Variation of Parameters (4A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Finding a Particular Solution
- Variation of Parameters

Variation of Parameter [c → u(x)]

$$y' + P(x)y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y' + P(x)y = Q(x)$$

$$y_p = u(x)\,y_1$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

$$y_1 = e^{-\int P(x)dx}$$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_p = u_1(x)\,y_1 + u_2(x)\,y_2$$

Young Won Lim
5/13/15

Variation of Parameter : Conditions

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

If the associated homogeneous solution can be solved → ***then, <u>always</u> a particular solution can be found***

No restriction

~~***constant coefficients***~~

~~***A constant** or*~~
~~***A polynomial** or*~~
~~***An exponential function** or*~~
~~***A sine and cosine functions** or*~~

Finding $u_1(x)$ & $u_2(x)$

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2''$$

$$y_p'' + P(x) y_p' + Q(x) y_p =$$

not a matrix notation

$$\begin{array}{l} + u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' \\ + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2'' \end{array} + P \begin{array}{l} + u_1' y_1 + u_1 y_1' \\ + u_2' y_2 + u_2 y_2' \end{array} + Q \begin{array}{l} + u_1 y_1 \\ + u_2 y_2 \end{array}$$

$$u_1 (y_1'' + P y_1' + Q y_1) = 0$$

$$u_2 (y_2'' + P y_2' + Q y_2) = 0$$

Finding $u_1(x)$ & $u_2(x)$ – a Wronskian matrix

not a matrix notation

$$y_p'' + P(x)y_p' + Q(x)y_p = \begin{bmatrix} +u_1''y_1 + u_1'y_1' \\ +u_2''y_2 + u_2'y_2' \end{bmatrix}\begin{bmatrix} +u_1'y_1' \\ +u_2'y_2' \end{bmatrix} + P\begin{bmatrix} +u_1'y_1 \\ +u_2'y_2 \end{bmatrix}$$

$$= \begin{bmatrix} +u_1''y_1 + u_1'y_1' \\ +u_2''y_2 + u_2'y_2' \end{bmatrix} + P\begin{bmatrix} +u_1'y_1 \\ +u_2'y_2 \end{bmatrix} + \begin{bmatrix} +u_1'y_1' \\ +u_2'y_2' \end{bmatrix}$$

$$= \frac{d}{dx}\begin{bmatrix} +u_1'y_1 \\ +u_2'y_2 \end{bmatrix} + P\begin{bmatrix} +u_1'y_1 \\ +u_2'y_2 \end{bmatrix} + \begin{bmatrix} +u_1'y_1' \\ +u_2'y_2' \end{bmatrix} = f(x)$$

$$0 \leftarrow 0 \qquad f(x)$$

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0 \\ u_1'y_1' + u_2'y_2' = f(x) \end{cases} \qquad \begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \qquad \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

Variation of Parameter : Wronskians

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_1}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{W_2}{W}$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$u_1'(x) = -\frac{y_2(x) f(x)}{W}$$

$$u_2'(x) = \frac{y_1(x) f(x)}{W}$$

Variation of Parameter : Particular Solutions

$$y'' + P(x)y' + Q(x)y = 0$$

$$y'' + P(x)y' + Q(x)y = f(x)$$

$$y_h = c_1 y_1 + c_2 y_2$$

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$$

$$y_p(x) = y_1 \int -\frac{y_2 f}{W}\, dx + y_2 \int \frac{y_1 f}{W}\, dx$$

$$y_p(x) = y_1 \int -\left(\frac{y_2 f}{y_1 y_2' - y_2 y_1'}\right) dx + y_2 \int \left(\frac{y_1 f}{y_1 y_2' - y_2 y_1'}\right) dx$$

Nonhomogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

$$a y'' + b y' + c y = g(x)$$

$$W = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = (m_2 - m_1) e^{(m_1+m_2)x}$$

$$u_1'(x) = -\frac{y_2(x) f(x)}{W} = -\frac{e^{m_2 x} f(x)}{(m_2-m_1) e^{(m_1+m_2)x}} = -\frac{f(x) e^{-m_1 x}}{(m_2-m_1)}$$

$$u_2'(x) = -\frac{y_1(x) f(x)}{W} = +\frac{e^{m_1 x} f(x)}{(m_2-m_1) e^{(m_1+m_2)x}} = +\frac{f(x) e^{-m_2 x}}{(m_2-m_1)}$$

Nonhomogeneous Second Order DEs with Constant Coefficients

$$a\frac{d^2 y}{dx^2} + b\frac{dy}{dx} + c\,y = x$$

$$a\,y'' + b\,y' + c\,y = x$$

$$u_1(x) = \frac{-1}{(m_2-m_1)}\int x e^{-m_1 x}\,dx = \frac{+1}{(m_2-m_1)m_1}\left(x e^{-m_1 x} + \frac{1}{m_1}e^{-m_1 x}\right) = \frac{+1}{(m_2-m_1)m_1}\left(x+\frac{1}{m_1}\right)e^{-m_1 x}$$

$$u_2(x) = \frac{+1}{(m_2-m_1)}\int x e^{-m_2 x}\,dx = \frac{-1}{(m_2-m_1)m_2}\left(x e^{-m_2 x} + \frac{1}{m_2}e^{-m_2 x}\right) = \frac{-1}{(m_2-m_1)m_2}\left(x+\frac{1}{m_2}\right)e^{-m_2 x}$$

$$y_p(x) \Leftarrow f(x)$$

a similar form as the input

$$y_{p1} = u_1 y_1 = \frac{+1}{(m_2-m_1)m_1}\left(x+\frac{1}{m_1}\right)e^{-m_1 x}\;e^{+m_1 x}$$

$$y_{p2} = u_2 y_2 = \frac{-1}{(m_2-m_1)m_2}\left(x+\frac{1}{m_2}\right)e^{-m_2 x}\,e^{-m_2 x}$$

Homogeneous Equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Associated Homogeneous Equation with constant coefficients

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Auxiliary Equation

$$m = m_1, m_2, \cdots, m_n$$

n solutions of the Auxiliary Equation

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

General Solutions of the Homogeneous Equation

Non-homogeneous Equation

$$a_n \frac{d^n y}{d x^n} + a_{n-1} \frac{d^{n-1} y}{d x^{n-1}} + \cdots + a_1 \frac{d y}{d x} + a_0 y = g(x)$$

$$y_h(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

k-th column

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W \qquad \begin{vmatrix} y_1 & 0 & \cdots & y_n \\ y_1^{(1)} & 0 & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & g(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = W_k \qquad u_k'(x) = \frac{W_k}{W}$$

$$y_p(x) = u_1 e^{m_1 x} + u_2 e^{m_2 x} + \cdots + u_n e^{m_n x}$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a