

Reduction of Orders (2A)

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Reduction of Orders

Finding another solution y_2 from the known y_1

Second Order EQ

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$y_1 = f(x)$ *known solution*
 $y_2 = u(x)f(x)$ *another solution to be found*

We know one solution

$$y_1(x) = e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

Suppose the other solution

$$y_2(x) = u(x)y_1(x) = u(x)e^{m_1 x}$$

Condition for $y_2(t)$ to be a solution



Find $u(x)$

Conditions for y_2 to be another solution

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$\begin{cases} y_2 = u y_1 \\ y_2' = u' y_1 + u y_1' \\ y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \end{cases}$$

$$a y_2'' + b y_2' + c y_2 = 0 \rightarrow a[u'' y_1 + 2u' y_1' + u y_1''] + b[u' y_1 + u y_1'] + c u y_1 = 0$$

$$a y_1'' + b y_1' + c y_1 = 0 \rightarrow u \underbrace{[a y_1'' + b y_1' + c y_1]}_{\text{Condition for } y_2(t) \text{ to be a solution}} + a[u'' y_1 + 2u' y_1'] + b[u' y_1] = 0$$

$$y_2(x) = u(x) y_1(x)$$



$$a u'' y_1 + u'[2a y_1' + b y_1] = 0$$

Condition for $y_2(t)$ to be a solution

Reduction of Order

We know one solution

$$y_1(x)$$

Suppose the other solution

$$y_2(x) = \boxed{u(x)} y_1(x)$$

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$a y''_2 + b y'_2 + c y_2 = 0$$

$$a u'' y_1 + u'[2a y_1' + b y_1] = 0$$

$$w(x) = u'(x)$$

$$a w' y_1 + w[2a y_1' + b y_1] = 0$$

$$w' y_1 = -w[2 y_1' + \frac{b}{a} y_1]$$

$$\frac{w'}{w} = -2 \frac{y_1'}{y_1} - \frac{b}{a}$$

$$\frac{1}{w} \frac{dw}{dx} = -2 \frac{1}{y_1} \frac{dy_1}{dx} - \frac{b}{a}$$

$$\int \frac{1}{w} \frac{dw}{dx} dx = - \int 2 \frac{1}{y_1} \frac{dy_1}{dx} dx - \int \frac{b}{a} dx$$

$$\ln|w| = -\ln|y_1|^2 - \frac{b}{a}x + c$$

$$\ln|w| + \ln|y_1|^2 = -\frac{b}{a}x + c$$

$$\ln|w y_1^2| = -\frac{b}{a}x + c$$

$$|w y_1^2| = C e^{-(b/a)x}$$

$$w y_1^2 = c_1 e^{-(b/a)x}$$

$$w = c_1 e^{-(b/a)x} / y_1^2$$

$$u' = c_1 e^{-(b/a)x} / y_1^2$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2$$

Another Solution y_2

We know one solution

$$y_1(x)$$

Suppose the other solution

$$y_2(x) = \boxed{u(x)} y_1(x)$$

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + c y = 0$$

$$a y'' + b y' + c y = 0$$

$$a y_2'' + b y_2' + c y_2 = 0$$



$$a u'' y_1 + u'[2a y_1' + b y_1] = 0$$

$$u = c_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2$$



$$y_2 = c_1 y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx + c_2 y_1 \quad (c_1=1, c_2=0)$$

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

General Solutions for the repeated roots case

$$y_2 = y_1 \int \frac{e^{-(b/a)x}}{y_1^2} dx$$

$$\begin{aligned}m_1 &= (-b + \sqrt{b^2 - 4ac})/2a \\m_2 &= (-b - \sqrt{b^2 - 4ac})/2a\end{aligned}$$



$$b^2 - 4ac = 0$$



$$\begin{aligned}m_1 &= -b/2a \\m_2 &= -b/2a\end{aligned}$$

$$e^{m_1 x} = e^{m_2 x} = e^{-\frac{b}{2a}x}$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_1^2 = e^{-\frac{b}{a}x}$$

$$y_2 = e^{-\frac{b}{2a}x} \int \frac{e^{-(b/a)x}}{e^{-(b/a)x}} dx = e^{-\frac{b}{2a}x} \int 1 dx$$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

$$y_1(x) = e^{-\frac{b}{2a}x}$$

$$y_2(x) = x e^{-\frac{b}{2a}x}$$

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

How can xy_1 be another solution to the known y_1

We know one solution

$$y_1(x)$$

Suppose the other solution

$$y_2(x) = \textcolor{violet}{x} y_1(x)$$

$$a \frac{d^2 y}{dx^2} + \textcolor{blue}{b} \frac{dy}{dx} + \textcolor{blue}{c} y = 0$$

$$\textcolor{blue}{a} y'' + \textcolor{blue}{b} y' + \textcolor{blue}{c} y = 0$$

$$(xy_1)' = y_1 + xy_1'$$

$$\textcolor{blue}{a}(xy_1)'' + \textcolor{blue}{b}(xy_1)' + \textcolor{blue}{c}(xy_1)$$

$$\begin{aligned} (xy_1)'' &= y_1' + y_1' + xy_1'' \\ &= 2y_1' + xy_1'' \end{aligned}$$

$$\begin{aligned} &= \textcolor{blue}{a}(2y_1' + xy_1'') + \textcolor{blue}{b}(y_1 + xy_1') + \textcolor{blue}{c}(xy_1) \\ &= x[\textcolor{blue}{a}y_1'' + \textcolor{blue}{b}y_1' + \textcolor{blue}{c}y_1] + 2\textcolor{blue}{a}y_1' + \textcolor{blue}{b}y_1 \\ &= 2\textcolor{blue}{a}y_1' + \textcolor{blue}{b}y_1 = 0 \end{aligned}$$

$$y_2(x) = \textcolor{violet}{x} y_1(x)$$



$$[2\textcolor{blue}{a}y_1' + \textcolor{blue}{b}y_1] = 0$$

$$y_2(x) = u(x)y_1(x)$$



$$\textcolor{blue}{a}u''y_1 + u'[2\textcolor{blue}{a}y_1' + \textcolor{blue}{b}y_1] = 0$$

Condition for xy_1 to be a solution

Example: Finding $y_2 \leftarrow y_1$ (1)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x} \quad Find \quad y_2(x)$$

$$y_2 = u y_1$$

$$y_2' = u'y_1 + u y_1'$$

$$y_2'' = u''y_1 + u'y_1' + u'y_1' + u y_1''$$

$$y_2 = ue^{-2x}$$

$$y_2' = u'e^{-2x} - u2e^{-2x}$$

$$y_2'' = u''e^{-2x} - u'2e^{-2x} - u'2e^{-2x} + u4e^{-2x}$$

$$y_2'' - 4 \cdot y_2 = 0$$



$$(u''e^{-2x} - u'2e^{-2x} - u'2e^{-2x} + u4e^{-2x}) - 4ue^{-2x} = 0$$

$$(u''e^{-2x} - 4u'e^{-2x}) + u(4e^{-2x} - 4e^{-2x}) = 0 \quad \Rightarrow \quad (u''e^{-2x} - 4u'e^{-2x}) = 0$$

$$1 \cdot y'' - 4 \cdot y = 0$$

$$au''y_1 + 2u'ay_1' = 0$$

$$y_2(x) = u(x)y_1(x)$$



Condition for $y_2(t)$ to be a solution

$$au''y_1 + u'[2ay_1' + by_1] = 0$$

Example: Finding $y_2 \leftarrow y_1(2)$

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

Find $y_2(x)$

$$(u''e^{-2x} - 4u'e^{-2x}) + u(4e^{-2x} - 4e^{-2x}) = 0 \rightarrow (u''e^{-2x} - 4u'e^{-2x}) = 0$$

$$1 \cdot y'' - 4 \cdot y = 0$$

$$au''y_1 + 2u'a y_1' = 0$$

$$(u'' - 4u')e^{-2x} = 0 \rightarrow u'' - 4u' = 0$$

$$w = u'$$

$$u'' - 4u' = 0$$

$$w' - 4w = 0$$

$$\frac{w'}{w} = 4$$

$$\int \frac{1}{w} \frac{dw}{dx} dx = \int 4 dx + c$$

$$\int \frac{1}{w} dw = \int 4 dx + c$$

$$\ln|w| = 4x + c$$

$$|w| = e^{4x+c}$$

$$w = c_1 e^{4x}$$

$$u' = c_1 e^{4x}$$

$$u = \frac{1}{4} c_1 e^{4x} + c_2$$

Example: Finding $y_2 \leftarrow y_1$ (3)

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

Find $y_2(x)$

$$u' = c_1 e^{4x} \quad \rightarrow \quad u = \frac{1}{4} c_1 e^{4x} + c_2$$

$$y = k_1 y_1 + k_2 y_2$$

$$= k_1 e^{-2x} + k_2 \left(\frac{1}{4} c_1 e^{+4x} + c_2 \right) e^{-2x}$$

$$= k_1 e^{-2x} + k_2 \frac{1}{4} c_1 e^{+2x} + k_2 c_2 e^{-2x}$$

$$= k_1 e^{-2x} + k_2 c_2 e^{-2x} + k_2 \frac{1}{4} c_1 e^{+2x}$$

$$y = (k_1 + k_2 c_2) e^{-2x} + \left(k_2 \frac{1}{4} c_1 \right) e^{+2x}$$

$$\begin{cases} (k_1) e^{-2x} + \left(k_2 \frac{1}{4} c_1 \right) e^{+2x} & c_1 \neq 0 \quad c_2 = 0 \\ (k_1 + k_2 c_2) e^{-2x} & c_1 = 0 \quad c_2 \neq 0 \end{cases}$$

$$k_1 e^{-2x} + k_2 e^{+2x} \quad c_1 = 4 \quad c_2 = 0$$

Example: Finding $y_2 \leftarrow y_1(4)$

$$1 \cdot y'' + 0 \cdot y' - 4 \cdot y = 0$$

$$y_1(x) = e^{-2x}$$

Find $y_2(x)$

$$y = (k_1 + k_2 c_2) e^{-2x} + \left(k_2 \frac{1}{4} c_1 \right) e^{+2x}$$

$$\begin{cases} (k_1) e^{-2x} + \left(k_2 \frac{1}{4} c_1 \right) e^{+2x} & c_1 \neq 0 \quad c_2 = 0 \\ (k_1 + k_2 c_2) e^{-2x} & c_1 = 0 \quad c_2 \neq 0 \end{cases}$$

$$k_1 e^{-2x} + k_2 e^{+2x} \quad c_1 = 4 \quad c_2 = 0$$

$$C_1 e^{-2x} + C_2 e^{+2x}$$

$$\begin{aligned} W(e^{-2x}, e^{+2x}) &= \begin{vmatrix} e^{-2x} & e^{+2x} \\ -2e^{-2x} & +2e^{+2x} \end{vmatrix} \\ &= 2 - (-2) = 4 \end{aligned}$$

$$W(e^{-2x}, e^{+2x}) \neq 0$$

linearly independent



Fundamental Set of Solutions

$$\{e^{-2x}, e^{+2x}\}$$

References

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