Linear Equations (2A)

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Linear First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x)\,y = g(x)$$

$$a_1(x)\,y' + a_0(x)\,y = g(x)$$

$$y = f(x)$$

$$\frac{dy}{dx} + \frac{a_0(x)}{a_1(x)}\,y = \frac{g(x)}{a_1(x)}$$

$$y' + \frac{a_0(x)}{a_1(x)}\,y = \frac{g(x)}{a_1(x)}$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y' + P(x)\,y = Q(x)$$

Standard Form of First Order ODEs

Homogeneous and Particular Solutions

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

total solution

$$y = y_h + y_p$$

The Homogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

the common part of the solutions of many different differential equations whose homogeneous DE's are the same

The Nonhomogeneous Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

particular solution

$$y_p = f_p(x)$$

the particular solution of a specific differential equation, excluding common part of the solution

Three Different Linear ODEs (2)

$$\frac{dy}{dx} + P(x)\,y = Q(x) \quad \longleftarrow \boxed{y_1} \longrightarrow \quad y' + P(x)\,y = Q(x) \qquad \textbf{EQ 1}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \longleftarrow \boxed{y_h} \longrightarrow \quad y' + P(x)\,y = 0$$

$$\frac{d[y_1+y_h]}{dx} + P(x)[y_1+y_h] = Q(x) \qquad [y_1+y_h]' + P(x)[y_1+y_h] = Q(x)$$

$y_1 + y_h$ ***solution of EQ 1***

$$\frac{dy}{dx} + P(x)\,y = R(x) \quad \longleftarrow \boxed{y_2} \longrightarrow \quad y' + P(x)\,y = R(x) \qquad \textbf{EQ 2}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \longleftarrow \boxed{y_h} \longrightarrow \quad y' + P(x)\,y = 0$$

$$\frac{d[y_2+y_h]}{dx} + P(x)[y_2+y_h] = R(x) \qquad [y_2+y_h]' + P(x)[y_2+y_h] = R(x)$$

$y_2 + y_h$ ***solution of EQ 2***

$$\frac{dy}{dx} + P(x)\,y = S(x) \quad \longleftarrow \boxed{y_3} \longrightarrow \quad y' + P(x)\,y = S(x) \qquad \textbf{EQ 3}$$

$$\frac{dy}{dx} + P(x)\,y = 0 \quad \longleftarrow \boxed{y_h} \longrightarrow \quad y' + P(x)\,y = 0$$

$$\frac{d[y_3+y_h]}{dx} + P(x)[y_3+y_h] = S(x) \qquad [y_3+y_h]' + P(x)[y_3+y_h] = S(x)$$

$y_3 + y_h$ ***solution of EQ 3***

Three Different Linear ODEs (2)

EQ 1

$$\frac{dy}{dx} + P(x)y = Q(x) \longleftarrow y_1$$

$$\frac{dy}{dx} + P(x)y = 0 \longleftarrow y_h$$

$$\frac{dy}{dx} + P(x)y = Q(x) \longleftarrow y_1 + y_h$$

EQ 2

$$\frac{dy}{dx} + P(x)y = R(x) \longleftarrow y_2$$

$$\frac{dy}{dx} + P(x)y = 0 \longleftarrow y_h$$

$$\frac{dy}{dx} + P(x)y = R(x) \longleftarrow y_2 + y_h$$

EQ 3

$$\frac{dy}{dx} + P(x)y = S(x) \longleftarrow y_3$$

$$\frac{dy}{dx} + P(x)y = 0 \longleftarrow y_h$$

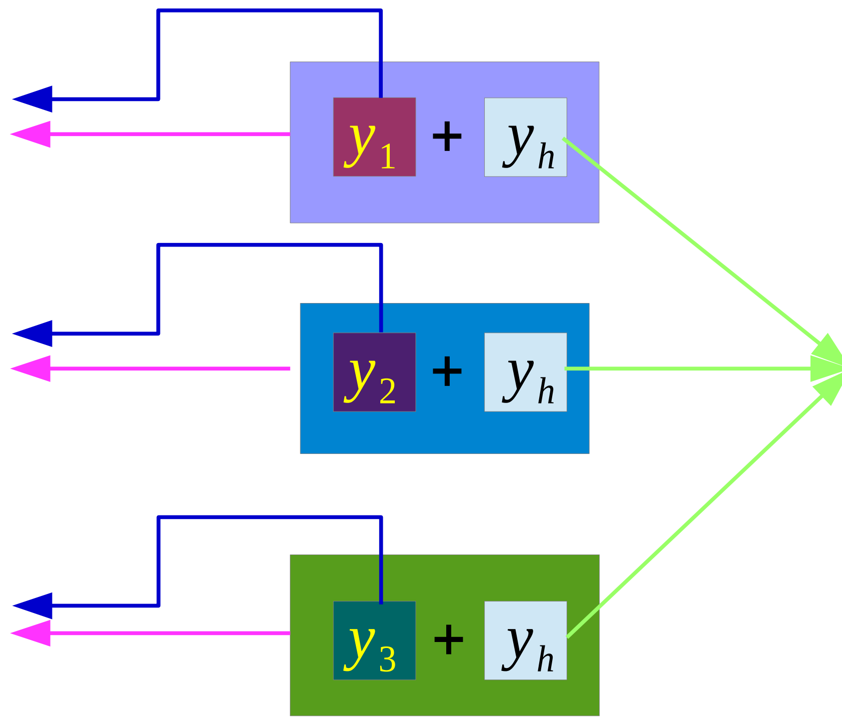
$$\frac{dy}{dx} + P(x)y = S(x) \longleftarrow y_3 + y_h$$

Three Different Linear ODEs (3)

EQ 1 $\frac{dy}{dx} + P(x)y = Q(x)$

EQ 2 $\frac{dy}{dx} + P(x)y = R(x)$

EQ 3 $\frac{dy}{dx} + P(x)y = S(x)$



$$\frac{dy}{dx} + P(x)y = 0$$

$$\begin{cases} (y_1 + y_h) + y_h \\ (y_2 + y_h) + y_h \\ (y_3 + y_h) + y_h \end{cases}$$

similarly

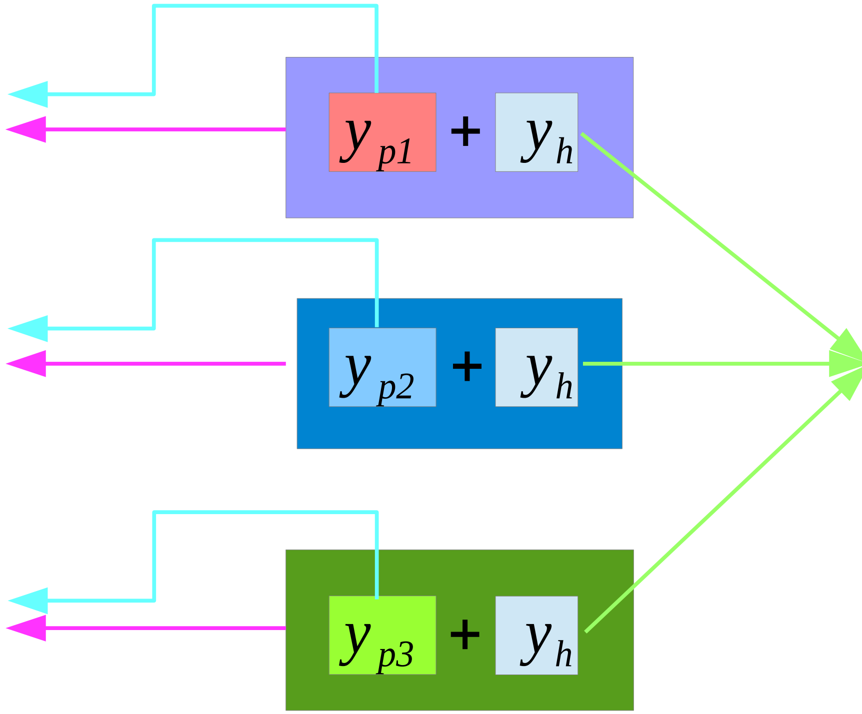
$$\begin{cases} y_1 + k \cdot y_h \\ y_2 + k \cdot y_h \\ y_3 + k \cdot y_h \end{cases}$$

Three Different Linear ODEs (4)

EQ 1 $\frac{dy}{dx} + P(x)y = Q(x)$

EQ 2 $\frac{dy}{dx} + P(x)y = R(x)$

EQ 3 $\frac{dy}{dx} + P(x)y = S(x)$



$\frac{dy}{dx} + P(x)y = \mathbf{0}$

y_p ***particular solution***

excluding any homogeneous solution

y_h ***homogeneous solution***

Total Solution

Standard Form of First Order ODEs

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y' + P(x)\,y = Q(x)$$

$$y = y_p$$
$$y = y_h + y_p$$

$$y = y_p$$
$$y = y_h + y_p$$

total solution

$$y = y_h + y_p$$

$$\frac{dy_h}{dx} + P(x)\,y_h = 0$$

$$y_h' + P(x)\,y_h = 0$$

homogeneous solution

+

+

$$\frac{dy_p}{dx} + P(x)\,y_p = Q(x)$$

$$y_p' + P(x)\,y_p = Q(x)$$

particular solution

||

||

$$\frac{d[y_h + y_p]}{dx} + P(x)[y_h + y_p] = Q(x)$$

$$[y_h + y_p]' + P(x)[y_h + y_p] = Q(x)$$

Solving the Homogeneous DE

$$\frac{dy}{dx} + P(x)y = 0$$

$$y' + P(x)y = 0$$

homogeneous solution

$$y_h = f_h(x)$$

$$\frac{dy}{dx} = -P(x)y \qquad y' = -P(x)y$$

$$\frac{1}{y}\frac{dy}{dx} = -P(x) \qquad \frac{1}{y}y' = -P(x)$$

not a ratio

$$\int \frac{1}{y}\frac{dy}{dx}dx = -\int P(x)dx \qquad \int \frac{1}{y}y'dx = -\int P(x)dx \qquad dy = \frac{df}{dx}dx$$

$$\int \frac{1}{y}dy = -\int P(x)dx \qquad \int \frac{1}{y}dy = -\int P(x)dx$$

$$\ln|y| = -\int P(x)dx + C \qquad \ln|y| = -\int P(x)dx + C$$

$$|y| = e^{-\int P(x)dx + C} \qquad |y| = e^{-\int P(x)dx + C}$$

$$y = \pm e^{-\int P(x)dx} \cdot e^{C} = \pm e^{C} \cdot e^{-\int P(x)dx}$$

$$y = c\,e^{-\int P(x)dx} \qquad y = c\,e^{-\int P(x)dx}$$

Integrating Factor

$$\frac{dy}{dx} + P(x)\,y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)\,y_1$$

$$\frac{d\,y_p}{dx} + P(x)\,y_p = Q(x)$$

$$y' + P(x)\,y = 0$$

$$y = c\,e^{-\int P(x)dx}$$

$$y_h = c\,y_1$$

$$y_1 = e^{-\int P(x)dx}$$

$$y_p = u(x)\,y_1$$

$$y_p' + P(x)\,y_p = Q(x)$$

homogeneous solution

$$y_h = f_h(x)$$

Integrating factor

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

particular solution

$$y_p = f_p(x)$$

Solving the Non-homogeneous DE (1)

$$\frac{d y_p}{d x} + P(x) y_p = Q(x)$$

$$y_p = u(x) y_1$$

$$\left\{\frac{du}{dx}\cdot y_1 + u\cdot\frac{dy_1}{dx}\right\} + P(x) u y_1 = Q(x)$$

$$u\cdot\left\{\frac{dy_1}{dx} + P(x) y_1\right\} + \frac{du}{dx}\cdot y_1 = Q(x)$$

$$\frac{du}{dx} dx = \frac{Q(x)}{y_1(x)} dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} dx + c\right]\cdot y_1$$

$$y_p' + P(x) y_p = Q(x)$$

$$y_p = u(x) y_1$$

$$\{u'\cdot y_1 + u\cdot y_1'\} + P(x) u y_1 = Q(x)$$

$$u\cdot\{y_1' + P(x) y_1\} + u'\cdot y_1 = Q(x)$$

$$u' dx = \frac{Q(x)}{y_1(x)} dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)} dx + c$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)} dx + c\right]\cdot y_1$$

$$y_p = f_p(x)$$

$$y_h(x) = c\cdot y_1$$

$$\left\{\frac{d y_h}{dx}\cdot + P(x) y_h\right\} = 0$$

$$c\left\{\frac{d y_1}{dx}\cdot + P(x) y_1\right\} = 0$$

$$y_p(x) = u\cdot y_1$$

$$y_h(x) = c\cdot y_1$$

excluding homogeneous parts of the solution

Solving the Non-homogeneous DE (2)

$$\frac{dy_p}{dx} + P(x)\,y_p = Q(x)$$

$$y_p' + P(x)\,y_p = Q(x)$$

$$y_p = u(x)\,y_1$$

$$\frac{d}{dx}\{u\cdot y_1\} + P(x)\,u\cdot y_1 = Q(x)$$

$$(u\cdot y_1)' + P(x)\,u\cdot y_1 = Q(x)$$

$$\frac{du}{dx}\cdot y_1 = Q(x)$$

$$u'\cdot y_1 = Q(x)$$

$$y_1 = e^{-\int P(x)dx}$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx$$

Integrating factor

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\,dx\right]\cdot y_1$$

$$y_p(x) = \left[\int \frac{Q(x)}{y_1(x)}\,dx\right]\cdot y_1$$

$$\frac{1}{y_1} = e^{+\int P(x)dx}$$

$$y_p(x) = \left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]\cdot y_1 = \left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]\cdot e^{-\int P(x)dx}$$

$$y(x) = y_h(x) + y_p(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]$$

Examine the role of u(x) – (1)

$$\frac{dy_p}{dx} + 2y_p = x$$

$$Q(x) = x$$

$$y_1 = e^{-\int 2\,dx} = e^{-2x}$$

$$y_h = c\,e^{-2x}$$

$$\frac{dy_h}{dx} + 2y_h = -2c\,e^{-2x} + 2c\,e^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx = \int \frac{x}{e^{-2x}}\,dx = \int x\,e^{+2x}\,dx$$

$$= \frac{1}{2}\left(x\,e^{+2x} - \frac{1}{2}e^{2x}\right) = \frac{1}{2}\left(x - \frac{1}{2}\right)e^{+2x}$$

(a general function of Q(x)) / y1

$$y_p(x) = u(x)\,e^{-2x} = \frac{1}{2}\left(x - \frac{1}{2}\right)e^{+2x}e^{-2x} = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

$$\frac{dy_p}{dx} + 2y_p = \frac{1}{2} + 2\frac{1}{2}\left(x - \frac{1}{2}\right) = x$$

a general function of x

Examine the role of u(x) – (2)

$$\frac{dy_p}{dx} + 2y_p = e^x$$

$$Q(x) = e^x$$

$$y_1 = e^{-\int 2\,dx} = e^{-2x}$$

$$y_h = c\,e^{-2x}$$

$$\frac{dy_h}{dx} + 2y_h = -2c\,e^{-2x} + 2c\,e^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\,dx = \int \frac{e^x}{e^{-2x}}\,dx = \int e^x e^{+2x}\,dx$$

$$= \frac{1}{3} e^x e^{+2x}$$

(a general function of $Q(x)$) / y1

$$y_p(x) = u(x)\,e^{-2x} = \frac{1}{3} e^x e^{+2x} e^{-2x} = \frac{1}{3} e^x$$

$$\frac{dy_p}{dx} + 2y_p = \frac{1}{3} e^x + 2\frac{1}{3} e^x = e^x$$

a general function of e^x

Examine the role of u(x) – (3)

$$\frac{d y_p}{d x} + 2 y_p = \cos(x)$$

$$Q(x) = \cos(x)$$

$$= \tfrac{1}{5}(\cos(x) - 2\sin(x)) + \tfrac{1}{5}(2\sin(x) + 4\cos(x))$$

$$y_1 = e^{-\int 2dx} = e^{-2x}$$

$$y_h = c e^{-2x}$$

$$\frac{d y_h}{d x} + 2 y_h = -2ce^{-2x} + 2ce^{-2x} = 0$$

$$u(x) = \int \frac{Q(x)}{y_1(x)}\, dx = \int \frac{\cos(x)}{e^{-2x}}\, dx = \int \cos(x) e^{+2x}\, dx$$

$$= \frac{1}{5}(\sin(x) + 2\cos(x)) e^{+2x}$$

(a general function of Q(x)) / y1

$$y_p(x) = u(x) e^{-2x}$$

$$= \frac{1}{5}(\sin(x) + 2\cos(x)) e^{+2x} e^{-2x}$$

$$= \frac{1}{5}(\sin(x) + 2\cos(x))$$

a general function of cos(x)

$$\frac{d y_p}{d x} + 2 y_p = \cos(x)$$

Higher Order Homogeneous Equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Associated Homogeneous Equation with constant coefficients

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Auxiliary Equation

$$m = m_1, m_2, \cdots, m_n$$

n solutions of the Auxiliary Equation

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

General Solutions of the Homogeneous Equation

See Variations of Parameter Techniques

Total Solution

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

$$y_1 = e^{-\int P(x)\,dx}$$

total solution

$c\,y_1 + u(x)\,y_1$

$u(x)\,y_1$

homogeneous solution

particular solution

$$c\,e^{-\int P(x)\,dx} + e^{-\int P(x)\,dx} \cdot \left[\int \left\{Q(x) \cdot e^{+\int P(x)\,dx}\right\} dx\right]$$

Integrating factor

$c\,y_1$

$$\frac{dy}{dx} + P(x)\,y = 0$$

The solution of an ODE (1)

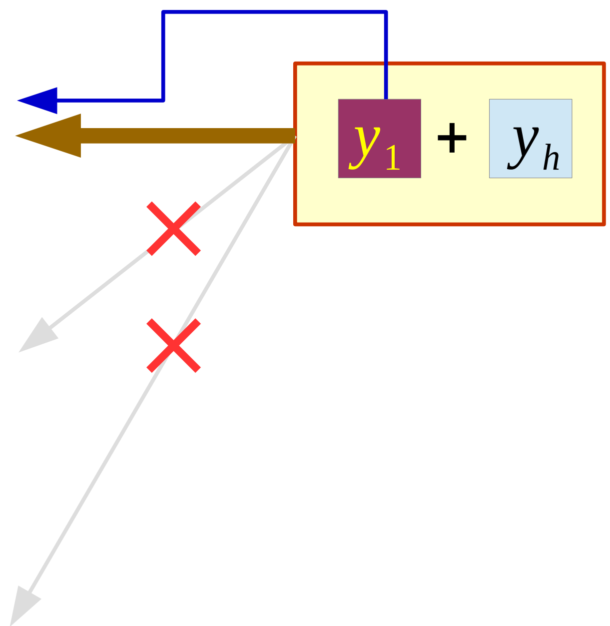
if there is any solution, it is in the form of S

A $$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y(x) = c e^{-\int P(x)dx} + e^{-\int P(x)dx} \cdot \left[\int \left\{ Q(x) \cdot e^{+\int P(x)dx} \right\} dx \right]$$ **S**

B is the solution of A ODE only

EQ 1 $$\frac{dy}{dx} + P(x)y = Q(x)$$



EQ 2 $$\frac{dy}{dx} + P(x)y = R(x)$$

EQ 3 $$\frac{dy}{dx} + P(x)y = S(x)$$

The solution of an ODE (2)

if there is any solution, it is in the form of S

A $$\frac{dy}{dx} + P(x)y = Q(x)$$ ⇄ $$y(x) = c\,e^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]$$ **S**

B is the solution of A ODE only

$$e^{+\int P(x)dx} \times \quad y(x) = \left(c\,e^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]\right) \quad \times\ e^{+\int P(x)dx}$$

$$\left\{e^{+\int P(x)dx}\right\}\cdot y(x) = c + \left[\int \left\{Q(x)\cdot e^{+\int P(x)dx}\right\} dx\right]$$

$$\frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\}\cdot y(x)\right] = Q(x)\cdot\left\{e^{+\int P(x)dx}\right\}$$

$$\left[\left\{e^{+\int P(x)dx}\right\}\cdot\frac{dy}{dx} + \left\{e^{+\int P(x)dx}\right\}P(x)\cdot y(x)\right] = Q(x)\cdot\left\{e^{+\int P(x)dx}\right\}$$

$$\frac{dy}{dx} + P(x)y = Q(x)$$

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Method of Solving First Order ODEs

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$y' + P(x)y = Q(x)$$

$$y = f(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\cdot\left[\frac{dy}{dx} + P(x)y\right] = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$y_1 = e^{-\int P(x)dx} \qquad \frac{1}{y_1} = e^{+\int P(x)dx}$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \left[\left\{e^{+\int P(x)dx}\right\}P(x)\right]y = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\cdot P(x) = \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\} \qquad f'(g(x))g'(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\}y = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\left\{e^{+\int P(x)dx}\right\}\frac{dy}{dx} + \frac{d}{dx}\left\{e^{+\int P(x)dx}\right\}y = \frac{d}{dx}\left[e^{+\int P(x)dx}\cdot y\right]$$

$$f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right] = \left\{e^{+\int P(x)dx}\right\}Q(x)$$

$$\int \frac{d}{dx}\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right] dx = \int \left\{e^{+\int P(x)dx}\right\}Q(x)\, dx + c$$

$$\left[\left\{e^{+\int P(x)dx}\right\}\cdot y\right] = \int \left\{e^{+\int P(x)dx}\right\}Q(x)\, dx + c$$

$$y(x) = c e^{-\int P(x)dx} + e^{-\int P(x)dx}\cdot\left[\int \left\{Q(x)\right\}\cdot\left\{e^{+\int P(x)dx}\right\} dx\right]$$

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Linear First Order ODEs – Summary

Linear First-Order ODE

$$y' + P y = Q \qquad P(x),\ Q(x)$$

$$y' + P y = 0 \qquad P(x),\ Q(x) = 0$$

$$\frac{dy}{dx} = -P \cdot y \;\Rightarrow\; \frac{dy}{y} = -P \cdot dx \;\Rightarrow\; \ln y = -\int P\, dx + C \;\Rightarrow\; y = e^{-\int P\, dx + C}$$

$$y = A e^{-I} \qquad \left(I = e^{-\int P\, dx},\ A = e^{+C} \right) \;\Rightarrow\; y e^{I} = A$$

$$y' + P y = Q \qquad P(x),\ Q(x) \neq 0$$

$$\frac{d}{dx}(y e^{I}) = y' e^{I} + y e^{I} \frac{dI}{dx} = y' e^{I} + y e^{I} P = e^{I}(y' + P y) = e^{I} Q \qquad (\neq 0)$$

$$\frac{d}{dx}(y e^{I}) = e^{I} Q \;\Rightarrow\; y e^{I} = \int e^{I} Q\, dx + c \;\Rightarrow\; y = e^{-I}\left(\int e^{I} Q\, dx + c \right)$$

$$y = e^{-I}\left(\int e^{I} Q\, dx + c \right) \qquad \left(I = e^{-\int P\, dx} \right)$$

Standard Form

$$\frac{dy}{dx} + P(x)y = f(x)$$ $P(x),\ f(x)$

$$y_c' + P y_c = 0$$ $P(x),\ f(x) = 0$

$$y_g = y_c + y_p$$

general solution = homogeneous solution + particular solution

$$\frac{d}{dx}\left[y_c + y_p\right] + P(x)\left[y_c + y_p\right] = f(x)$$

$$\left[y_c' + P(x)y_c\right] + \left[y_p' + P(x)y_p\right] = f(x)$$

$$y_c' + P y_c = 0 \qquad y_p' + P y_p = f(x)$$

The Homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_c' + P y_c = 0 \qquad P(x),\ f(x) = 0$$

$$\frac{dy_c}{dx} = -P \cdot y_c \Rightarrow \frac{dy_c}{y_c} = -P \cdot dx \Rightarrow \ln y_c = -\int P\, dx + C \Rightarrow y_c = e^{-\int P\, dx + C}$$

$$y_c = c\, y_1 \qquad \left(y_1 = e^{-\int P\, dx},\ c = e^{+C} \right)$$

homogeneous solution

$$\frac{dy_c}{dx} + P \cdot y_c = c\left(\frac{d}{dx}\left(e^{-\int P\, dx}\right) + P \cdot e^{-\int P\, dx} \right) = c\left(-P \cdot e^{-\int P\, dx} + P \cdot e^{-\int P\, dx}\right) = 0$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_p' + P y_p = f(x) \qquad P(x),\ f(x) \neq 0$$

$$y_c = c\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ c = e^{+C}\ )$$ homogeneous solution

$$y_p = u\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ u(x) \neq const\ )$$ particular solution

$$\frac{d}{dx}(y_1 u) + P\cdot y_1 u = f(x) \qquad \left[\frac{dy_1}{dx} + P\cdot y_1\right]u + y_1\frac{du}{dx} = f(x)$$

$$y_1\frac{du}{dx} = f(x) \qquad du = \frac{f(x)}{y_1(x)}dx \qquad u = \int \frac{f(x)}{y_1(x)}\,dx$$

$$y_p = u\,y_1 \qquad y_p = y_1\int \frac{f(x)}{y_1(x)}\,dx \quad = e^{-\int P\,dx}\int f(x)e^{+\int P\,dx}\,dx$$

$$y_p = u\,y_1 \qquad (\ y_1 = e^{-\int P\,dx},\ u = \int f(x)e^{+\int P\,dx}\,dx\ )$$ particular solution

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$y_p' + P y_p = f(x) \qquad P(x),\ f(x) \neq 0$$

$$y_p = u y_1 \qquad \left(y_1 = e^{-\int P\,dx},\ u = \int f(x) e^{+\int P\,dx}\,dx \right)$$ particular solution

$$y_g = y_c + y_p = c e^{-\int P\,dx} + e^{-\int P\,dx} \int f(x) e^{+\int P\,dx}\,dx$$ general solution

$$y_g e^{+\int P\,dx} = c + \int f(x) e^{+\int P\,dx}\,dx \qquad \frac{d}{dx}\left(y_g e^{+\int P\,dx}\right) = f(x) e^{+\int P\,dx}$$

$$\left(\frac{dy_g}{dx}\right) e^{+\int P\,dx} + \left(P y_g\right) e^{+\int P\,dx} = f(x) e^{+\int P\,dx}$$

$$\frac{dy_g}{dx} + P y_g = f(x)$$

The Non-homogeneous DE

$$\frac{dy}{dx} + P(x)y = f(x) \qquad P(x),\ f(x)$$

$$\left(\frac{dy_g}{dx}\right)e^{+\int P\,dx} + \left(P y_g\right)e^{+\int P\,dx} = f(x)e^{+\int P\,dx}$$

integration factor $e^{+\int P\,dx}$

$$\frac{d}{dx}\left(y_g e^{+\int P\,dx}\right) = f(x)e^{+\int P\,dx}$$

$$y_g e^{+\int P\,dx} = c + \int f(x)e^{+\int P\,dx}\,dx$$

$$y_g = y_c + y_p = c e^{-\int P\,dx} + e^{-\int P\,dx}\int f(x)e^{+\int P\,dx}\,dx$$

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