ODE Background: Partial Derivative (3A)

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Prerequisite to First Order ODEs

Partial Derivatives

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variable $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Partial Derivatives Notations

Function of one variable $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Function of two variables $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x \qquad \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

treating y as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y \qquad \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

treating x as a constant

Higher-Order & Mixed Partial Derivatives

Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

Third-order Partial Derivatives

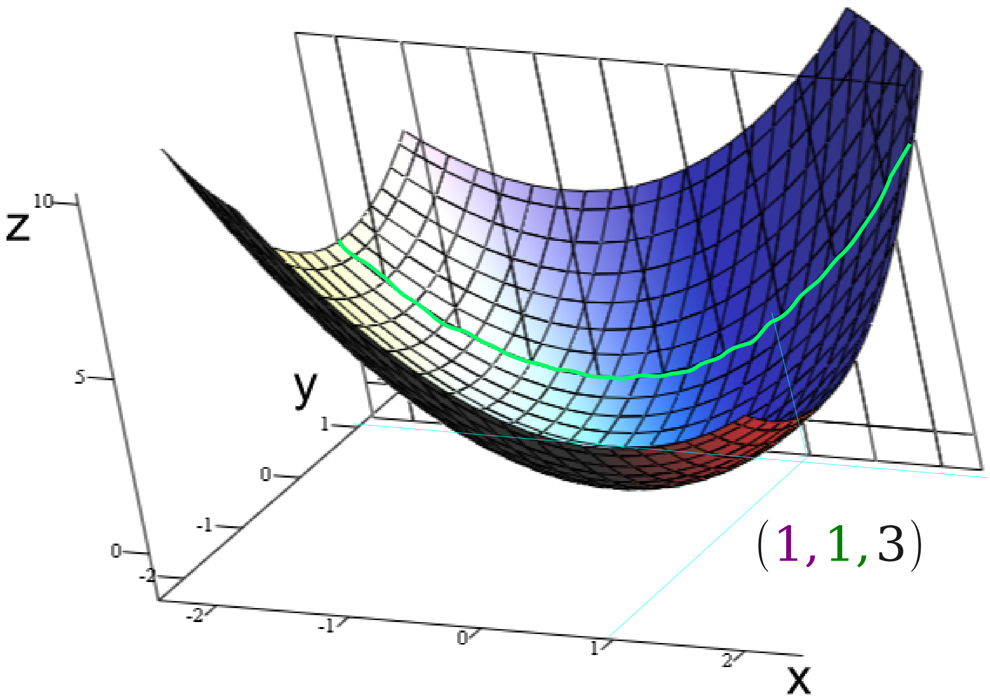
$$\frac{\partial^3 z}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 z}{\partial x^2}\right) \qquad \frac{\partial^3 z}{\partial y^3} = \frac{\partial}{\partial y}\left(\frac{\partial^2 z}{\partial y^2}\right)$$

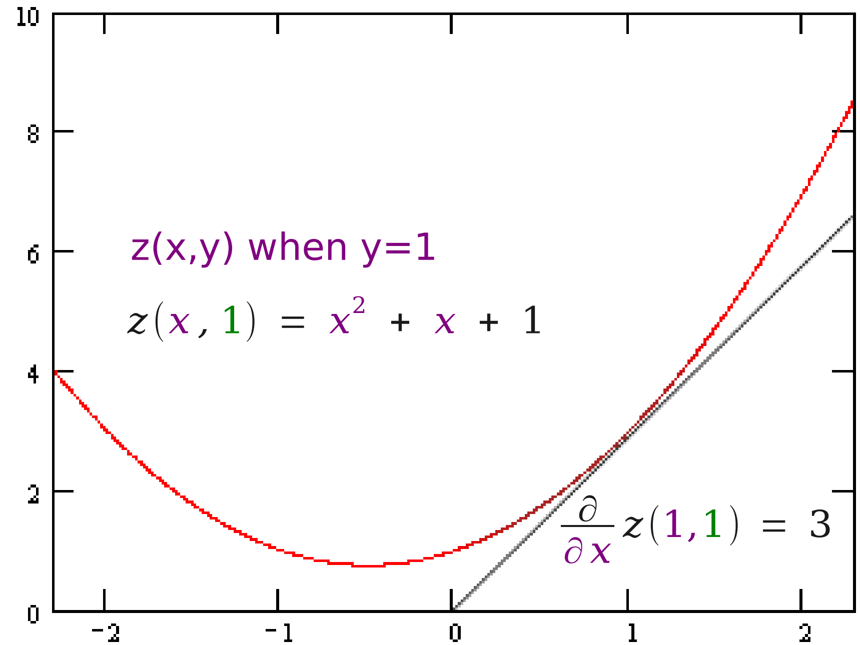
Mixed Partial Derivatives

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \overset{?}{=} \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} \iff \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}$$ all defined and continuous

Partial Derivative Examples (1)





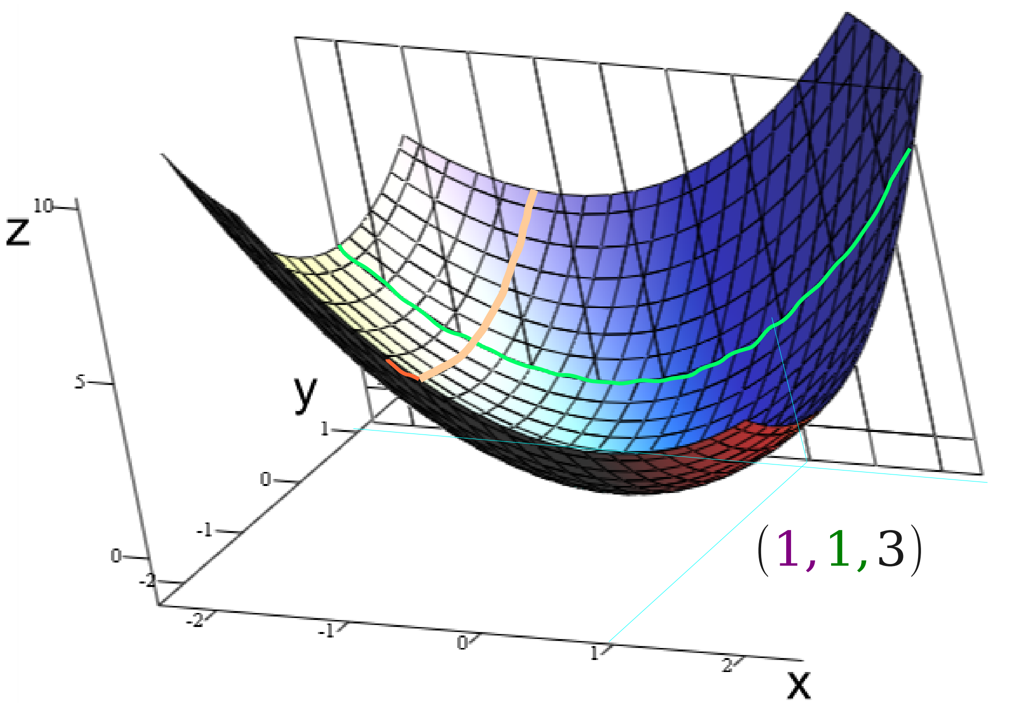
tangent at x=1 of the function z(x,1)

$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$$

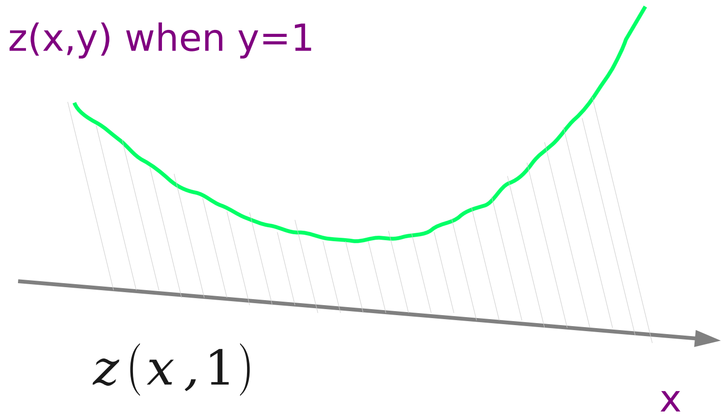
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$$

http://en.wikipedia.org/wiki/Partial_derivative

Partial Derivative Examples (2)



z(x,y) when y=1

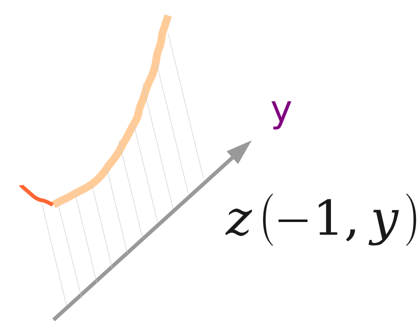


$$z = x^2 + xy + y^2 \quad \Rightarrow \quad \frac{\partial z}{\partial x} = 2x + y$$

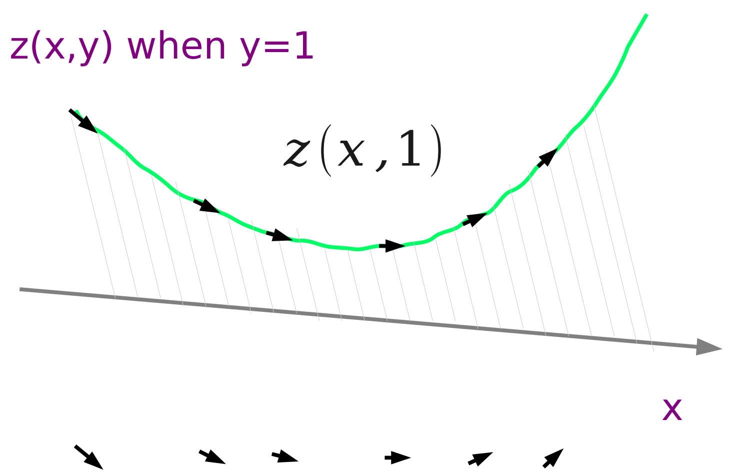
$$z = x^2 + xy + y^2 \quad \Rightarrow \quad \frac{\partial z}{\partial y} = x + 2y$$

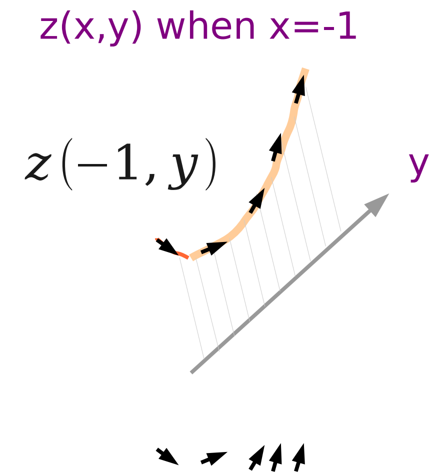
http://en.wikipedia.org/wiki/Partial_derivative

z(x,y) when x=-1



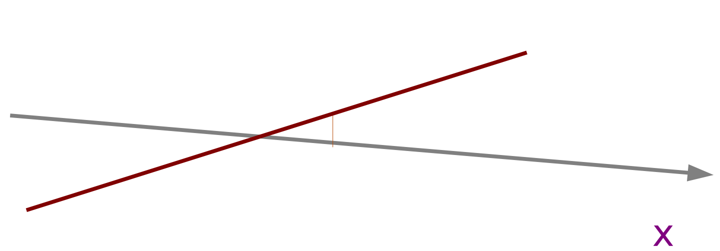
Partial Derivative Examples (3)

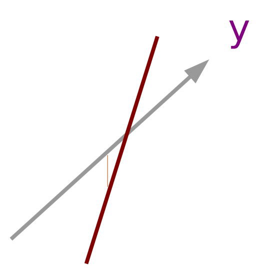




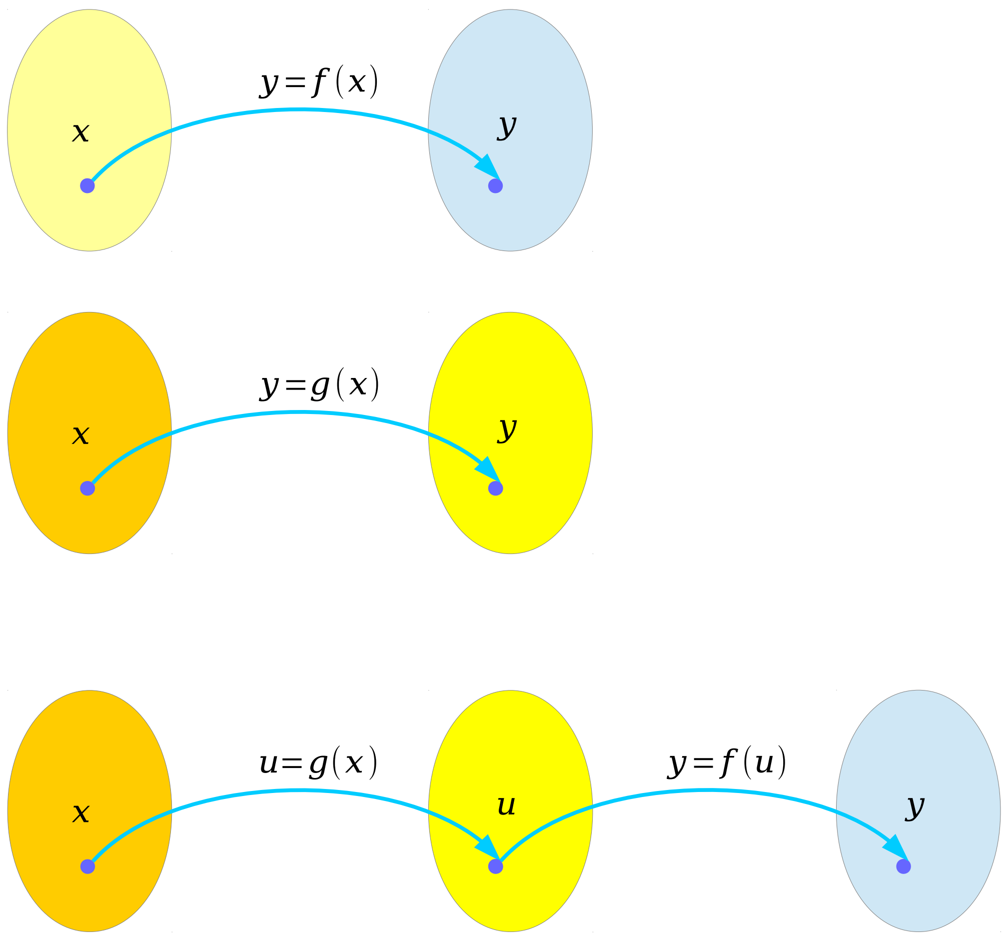
$$\frac{\partial z}{\partial x} = 2x + y \Rightarrow 2x + 1$$

$$\frac{\partial z}{\partial y} = -1 + 2y \Rightarrow x + 2y$$



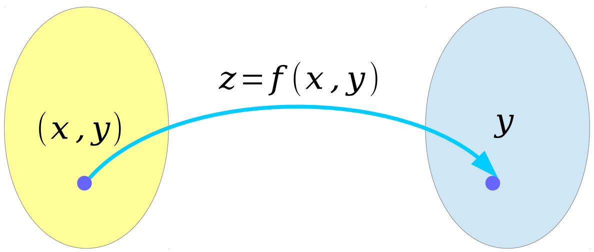


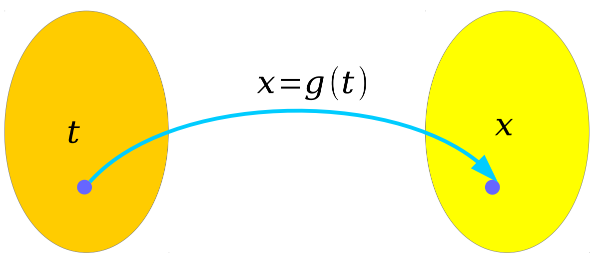
Chain Rule

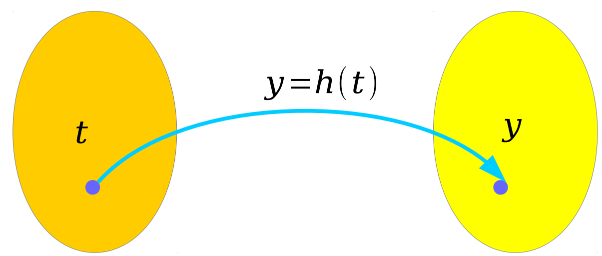


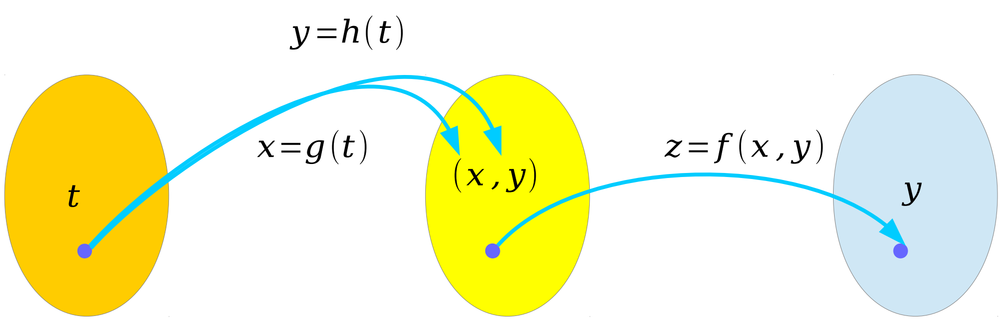
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Chain Rule and Partial Differentiation









$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain Rule and Total Differentials

$$z = f(x, y)$$

$$x(t)$$

$$y(t)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dx = \frac{dx}{dt} dt$$

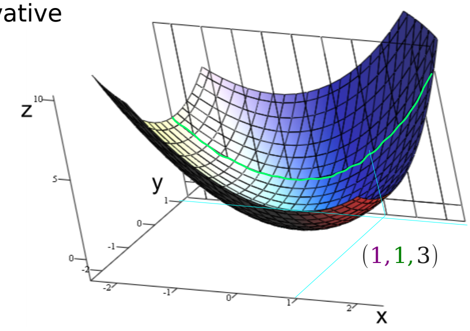
$$dy = \frac{dy}{dt} dt$$

$$\frac{dz}{dt} dt = \frac{\partial z}{\partial x} \frac{dx}{dt} dt + \frac{\partial z}{\partial y} \frac{dy}{dt} dt$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

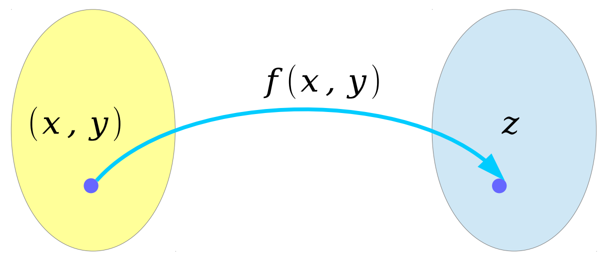
Parameterized Function of Two Variables

http://en.wikipedia.org/wiki/Partial_derivative



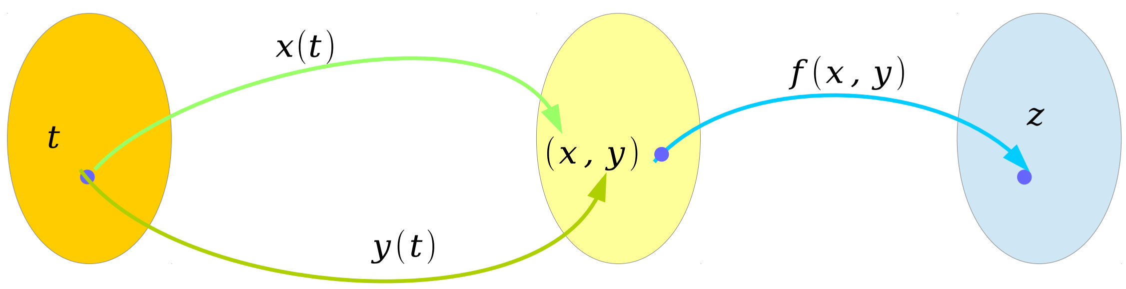
$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$

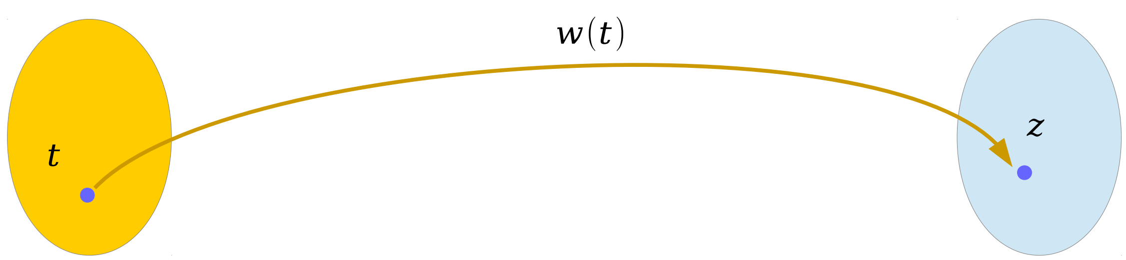
$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$



$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

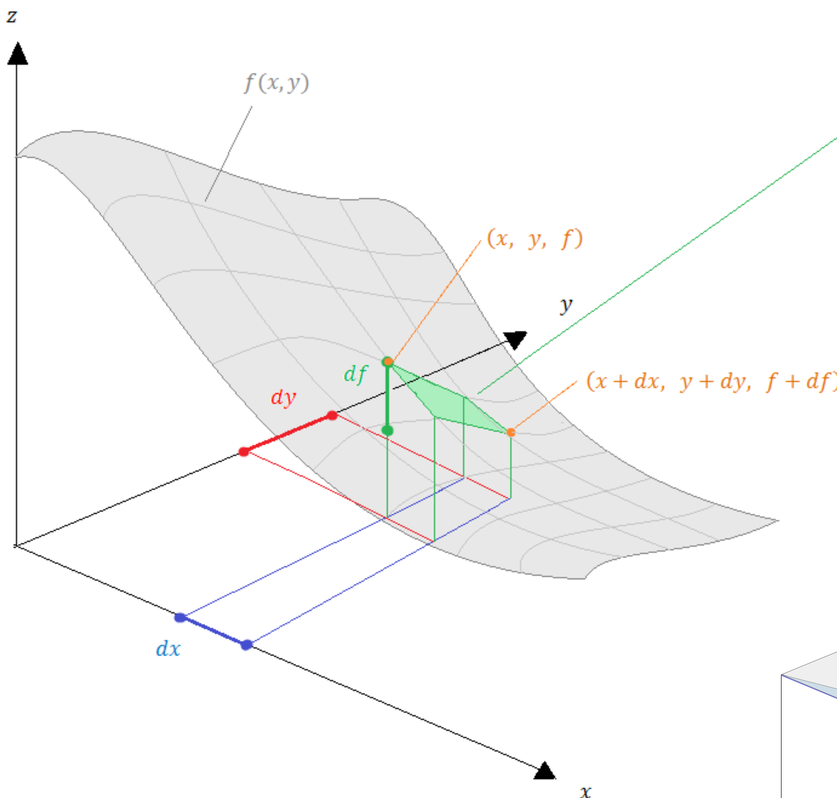


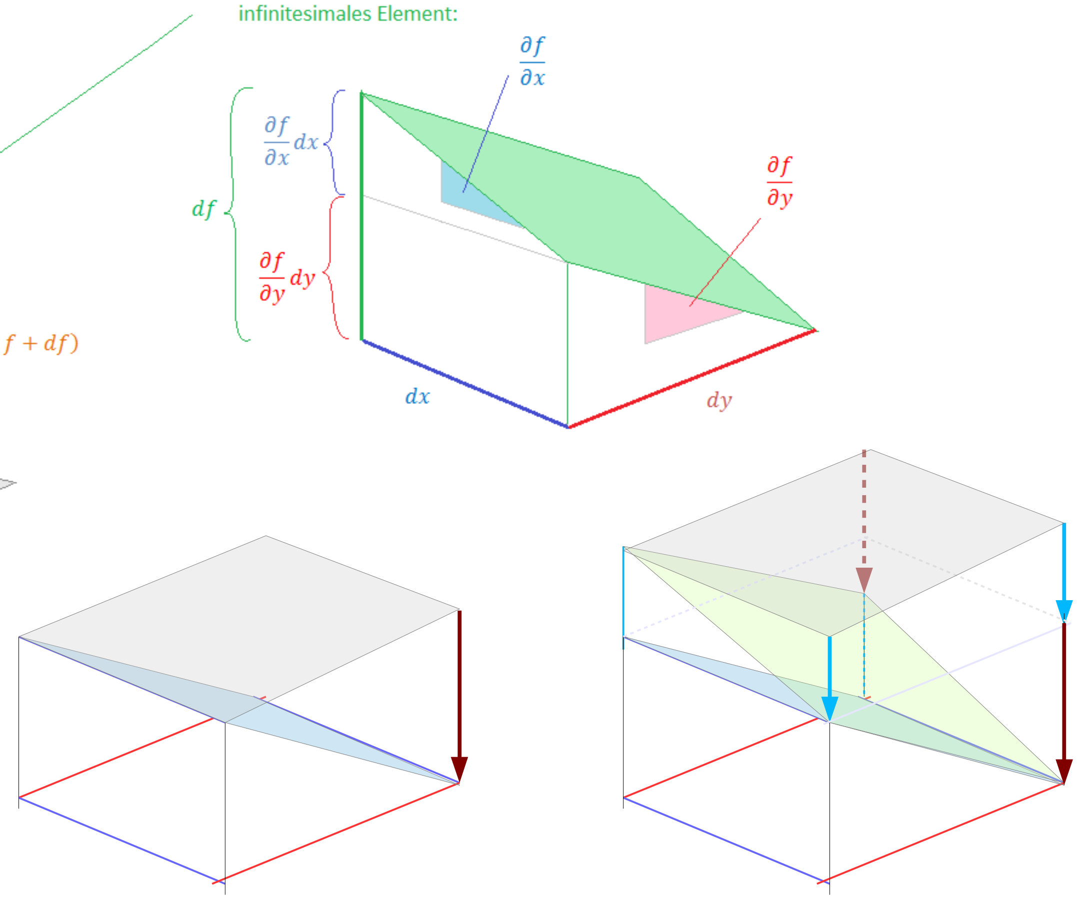


$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

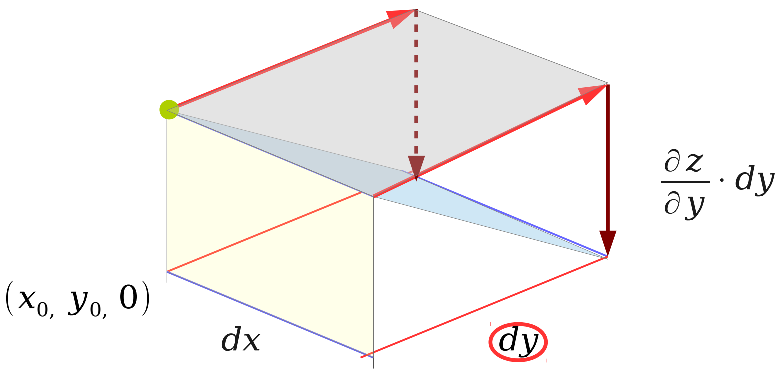
$$\frac{dw}{dt}(t) = \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Total Differential





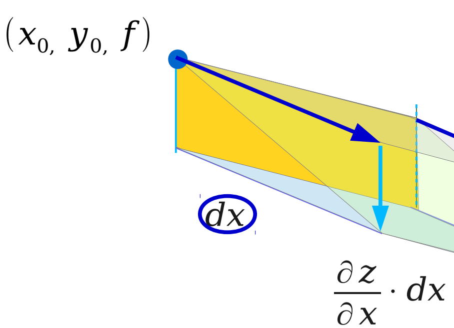
http://de.wikipedia.org/wiki/Totales_Differential
Muhammet Cakir

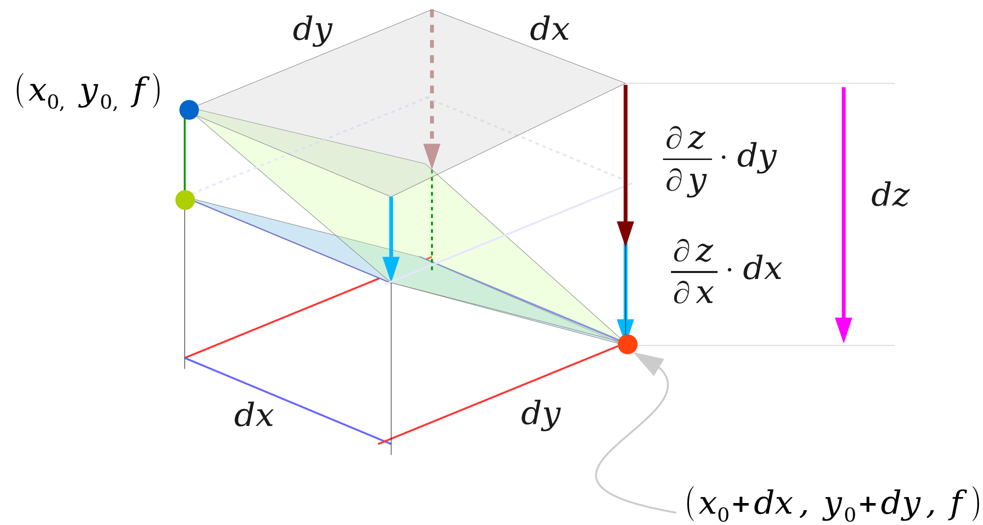


$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

differential, or
total differential





Integrating Gaussian Function (1)

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)$$

$$= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

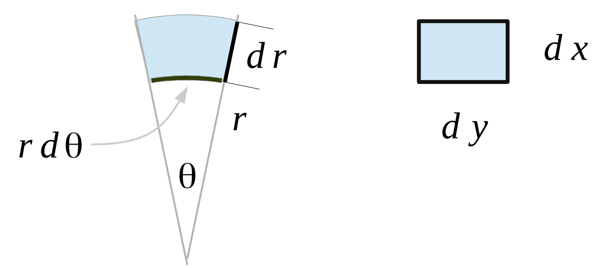
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2}\,dx\,dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= \left(\int_{0}^{2\pi} d\theta\right)\left(\int_{0}^{+\infty} e^{-r^2}\,r\,dr\right)$$

$$= 2\pi\left(\int_{0}^{+\infty} e^{-r^2}\,r\,dr\right)$$



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

Integrating Gaussian Function (2)

$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_{0}^{2\pi}\int_{0}^{+\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= 2\pi\left(\int_{0}^{+\infty} e^{-r^2}\,r\,dr\right) \qquad s = -r^2 \qquad ds = -2r\,dr$$

$$= \pi\left(\int_{-\infty}^{0} e^{s}\,ds\right)$$

$$= \pi\left[e^{s}\right]_{-\infty}^{0}$$

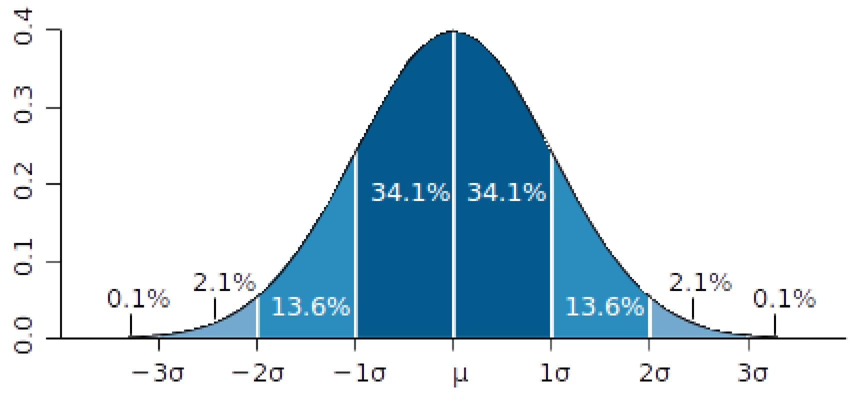
$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 = \pi\left[e^{0} - e^{-\infty}\right] = \pi$$

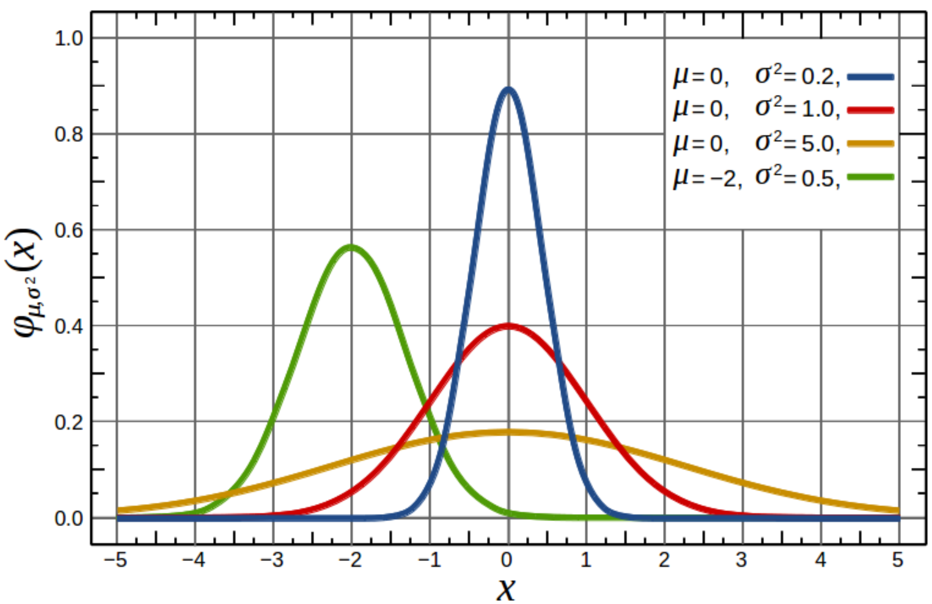
$$\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right) = \sqrt{\pi}$$

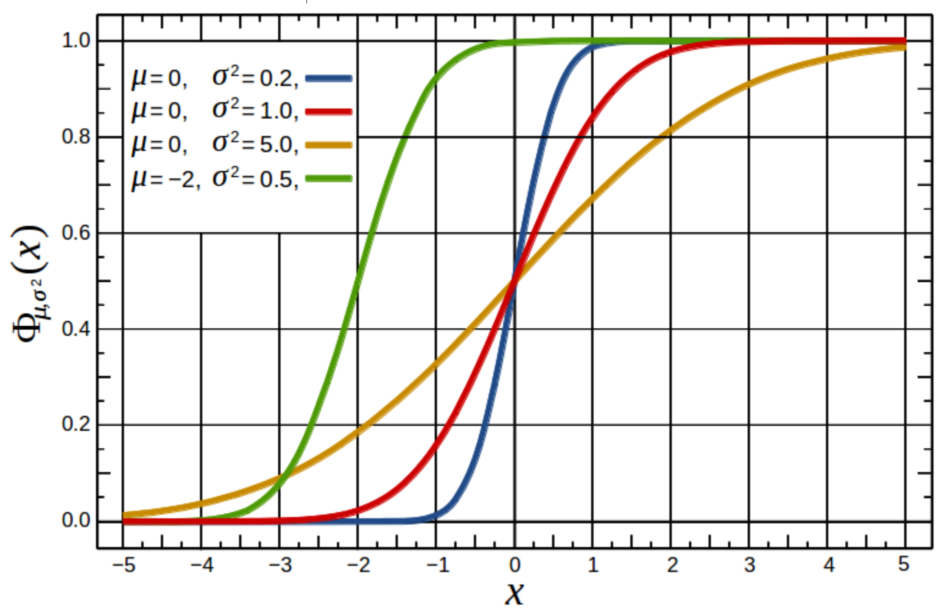
Normal Distribution

http://en.wikipedia.org/wiki/Derivative

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$







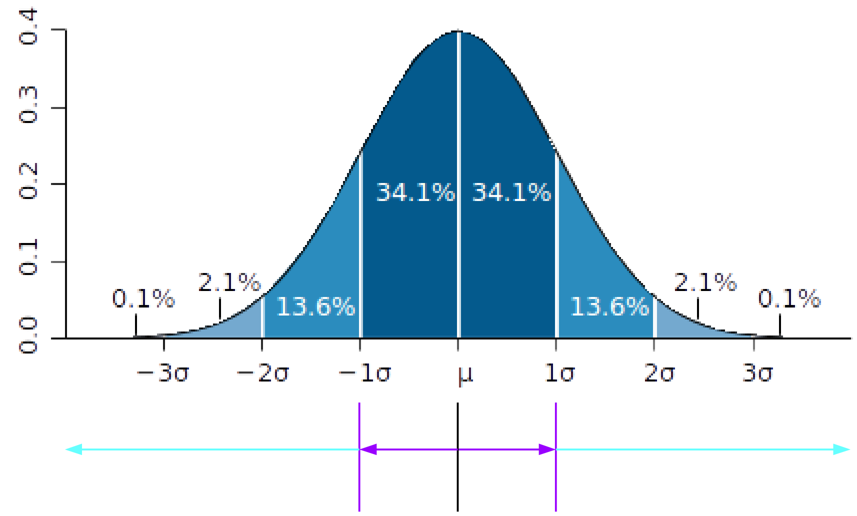
Error Function

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{+x} e^{-x^2} dx$$

$$errc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{+\infty} e^{-x^2} dx$$

$$erf(x) + erfc(x) = 1$$

http://en.wikipedia.org/wiki/Derivative



Error Function Table

x	$erf(x)$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, “Mathematical Methods in the Physical Sciences”
[3] E. Kreyszig, “Advanced Engineering Mathematics”
[4] D. G. Zill, W. S. Wright, “Advanced Engineering Mathematics”
[5] www.chem.arizona.edu/~salzmanr/480a