_{x_0}^{x_0} G(x, t)f(t)dt = 0 \\ y_p'(x_0) = \int_{x_0}^{x_0} \left[ \frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)} \right] f(t) dt = 0 \end{cases}$$

# Green's Function



$$y_p = u_1(x)y_1 + u_2(x)y_2 = \int_{x_0}^x G(x, t)f(t)dt = \int_{x_0}^x \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right] f(t)dt$$

# General Solutions of the Initial Value Problem

$$y'' + 5y' + 6y = f(x)$$

$$y(x_0) = 0$$

$$y'(x_0) = 0$$

$$m^2 + 5m + 6 = (m+2)(m+3) = 0$$

$$m = -2, -3$$

$$y_1 = e^{-2t} \quad y_2 = e^{-3t}$$

$$W(t) = \begin{vmatrix} e^{-2t} & e^{-3t} \\ -2e^{-2t} & -3e^{-3t} \end{vmatrix} = -e^{-5t}$$

$$G(x, t) = \left[ \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)} \right]$$

$$= \left[ \frac{e^{-2t}e^{-3x} - e^{-2x}e^{-3t}}{-e^{-5t}} \right]$$

$$= [-e^{3t}e^{-3x} + e^{-2x}e^{+2t}]$$

$$= [e^{-2(x-t)} - e^{-3(x-t)}]$$

$$= h(x-t)$$

$$y_p = \int_{x_0}^x h(x-t)f(t)dt$$

# Impulse Response by the Green's function

$$y'' + 3y' + 2y = x'$$

$$\begin{aligned} y'' + 3y' + 2y &= x \\ y(x_0) &= 0 \\ y'(x_0) &= 0 \end{aligned}$$

$$m^2 + 3m + 2 = (m+1)(m+2) = 0$$

$$m = -1, -2$$

$$y_1 = e^{-t} \quad y_2 = e^{-2t}$$

$$W(t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}$$

$$G(\textcolor{violet}{x}, \textcolor{teal}{t}) = \left[ \frac{y_1(\textcolor{teal}{t})y_2(\textcolor{violet}{x}) - y_1(\textcolor{violet}{x})y_2(\textcolor{teal}{t})}{W(\textcolor{teal}{t})} \right]$$

$$= \left[ \frac{e^{-\textcolor{teal}{t}} e^{-2\textcolor{violet}{x}} - e^{-1\textcolor{violet}{x}} e^{-2\textcolor{teal}{t}}}{-e^{-3\textcolor{teal}{t}}} \right]$$

$$= [-e^{2\textcolor{teal}{t}} e^{-2\textcolor{violet}{x}} + e^{-\textcolor{violet}{x}} e^{+\textcolor{teal}{t}}]$$

$$= [e^{-(\textcolor{violet}{x}-\textcolor{teal}{t})} - e^{-2(\textcolor{violet}{x}-\textcolor{teal}{t})}]$$

$$= y_n(\textcolor{violet}{x}-\textcolor{teal}{t})$$

$$y_p = \int_{x_0}^x h(\textcolor{violet}{x}-\textcolor{teal}{t})f(\textcolor{teal}{t})dt$$

$$y_n(\textcolor{green}{t}) = [e^{-\textcolor{green}{t}} - e^{-2\textcolor{green}{t}}]$$

$$h(\textcolor{teal}{t}) = [Dy_n(t)]u(\textcolor{teal}{t})$$

$$= -e^{-\textcolor{green}{t}} + 2e^{-2\textcolor{green}{t}}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
- [3] E. Kreyszig, "Advanced Engineering Mathematics"
- [4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"
- [5] [www.chem.arizona.edu/~salzmanr/480a](http://www.chem.arizona.edu/~salzmanr/480a)