Separable Equations (1A)

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Types of First Order ODEs

First & Second Order IVPs (y=f(x))

***First Order** Initial Value Problem*

$$\frac{dy}{dx} = g(x, y)$$
$$y(x_0) = y_0$$

$$y' = g(x, y)$$
$$y(x_0) = y_0$$

***Second Order** Initial Value Problem*

$$\frac{d^2 y}{dx^2} = g(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

$$y'' = g(x, y, y')$$
$$y(x_0) = y_0$$
$$y'(x_0) = y_1$$

Types of First Order ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x) g_2(y)$$

$$y' = g_1(x) g_2(y)$$

$$y = f(x)$$

Linear Equations

$$a_1(x)\frac{dy}{dx} + a_0(x) y = g(x)$$

$$a_1(x) y' + a_0(x) y = g(x)$$

$$y = f(x)$$

Exact Equations

$$M(x, y) dx + N(x, y) dy = 0$$

$$\frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 0$$

$$z = f(x, y)$$

Separable First Order ODEs

Separable ODEs

A General Form of First Order Differential Equations

$$\frac{dy}{dx} = g(x, y)$$

$$y' = g(x, y)$$

Separable Equations

$$\frac{dy}{dx} = g_1(x)\, g_2(y)$$

$$y' = g_1(x)\, g_2(y)$$

$$y = f(x)$$

$$\frac{1}{g_2(y)} \frac{dy}{dx} = g_1(x)$$

$$p(y) \frac{dy}{dx} = q(x)$$

$$\frac{1}{g_2(y)}\, y' = g_1(x)$$

$$p(y)\, y' = q(x)$$

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Solving Separable ODEs (1)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\frac{dy}{dx}dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$p(y)y' = q(x)$$

$$p(y)y' = q(x)$$

$$p(y)y'dx = q(x)dx$$

$$p(y)\,dy = q(x)\,dx$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$y = f(x)$$

not a ratio

$$dy = \frac{df}{dx}dx$$

$$\int p(y)dy$$

includes a constant

$$\int q(x)dx$$

includes another constant

implicit function y

Solving Separable ODEs (2)

$$p(y)\frac{dy}{dx} = q(x)$$

$$p(y)\,y' = q(x)$$

given a composite function
p(y(x)) find y=f(x)

$$P(y) = \int p(y)\,dy + c_1$$

$$\frac{d}{dy}[P(y)] = p(y)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right]\cdot\frac{dy}{dx} = q(x)$$

$$\frac{d}{dy}\left[\int p(y)\,dy + c_1\right]\cdot y' = q(x)$$

$$\frac{d}{dy}[P(y)]\cdot\frac{dy}{dx} = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}\left[\int p(y)\,dy + c_1\right] = q(x)$$

$$\frac{d}{dx}[P(y)]\cdot = q(x)$$

$$\int p(y)dy = \int q(x)dx$$

$$\int p(y)dy = \int q(x)dx$$

$$p(y) = \frac{d}{dy}\left[\int p(y)\,dy + c_1\right]$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$\int p(y)\,dy = \int q(x)\,dx$$

$$P(y) = Q(x) + C$$

$$P(y) = Q(x) + C$$

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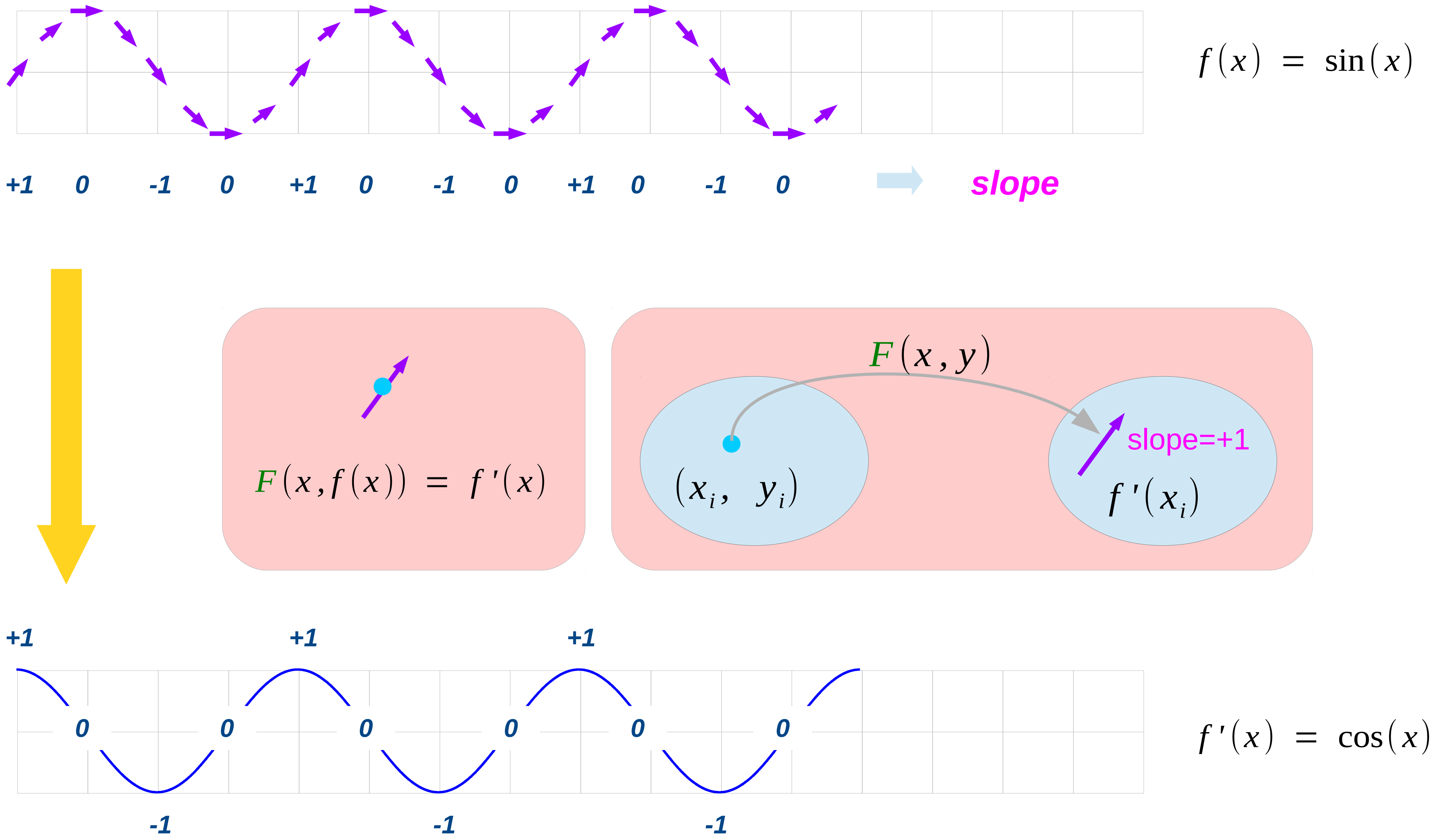
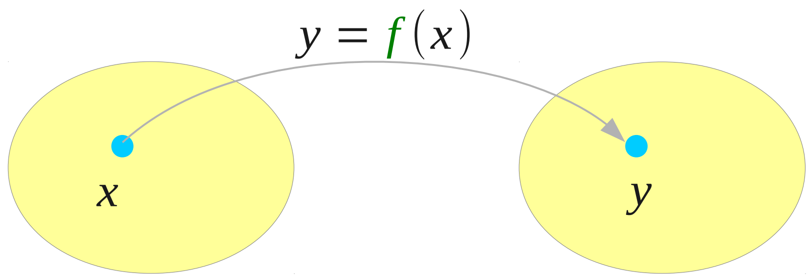
Direction Fields

Plot of F(x,y)=cos(x)

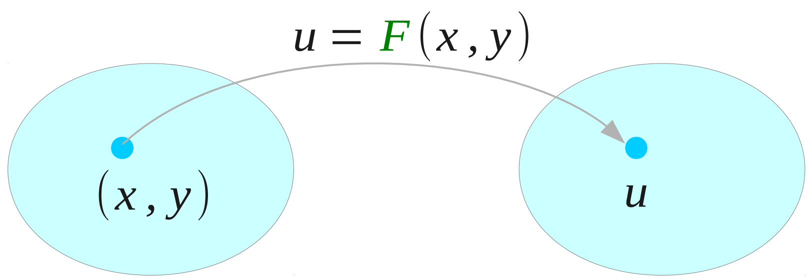


A function of two variables



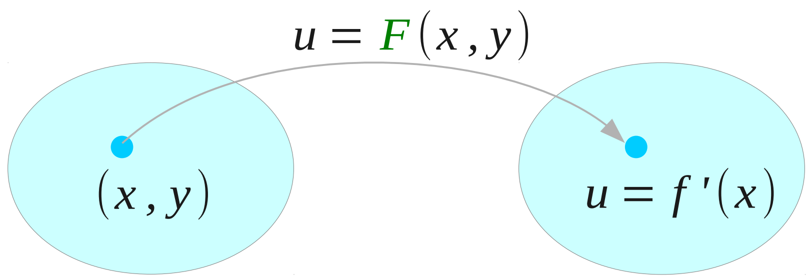
$y = f(x)$

f maps x to y



$u = F(x, y)$

F maps (x,y) to u



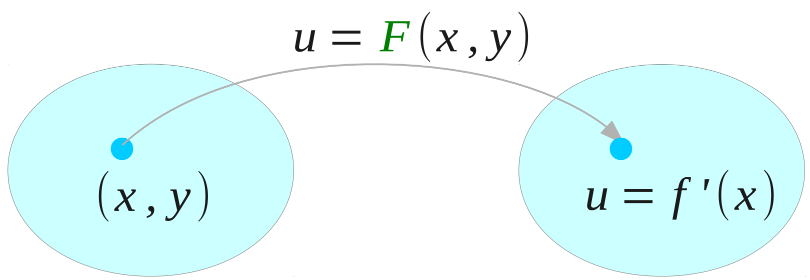
$u = F(x, y)$

F maps (x,y) to f'(x)

the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

Direction Field



$u = F(x, y)$ **F maps (x,y) to f'(x)**

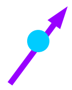
the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

$f'(x) = F(x, y)$ **F maps (x,y) to f'(x)**

Direction Field
Slope Field

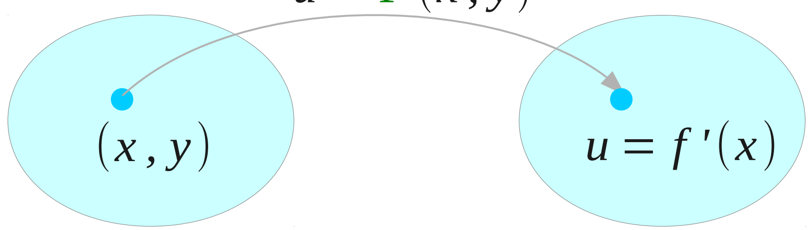
A 2-d plot representing f'(x) by a lineal element at (x, y)



The slope of the tangent line at (x, f(x))

Direction Field, First Order ODE

$u = F(x, y)$



$y = f(x)$

$u = F(x, y)$ **F maps (x,y) to u**

$f'(x) = F(x, y)$ **F maps (x,y) to f'(x)**

the derivative of f(x) at x

The slope of the tangent line at (x, f(x))

First Order ODE

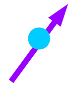
Find solution y=f(x)

$$\frac{dy}{dx} = g(x, y)$$

where the first derivative y' is given by some formula g(x,y) containing variable x, y

Direction Field
Slope Field

A 2-d plot of y'=f'(x) at (x, y)



$$g(x, y) = \frac{dy}{dx}$$

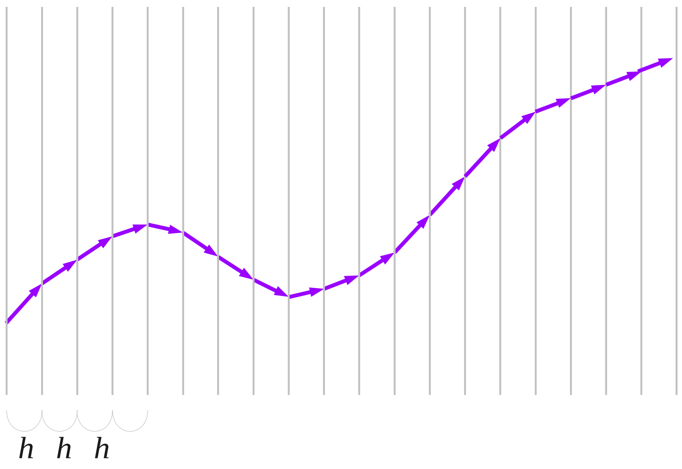
Now, it also can be viewed as a function

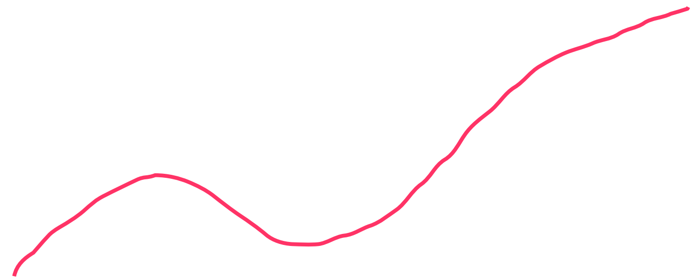
g maps (x,y) to f'(x)

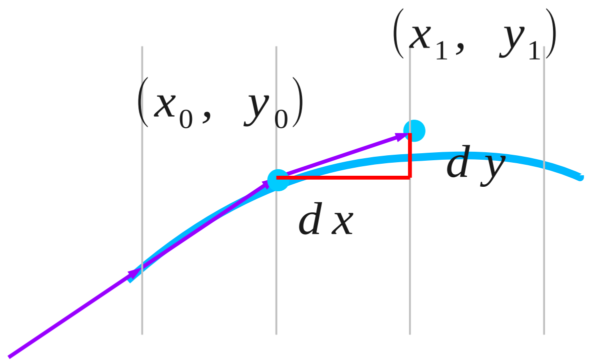
Euler's Method

$F(x, y) = f'(x)$

$y = f(x)$







$dy = f'(x)dx$

$x_1 - x_0 = dx = h$

$y_1 - y_0 = dy = f'(x)dx = F(x_0,\ y_0)dx$

$y_1 = y_0 + F(x_0, y_0)h$

$$y_{i+1} = y_i + F(x_i, y_i)h$$

Special Cases of Separable Equations

$$\frac{dy}{dx} = g_1(x)$$

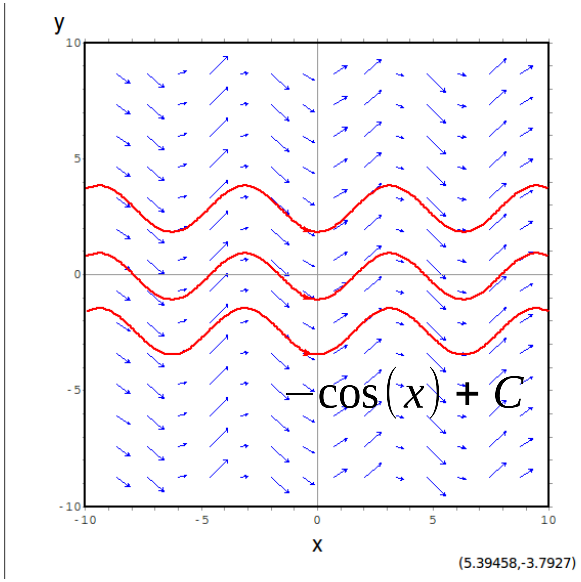
$$\frac{dy}{dx} = \sin(x)$$

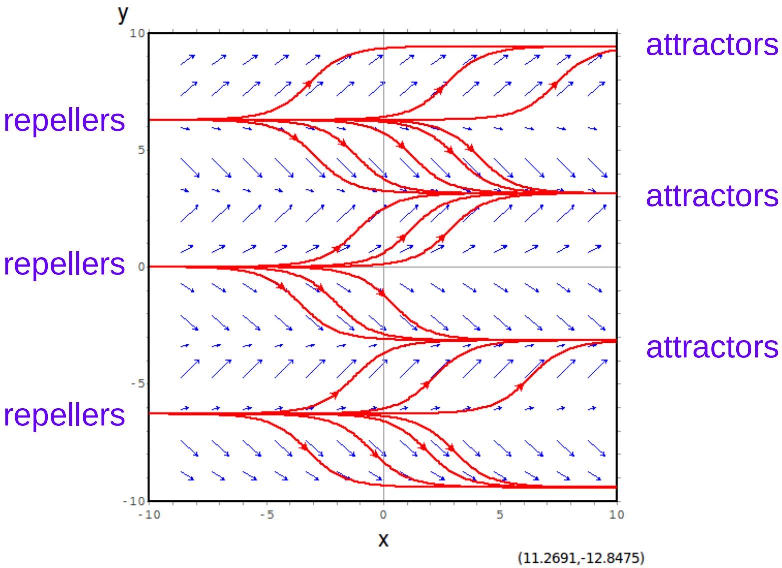
$$\frac{dy}{dx} = g_2(y)$$

$$\frac{dy}{dx} = \sin(y)$$

the only literal y

Autonomous ODEs





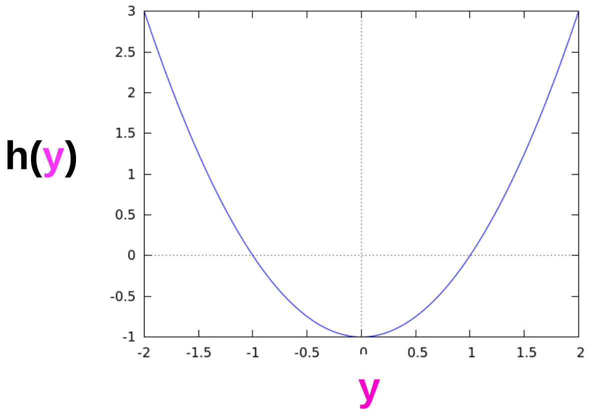
solution curves y(x): translated in the y direction

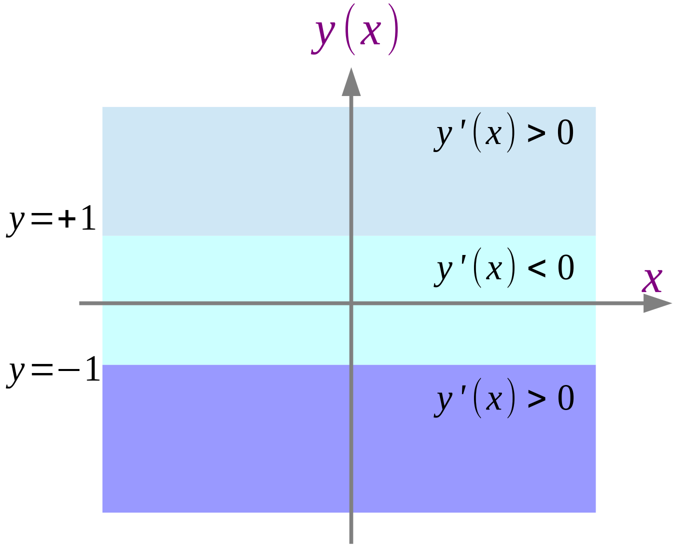
solution curves y(x): translated in the x direction

Autonomous First Order ODEs

$$\frac{dy}{dx} = (y-1)(y+1) \quad \Rightarrow \quad h(y)$$

$$y'(x) = h(y) > 0 \qquad y > +1$$
$$y'(x) = h(y) < 0 \qquad -1 < y < +1$$
$$y'(x) = h(y) > 0 \qquad y < -1$$

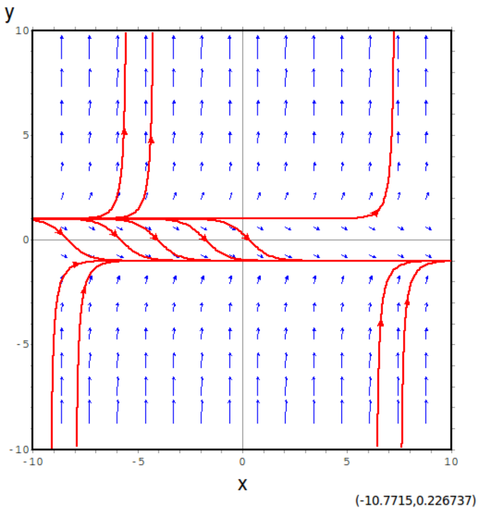




increasing $y(x)$

decreasing $y(x)$

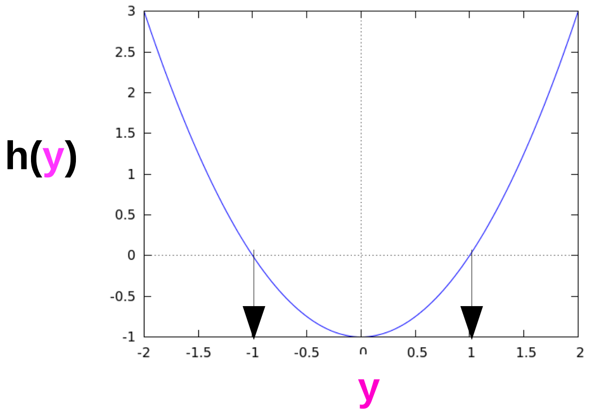
increasing $y(x)$



Critical Points of Autonomous ODEs

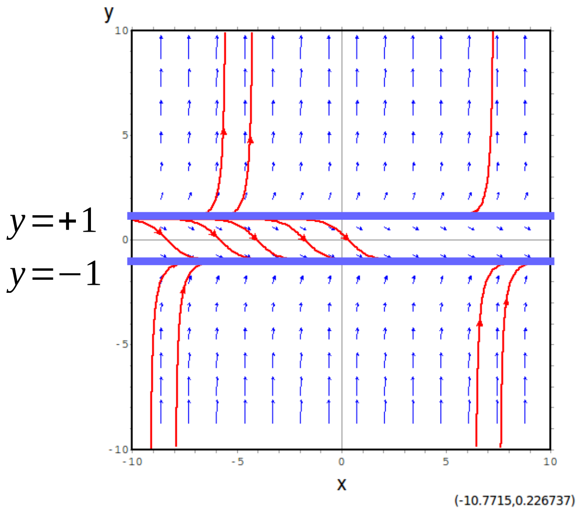
$$\frac{dy}{dx} = (y-1)(y+1) \quad \Rightarrow \quad h(y)$$

$$h(y) = 0$$
$$(y-1)(y+1) = 0 \quad \Rightarrow \quad \begin{cases} y = +1 \\ y = -1 \end{cases}$$



critical points (equilibrium, stationary points) $+1, \; -1$

constant solutions $\begin{cases} y = +1 \\ y = -1 \end{cases}$



Examples of Autonomous ODEs (1)

$$\frac{dy}{dx} = (y-1)(y+1)$$

$$\frac{1}{(y^2-1)}\frac{dy}{dx}dx = 1\,dx$$

assumed
$(y \neq -1) \wedge (y \neq +1)$

$$\frac{1}{(y^2-1)}dy = 1\,dx$$

$$\frac{1}{2}\left(\frac{1}{(y-1)} - \frac{1}{(y+1)}\right)dy = 1\,dx$$

$$\frac{1}{2}\left(\int \frac{1}{(y-1)}dy - \int \frac{1}{(y+1)}dy\right) = \int 1\,dx$$

$$\frac{1}{2}(\ln|y-1| - \ln|y+1|) = x + c_1$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2x + c_2$$

$$\ln\left|\frac{y-1}{y+1}\right| = 2x + c_2 \Rightarrow \left|\frac{y-1}{y+1}\right| = e^{2x+c_2} = c_3 e^{2x}$$

$$\frac{y-1}{y+1} = +c_3 e^{2x} \qquad \frac{y-1}{y+1} = -c_3 e^{2x}$$

$$y = \frac{1+c_3 e^{2x}}{1-c_3 e^{2x}} \qquad y = \frac{1-c_3 e^{2x}}{1+c_3 e^{2x}}$$

$+c_3 \Rightarrow c$ $\qquad$ $-c_3 \Rightarrow c$

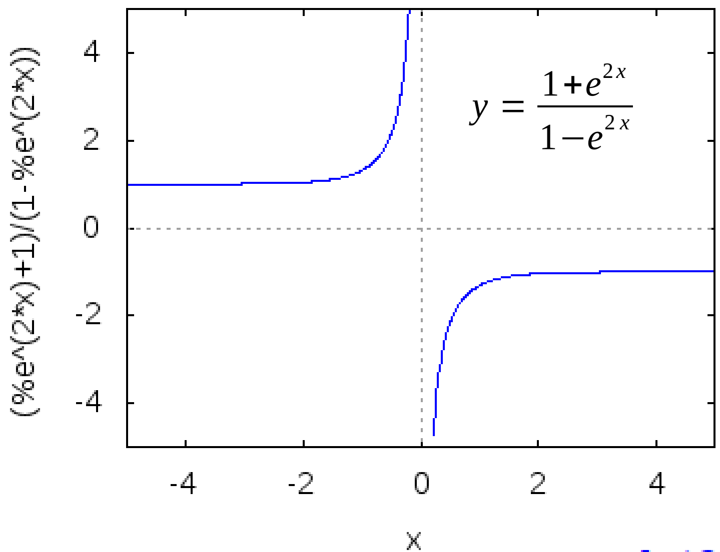
$$y = \frac{1+c\,e^{2x}}{1-c\,e^{2x}}$$

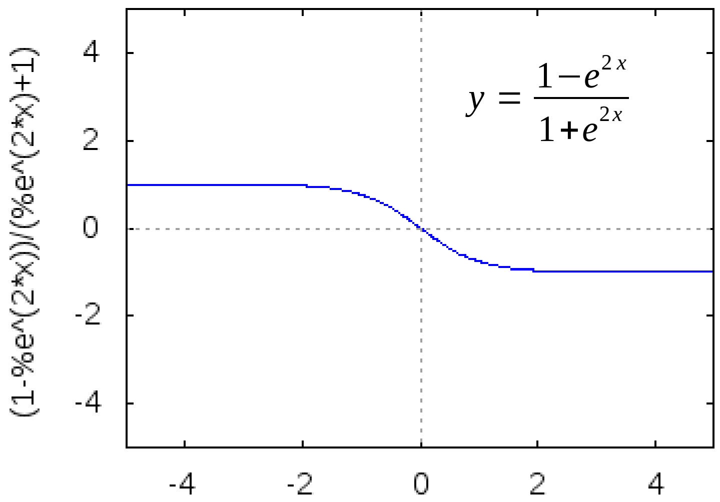
constant solutions
$y = +1$ $\quad$ $c \Rightarrow 0$
$y = -1$

: excluded in this method

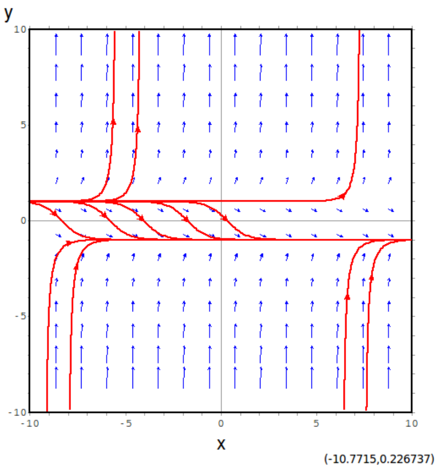
Examples of Autonomous ODEs (2)

```
wxplot2d([(1+%e^(2*x))/(1-%e^(2*x))], [x, -5, 5], [y, -5, 5]);
```






```
wxplot2d([(1-%e^(2*x))/(1+%e^(2*x))], [x, -5, 5], [y, -5, 5]);
```



References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a