

# ODE Background: Partial Derivative (3A)

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# *Prerequisite to First Order ODEs*

# Partial Derivatives

Function of one variable       $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Function of two variable       $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

treating  $y$  as a constant

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

treating  $x$  as a constant

# Partial Derivatives Notations

Function of one variable       $y = f(x)$

$$\frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Function of two variables       $z = f(x, y)$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = z_x = f_x$$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

treating  $y$  as a constant

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = z_y = f_y$$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

treating  $x$  as a constant

# Higher-Order & Mixed Partial Derivatives

## Second-order Partial Derivatives

$$\frac{\partial^2 z}{\partial \mathbf{x}^2} = \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial z}{\partial \mathbf{x}} \right)$$

$$\frac{\partial^2 z}{\partial \mathbf{y}^2} = \frac{\partial}{\partial \mathbf{y}} \left( \frac{\partial z}{\partial \mathbf{y}} \right)$$

## Third-order Partial Derivatives

$$\frac{\partial^3 z}{\partial \mathbf{x}^3} = \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial^2 z}{\partial \mathbf{x}^2} \right)$$

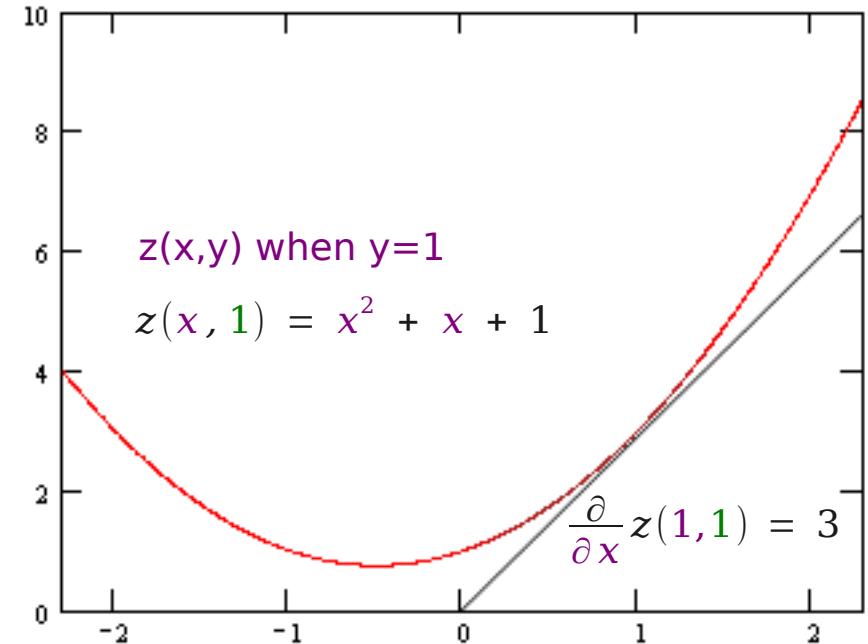
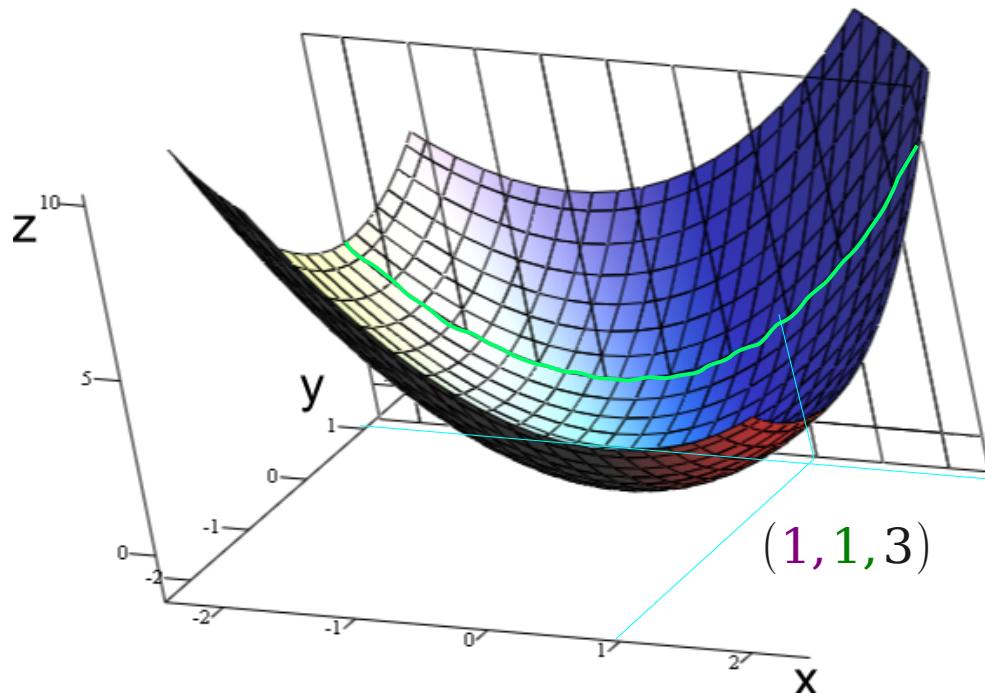
$$\frac{\partial^3 z}{\partial \mathbf{y}^3} = \frac{\partial}{\partial \mathbf{y}} \left( \frac{\partial^2 z}{\partial \mathbf{y}^2} \right)$$

## Mixed Partial Derivatives

$$\frac{\partial^2 z}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial}{\partial \mathbf{x}} \left( \frac{\partial z}{\partial \mathbf{y}} \right) \stackrel{?}{=} \frac{\partial^2 z}{\partial \mathbf{y} \partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}} \left( \frac{\partial z}{\partial \mathbf{x}} \right)$$

$$\frac{\partial^2 z}{\partial \mathbf{x} \partial \mathbf{y}} = \frac{\partial^2 z}{\partial \mathbf{y} \partial \mathbf{x}} \quad \Leftrightarrow \quad \frac{\partial z}{\partial \mathbf{x}}, \frac{\partial z}{\partial \mathbf{y}}, \frac{\partial^2 z}{\partial \mathbf{x} \partial \mathbf{y}}, \frac{\partial^2 z}{\partial \mathbf{y} \partial \mathbf{x}} \quad \text{all defined and continuous}$$

# Partial Derivative Examples (1)

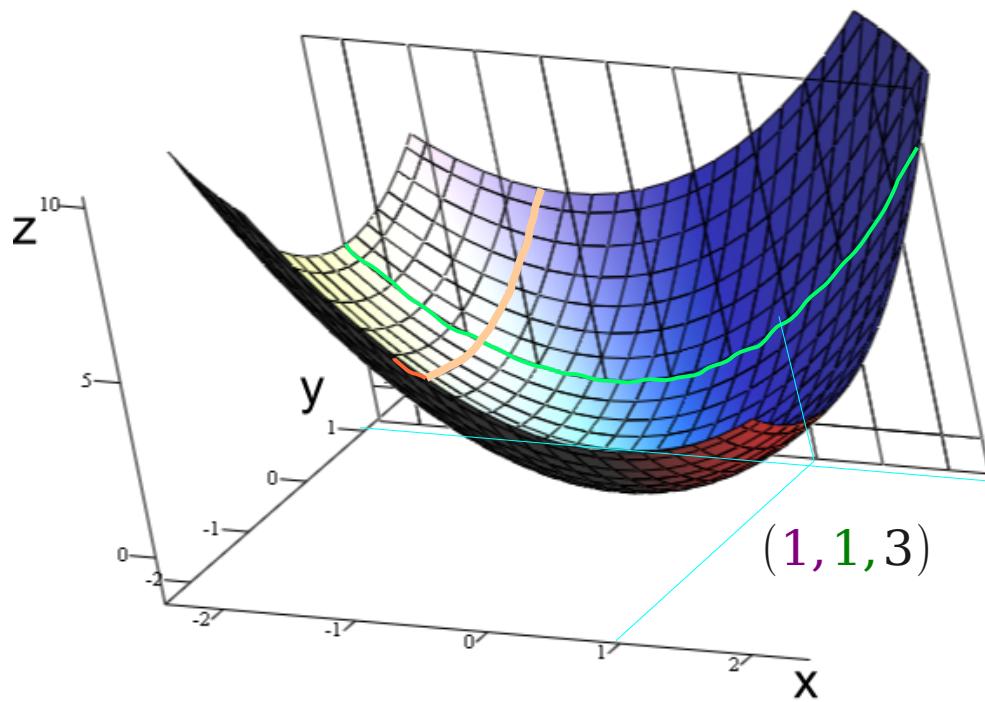


$$z = \textcolor{violet}{x}^2 + \textcolor{violet}{x}y + y^2 \rightarrow \frac{\partial z}{\partial x} = 2\textcolor{violet}{x} + y$$
$$z = x^2 + x\textcolor{green}{y} + \textcolor{green}{y}^2 \rightarrow \frac{\partial z}{\partial y} = x + 2\textcolor{green}{y}$$

[http://en.wikipedia.org/wiki/Partial\\_derivative](http://en.wikipedia.org/wiki/Partial_derivative)

tangent at  $x=1$  of the function  $z(x, 1)$

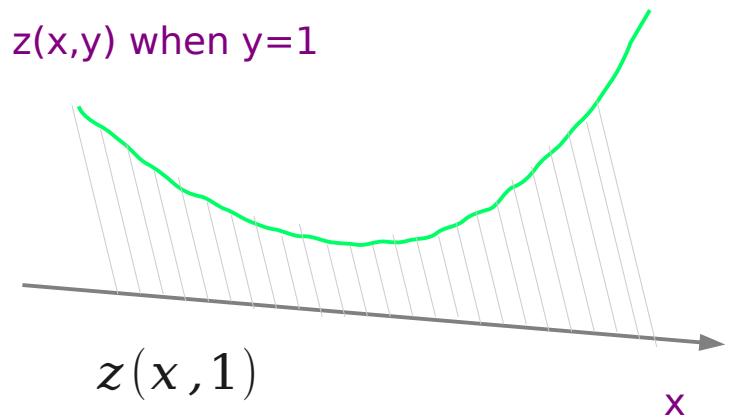
# Partial Derivative Examples (2)



$$z = \textcolor{violet}{x}^2 + \textcolor{violet}{x}y + y^2 \rightarrow \frac{\partial z}{\partial x} = 2\textcolor{violet}{x} + y$$
$$z = x^2 + x\textcolor{green}{y} + \textcolor{green}{y}^2 \rightarrow \frac{\partial z}{\partial y} = x + 2\textcolor{green}{y}$$

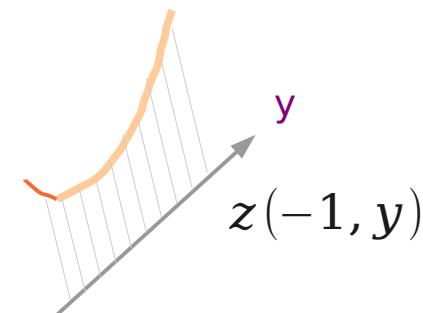
[http://en.wikipedia.org/wiki/Partial\\_derivative](http://en.wikipedia.org/wiki/Partial_derivative)

$z(x,y)$  when  $y=1$



$z(x, 1)$

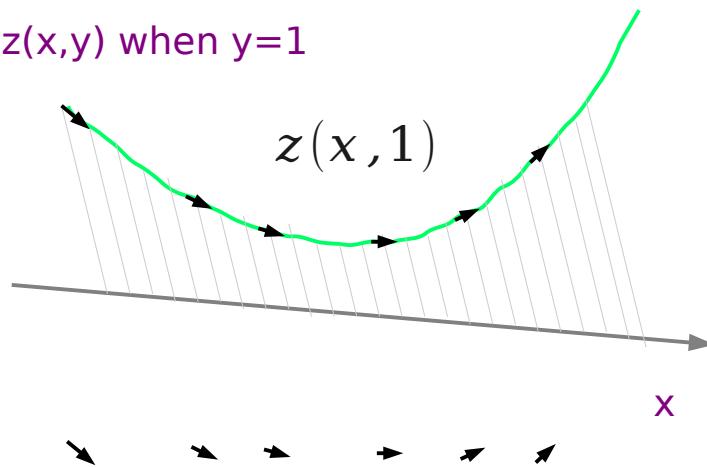
$z(x,y)$  when  $x=-1$



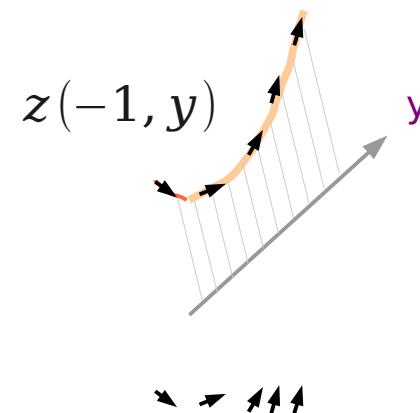
$z(-1, y)$

# Partial Derivative Examples (3)

$z(x,y)$  when  $y=1$

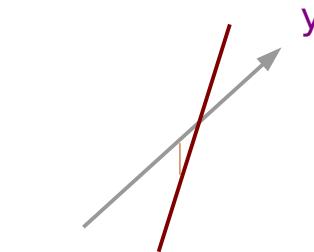
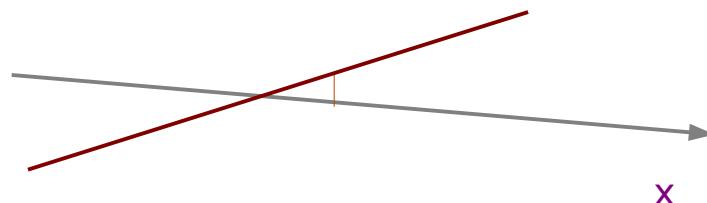


$z(x,y)$  when  $x=-1$

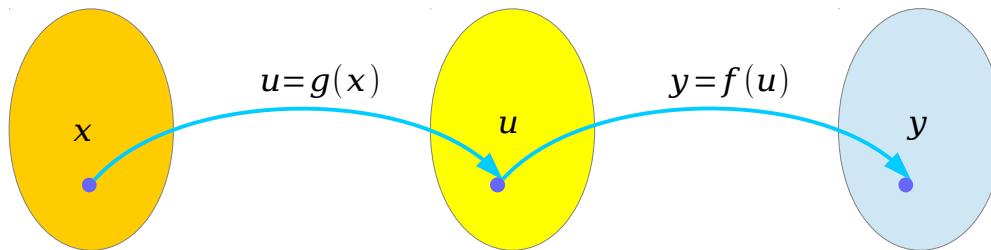
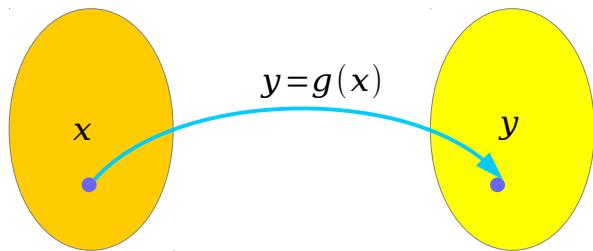
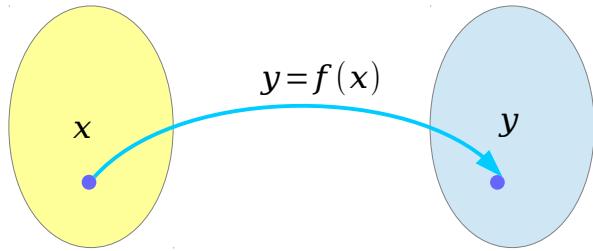


$$\frac{\partial z}{\partial x} = 2x + y \rightarrow 2x + 1$$

$$\frac{\partial z}{\partial y} = -1 + 2y \rightarrow x + 2y$$

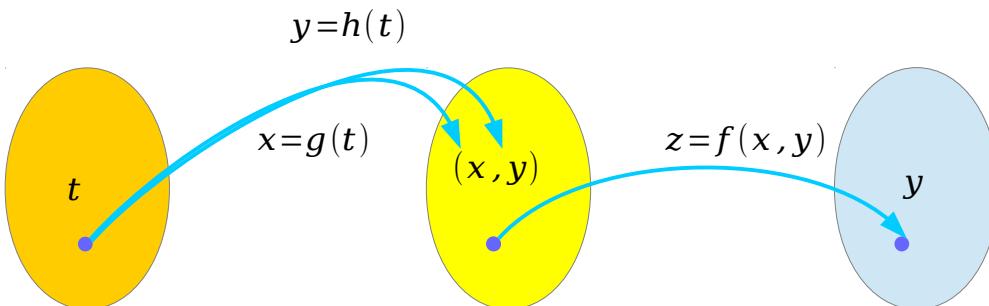
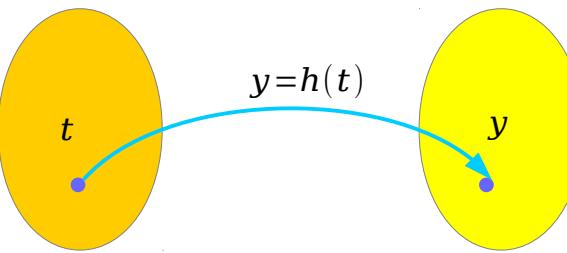
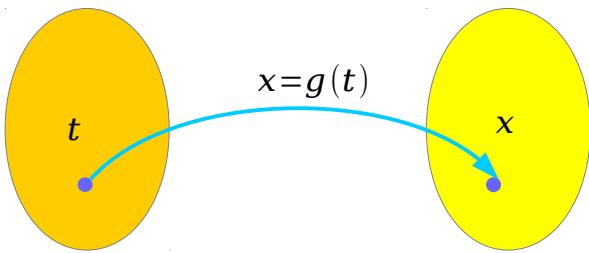
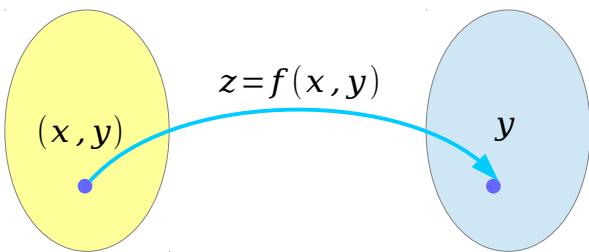


# Chain Rule



$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

# Chain Rule and Partial Differentiation



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

# Chain Rule and Total Differentials

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$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$x(t)$$

$$dx = \frac{dx}{dt} dt$$

$$y(t)$$

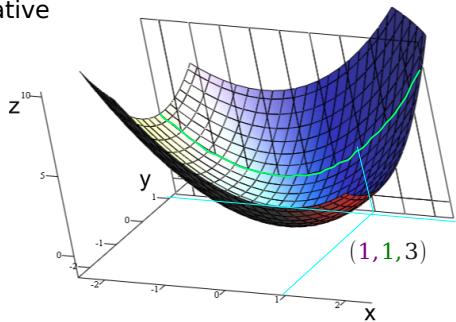
$$dy = \frac{dy}{dt} dt$$

$$\frac{dz}{dt} dt = \frac{\partial z}{\partial x} \frac{dx}{dt} dt + \frac{\partial z}{\partial y} \frac{dy}{dt} dt$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

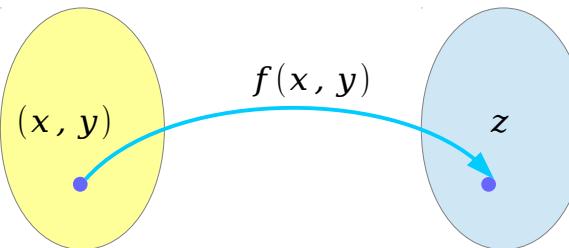
# Parameterized Function of Two Variables

[http://en.wikipedia.org/wiki/Partial\\_derivative](http://en.wikipedia.org/wiki/Partial_derivative)



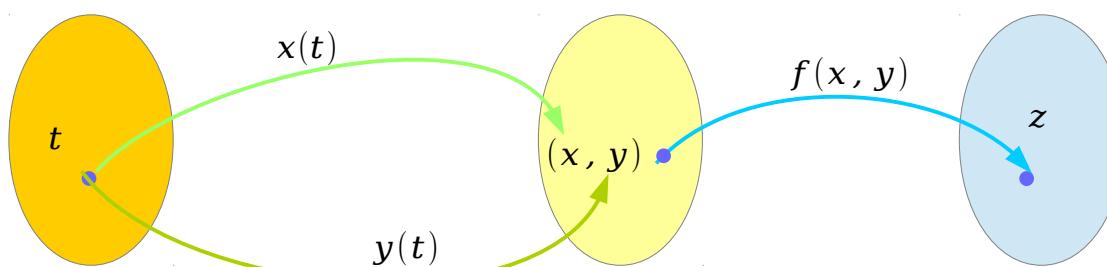
$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial x} = 2x + y$$

$$z = x^2 + xy + y^2 \Rightarrow \frac{\partial z}{\partial y} = x + 2y$$

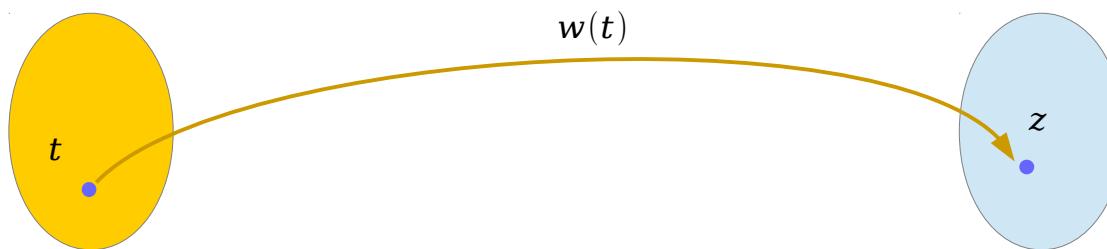


$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y}$$

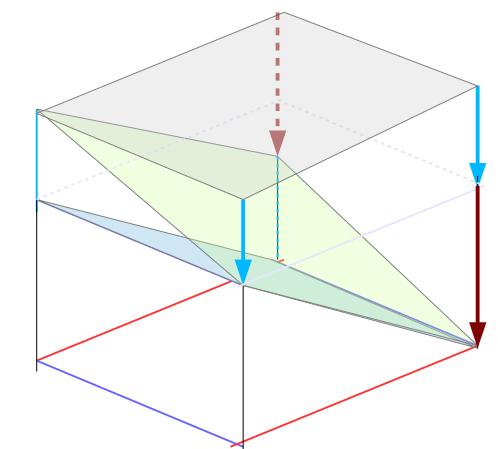
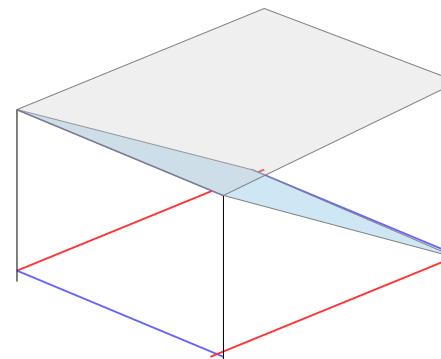
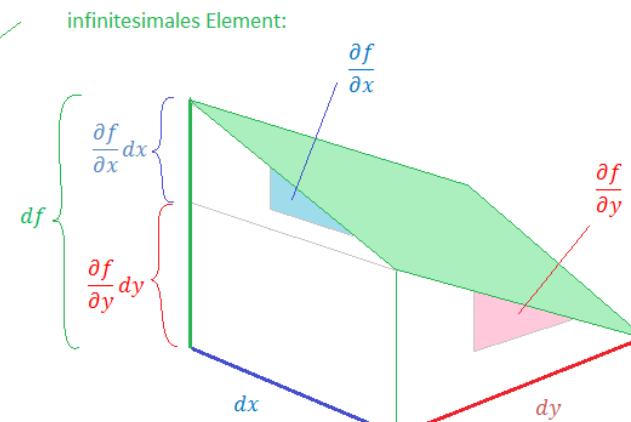
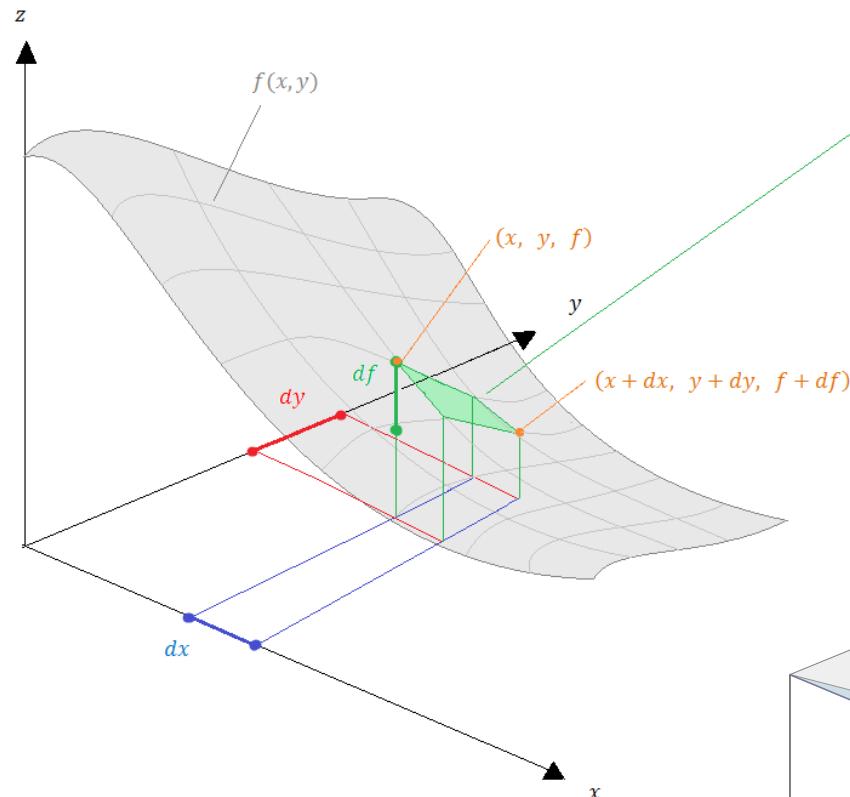


$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$



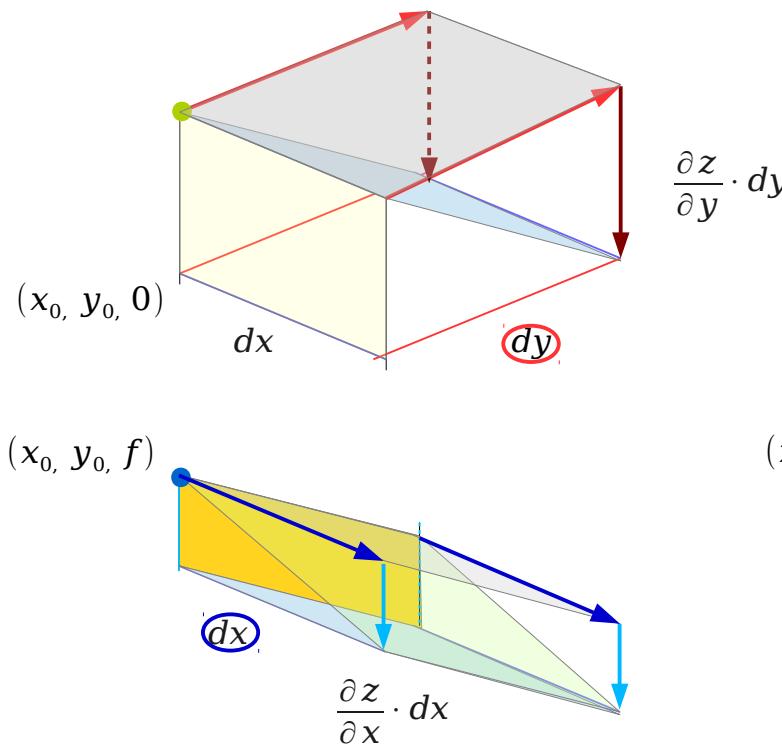
$$\frac{dw}{dt}(t) = \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

# Total Differential



[http://de.wikipedia.org/wiki/Totales\\_Differential](http://de.wikipedia.org/wiki/Totales_Differential)  
Muhammet Cakir

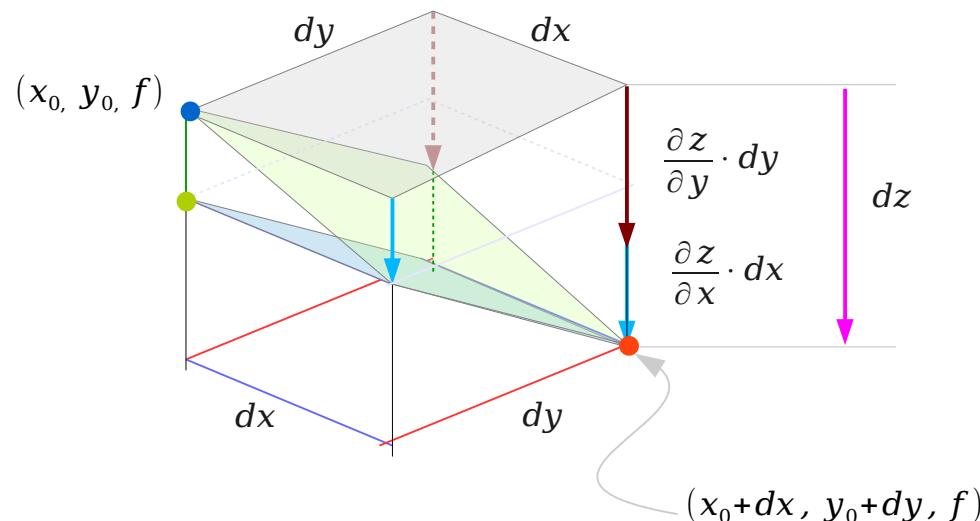
# Total Differential



$$z = f(x, y)$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

differential, or  
total differential



# Integrating Gaussian Function (1)

$$\left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 = \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right)$$

$$= \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) \left( \int_{-\infty}^{+\infty} e^{-y^2} dy \right)$$

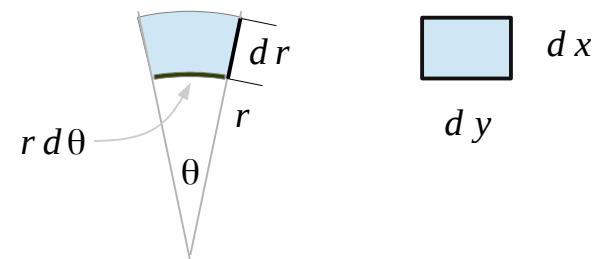
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-x^2} e^{-y^2} dx dy$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy$$

$$= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r dr d\theta$$

$$= \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{+\infty} e^{-r^2} r dr \right)$$

$$= 2\pi \left( \int_0^{+\infty} e^{-r^2} r dr \right)$$



$$x = r \cos \theta$$

$$x^2 + y^2 = r^2$$

$$y = r \sin \theta$$

# Integrating Gaussian Function (2)

$$\begin{aligned} \left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r dr d\theta \\ &= 2\pi \left( \int_0^{+\infty} e^{-r^2} r dr \right) \quad s = -r^2 \quad ds = -2r dr \\ &= \pi \left( \int_{-\infty}^0 e^s ds \right) \\ &= \pi [e^s]_{-\infty}^0 \end{aligned}$$

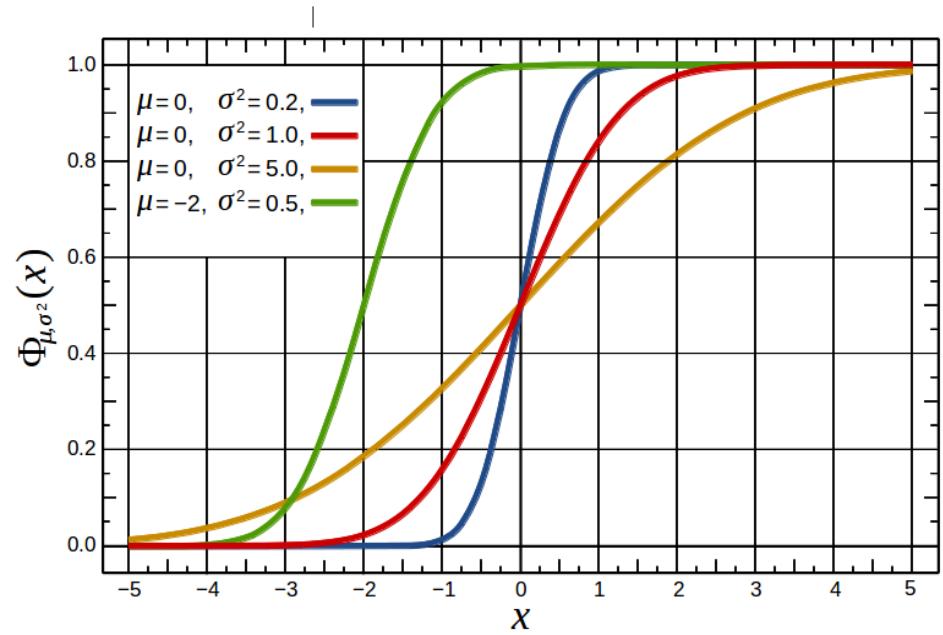
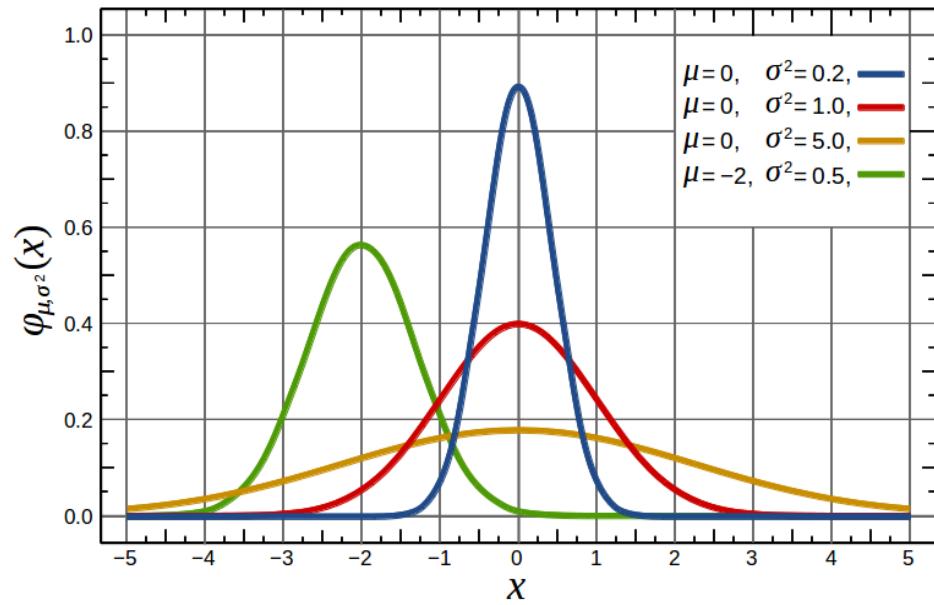
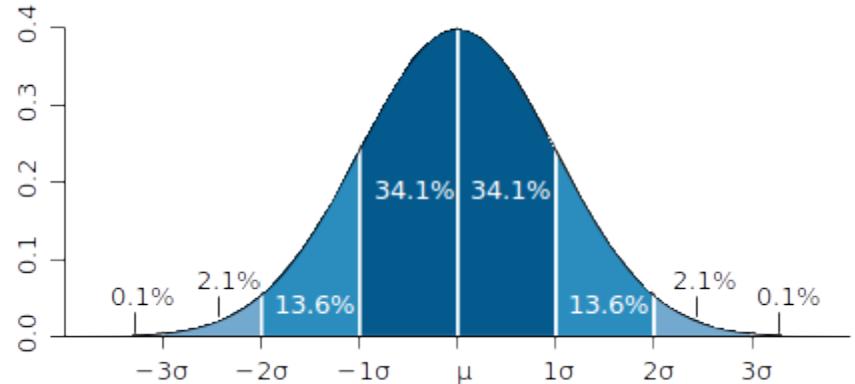
$$\left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2 = \pi [e^0 - e^{-\infty}] = \pi$$

$$\left( \int_{-\infty}^{+\infty} e^{-x^2} dx \right) = \sqrt{\pi}$$

# Normal Distribution

<http://en.wikipedia.org/wiki/Derivative>

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$



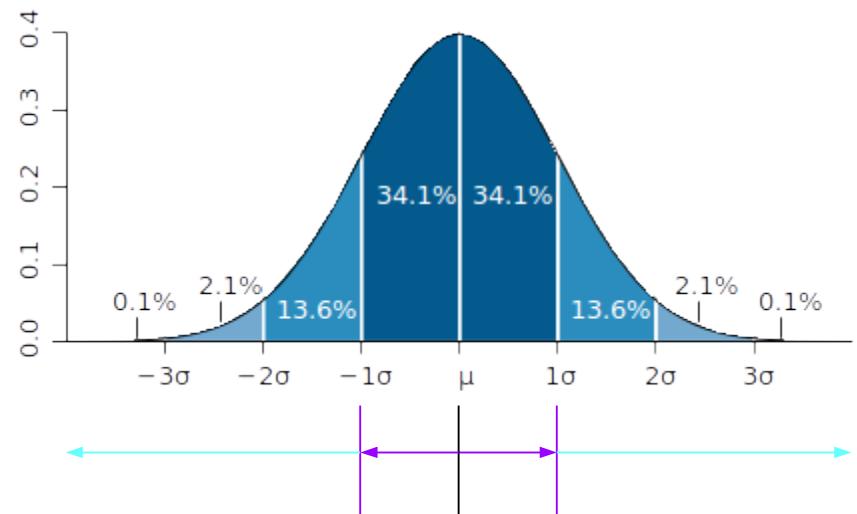
# Error Function

<http://en.wikipedia.org/wiki/Derivative>

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{+x} e^{-x^2} dx$$

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-x^2} dx$$

$$\operatorname{erf}(x) + \operatorname{erfc}(x) = 1$$



## Error Function Table

$x$	$\operatorname{erf}(x)$

## References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
- [3] E. Kreyszig, "Advanced Engineering Mathematics"
- [4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"
- [5] [www.chem.arizona.edu/~salzmanr/480a](http://www.chem.arizona.edu/~salzmanr/480a)