

# Higher Order ODE's (3A)

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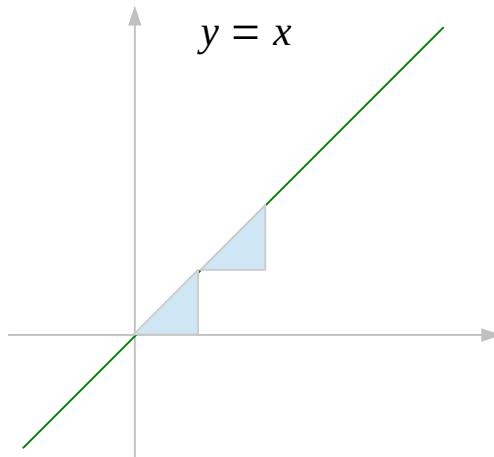
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# The Properties of a Line Equation (1)



$$f(1) = 1$$

$$f(2) = 2$$

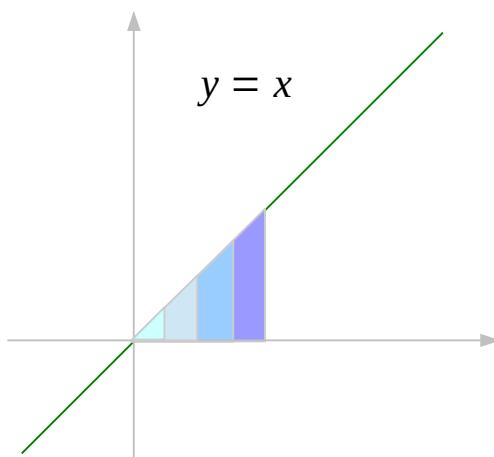
$$f(3) = 3$$

$$f(2) = f(1+1) = f(1) + f(1) = 2$$

$$f(3) = f(2+1) = f(2) + f(1) = 3$$

$$f(4) = f(3+1) = f(3) + f(1) = 4$$

*multiples of a unit*



$$f(0.1) = 0.1$$

$$f(0.5) = 0.5$$

$$f(1.5) = 1.5$$

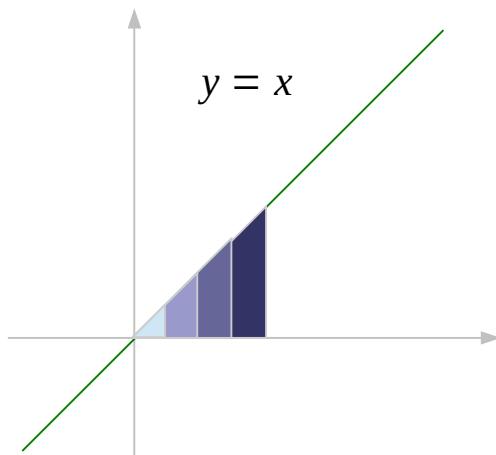
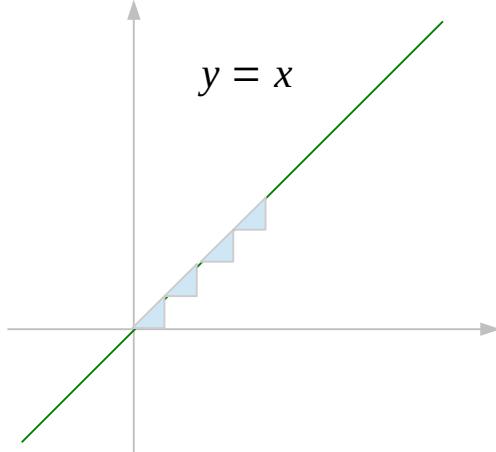
$$f(0.1) = f(0.1 \cdot 1) = 0.1 \cdot f(1) = 0.1$$

$$f(0.5) = f(0.5 \cdot 1) = 0.5 \cdot f(1) = 0.5$$

$$f(1.5) = f(1.5 \cdot 1) = 1.5 f(1) = 1.5$$

*fractions of a unit*

# The Properties of a Line Equation (2)



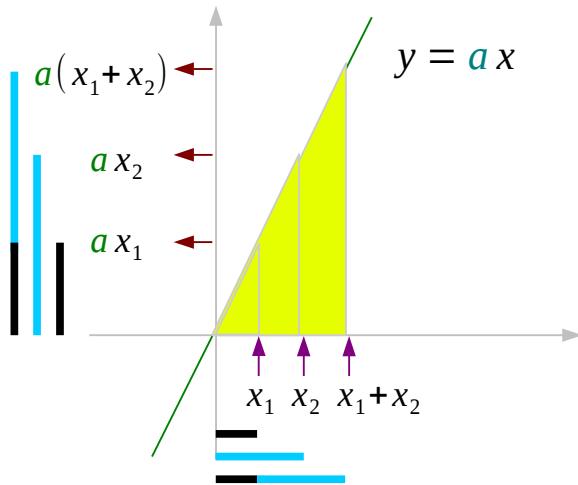
multiples of a unit			
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f(1) = 1$	$f(2) = 2$	$f(3) = 3$	$f(4) = 4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f(0.5) = 0.5$	$f(1.0) = 1.0$	$f(1.5) = 1.5$	$f(2.0) = 2.0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$f(0.1) = 0.1$	$f(0.2) = 0.2$	$f(0.3) = 0.3$	$f(0.4) = 0.4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

*fraction of a unit*

$f(\textcolor{teal}{x}_1 + x_2) = f(\textcolor{teal}{x}_1) + f(x_2)$

A diagram showing a horizontal line with vertical tick marks. Above the line, a blue arrow points from the equation  $f(\textcolor{teal}{x}_1 + x_2) = f(\textcolor{teal}{x}_1) + f(x_2)$  to the tick marks. Below the line, a blue arrow points from the equation  $f(\textcolor{violet}{k}x) = \textcolor{violet}{k}f(x)$  to a series of horizontal blue bars of increasing length, representing the scaling of the function.

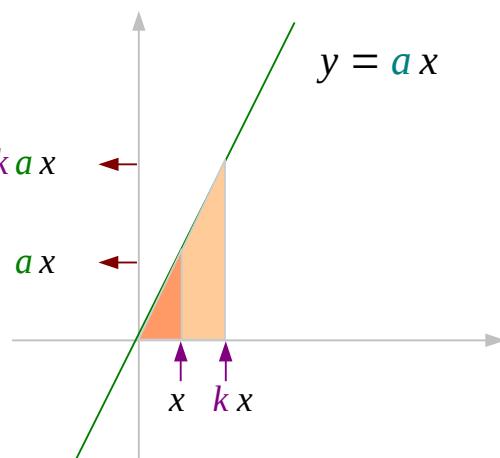
# Linearity Property



**Additivity**

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$a \cdot (x_1 + x_2) = a \cdot (x_1) + a \cdot (x_2)$$



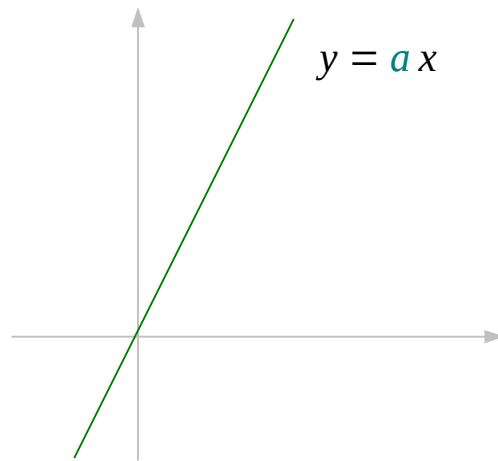
**Homogeneity**

$$f(kx) = kf(x)$$

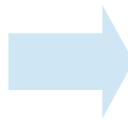
$$a \cdot (kx) = k(a \cdot x)$$

# Linearity & Affinity

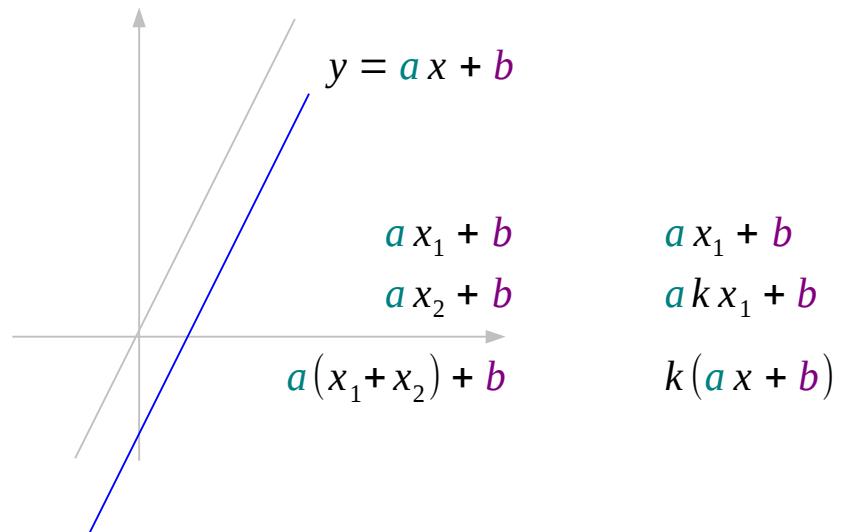
## Linearity



*Translation*



## Affinity



from linear + -ity.

from the Latin, affinis,  
"connected with"

$$f(\textcolor{teal}{x}_1 + x_2) = f(x_1) + f(x_2)$$
$$f(kx) = kf(x)$$

*Additivity*

*Homogeneity*

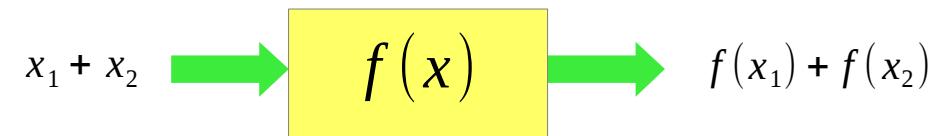
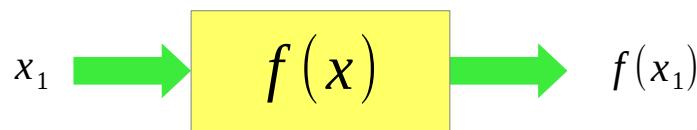
*Additivity*

*Homogeneity*

# Linear Map

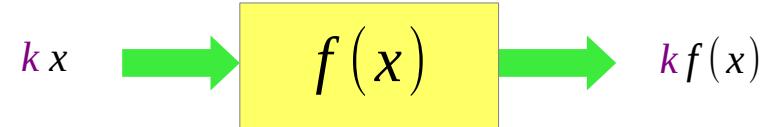
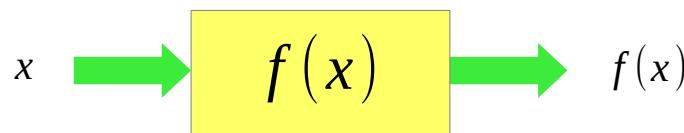
**Additivity**

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$



**Homogeneity**

$$f(kx) = kf(x)$$



# Linear Operators

**Additivity**

$$\frac{d}{dx}[f + g] = \frac{d f}{d x} + \frac{d g}{d x}$$

$$f(x) \xrightarrow{\frac{d}{d x}} f'(x)$$

$$f(x) + g(x) \xrightarrow{\frac{d}{d x}} f'(x) + g'(x)$$

$$g(x) \xrightarrow{\frac{d}{d x}} g'(x)$$

**Homogeneity**

$$\frac{d}{dx}[k f] = k \frac{d f}{d x}$$

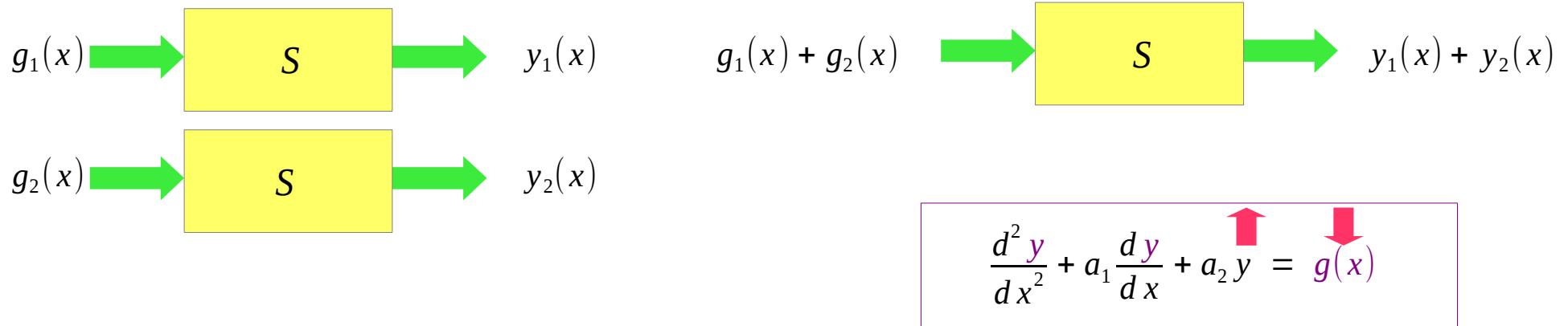
$$f(x) \xrightarrow{\frac{d}{d x}} f'(x)$$

$$k f(x) \xrightarrow{\frac{d}{d x}} k f'(x)$$

# Linear Systems

## Additivity

$$S\{g_1(x) + g_2(x)\} = S\{g_1(x)\} + S\{g_2(x)\}$$



## Homogeneity

$$S\{kf(x)\} = kS\{f(x)\}$$



# Differential Operator

## Differential Operator

$$y(x) \xrightarrow{D} y'(x)$$

$$D = \frac{d}{dx}$$

$$D(y) = \frac{dy}{dx}$$

$$D(y(x)) = \frac{dy}{dx}$$

## N-th Order Differential Operator

$$y(x) \xrightarrow{L} L(y(x))$$

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

$$L(y) = \{a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\}(y)$$

$$L(y) = a_n(x)D^n(y) + a_{n-1}(x)D^{n-1}(y) + \cdots + a_1(x)D(y) + a_0(x)(y)$$

$$L(y) = a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y$$

# Examples

$f(x) \rightarrow \boxed{\frac{d}{dx}} \rightarrow f'(x)$ $f(x) \rightarrow \boxed{D} \rightarrow f'(x)$	$\frac{d}{dx} = f'(x)$ $Df = f'(x)$
	$D(Df) = \frac{d}{dx}\left(\frac{df}{dx}\right) = f''(x)$ $D^2f = \frac{d^2f}{dx^2} = f''(x)$

## Differential Operator : Linear

$$D(cf(x)) = cDf(x)$$

$$D(f(x) + g(x)) = Df(x) + Dg(x)$$

$$D(\alpha f(x) + \beta g(x)) = \alpha Df(x) + \beta Dg(x)$$

## n-th order Differential Operator

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \dots + a_1(x)D + a_0(x)$$

$$(D^2 + 2D + 1)f(x) = D^2f(x) + 2Df(x) + f(x) = f''(x) + 2f'(x) + f(x)$$

## n-th order Differential Equations using the Differential Operator

$$y'' + 5y' + 2y = 3x$$

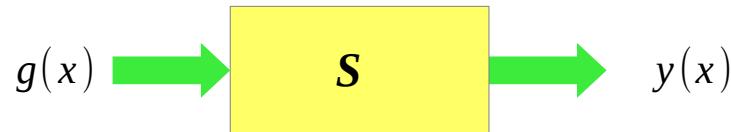
$$L(y) = 0 \quad \Rightarrow \quad y'' + 5y' + 2y = 0$$

$$D^2 + 5D + 2 = L$$

$$L(y) = g(x) \quad \Rightarrow \quad y'' + 5y' + 2y = 3x$$

# Linear System

*Linear System*



$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

$$(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x))(y(x)) = g(x)$$

$$L(y(x)) = g(x)$$

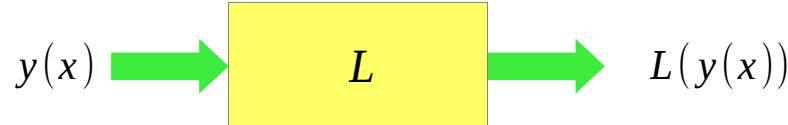
# Linear Differential Equations

$$\left\{ \begin{array}{l} a_n(x) \frac{d^n y}{d x^n} + a_{n-1}(x) \frac{d^{n-1} y}{d x^{n-1}} + \cdots + a_1(x) \frac{d y}{d x} + a_0(x) y = g(x) \\ a_n(x) \frac{d^n y}{d x^n} + a_{n-1}(x) \frac{d^{n-1} y}{d x^{n-1}} + \cdots + a_1(x) \frac{d y}{d x} + a_0(x) y = 0 \end{array} \right.$$

*Non-homogeneous Equation*

*Homogeneous Equation*

$$L = a_n(x) D^n + a_{n-1}(x) D^{n-1} + \cdots + a_1(x) D + a_0(x)$$



$$L(y) = a_n(x) \frac{d^n y}{d x^n} + a_{n-1}(x) \frac{d^{n-1} y}{d x^{n-1}} + \cdots + a_1(x) \frac{d y}{d x} + a_0(x) y$$

# Linear Differential Equations

## Linear Equation - Additivity

$$a_n(x) \frac{d^n y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_1}{dx} + a_0(x) y_1 = L(y_1)$$



$$a_n(x) \frac{d^n y_2}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_2}{dx} + a_0(x) y_2 = L(y_2)$$



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$$a_n(x) \frac{d^n(y_1+y_2)}{dx^n} + a_{n-1}(x) \frac{d^{n-1}(y_1+y_2)}{dx^{n-1}} + \cdots + a_1(x) \frac{d(y_1+y_2)}{dx} + a_0(x)(y_1+y_2) = L(y_1+y_2)$$



## Linear Equation - Homogeneity

---

$$a_n(x) \frac{d^n y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_1}{dx} + a_0(x) y_1 = L(y_1)$$



---

$$a_n(x) \frac{d^n k y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} k y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{d k y_1}{dx} + a_0(x) k y_1 = L(k y_1)$$



# Linear Differential Equation Solutions

## Linear Differential Equation Solution – Additivity

$$a_n(x) \frac{d^n y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_1}{dx} + a_0(x) y_1 = g_1(x)$$



$$a_n(x) \frac{d^n y_2}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_2}{dx} + a_0(x) y_2 = g_2(x)$$



---

$$a_n(x) \frac{d^n(y_1+y_2)}{dx^n} + a_{n-1}(x) \frac{d^{n-1}(y_1+y_2)}{dx^{n-1}} + \cdots + a_1(x) \frac{d(y_1+y_2)}{dx} + a_0(x)(y_1+y_2) = g_1(x) + g_2(x)$$

*Superposition*



## Linear Differential Equation Solution – Homogeneity

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$$a_n(x) \frac{d^n y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_1}{dx} + a_0(x) y_1 = g_1(x)$$



---

$$a_n(x) \frac{d^n k y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} k y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{d k y_1}{dx} + a_0(x) k y_1 = k g_1(x)$$



# Homogeneous Equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Associated  
Homogeneous Equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

Associated  
Homogeneous Equation  
with constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Auxiliary Equation

$$\downarrow \quad m = m_1, m_2, \dots, m_n$$

*n* solutions of the  
Auxiliary Equation

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

General Solutions of the  
Homogeneous Equation

# Non-homogeneous Equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$

Associated  
Homogeneous Equation

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0$$

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

Associated  
Homogeneous Equation  
with constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = 0$$

Particular Solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} + y_p(x)$$

General Solutions of the  
Non-homogeneous Equation

# Particular Solutions : Variation of Parameters

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$y_h(x) = c_1 e^{\textcolor{violet}{m}_1 x} + c_2 e^{\textcolor{violet}{m}_2 x} + \cdots + c_n e^{\textcolor{violet}{m}_n x}$$

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W$$

*k-th column*

$$\begin{vmatrix} y_1 & 0 & \cdots & y_n \\ y_1^{(1)} & 0 & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & g(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = W_k \quad u_k'(x) = \frac{W_k}{W}$$

$$y_p(x) = \textcolor{teal}{u}_1 e^{\textcolor{violet}{m}_1 x} + \textcolor{teal}{u}_2 e^{\textcolor{violet}{m}_2 x} + \cdots + \textcolor{teal}{u}_n e^{\textcolor{violet}{m}_n x}$$

## References

- [1] <http://en.wikipedia.org/>
- [2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
- [3] E. Kreyszig, "Advanced Engineering Mathematics"
- [4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"
- [5] [www.chem.arizona.edu/~salzmanr/480a](http://www.chem.arizona.edu/~salzmanr/480a)