Higher Order ODE's (3A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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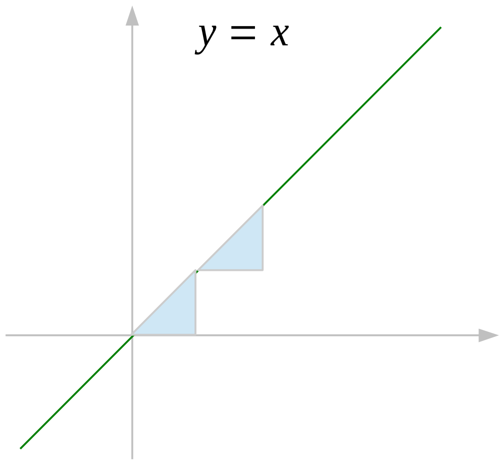
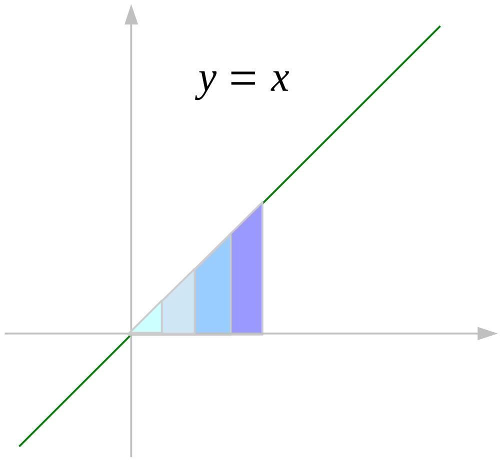
The Properties of a Line Equation (1)



$$f(1) = 1$$
$$f(2) = 2$$
$$f(3) = 3$$

$$f(2) = f(1+1) = f(1) + f(1) = 2$$
$$f(3) = f(2+1) = f(2) + f(1) = 3$$
$$f(4) = f(3+1) = f(3) + f(1) = 4$$

multiples of a unit

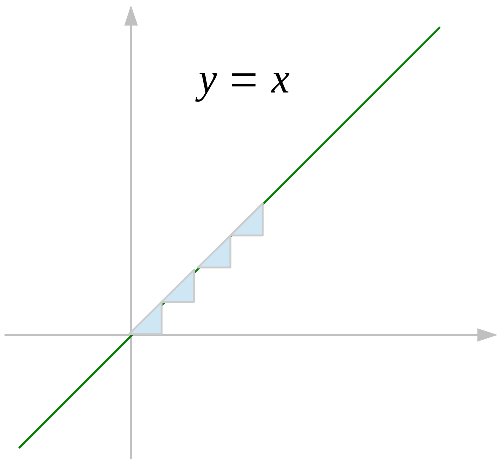


$$f(0.1) = 0.1$$
$$f(0.5) = 0.5$$
$$f(1.5) = 1.5$$

$$f(0.1) = f(0.1 \cdot 1) = 0.1 \cdot f(1) = 0.1$$
$$f(0.5) = f(0.5 \cdot 1) = 0.5 \cdot f(1) = 0.5$$
$$f(1.5) = f(1.5 \cdot 1) = 1.5 f(1) = 1.5$$

fractions of a unit

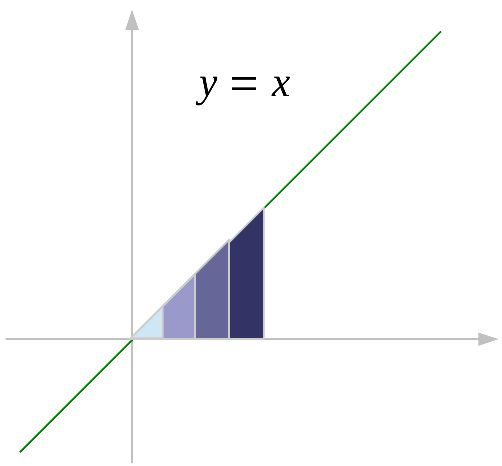
The Properties of a Line Equation (2)



multiples of a unit →

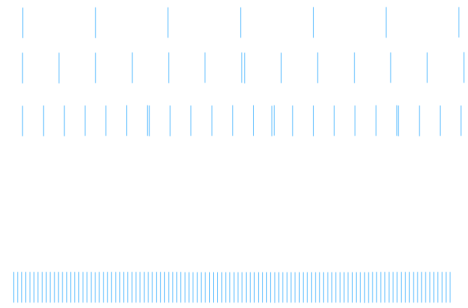
fractions of a unit ↓

⋮	⋮	⋮	⋮
$f(1) = 1$	$f(2) = 2$	$f(3) = 3$	$f(4) = 4$
⋮	⋮	⋮	⋮
$f(0.5) = 0.5$	$f(1.0) = 1.0$	$f(1.5) = 1.5$	$f(2.0) = 2.0$
⋮	⋮	⋮	⋮
$f(0.1) = 0.1$	$f(0.2) = 0.2$	$f(0.3) = 0.3$	$f(0.4) = 0.4$
⋮	⋮	⋮	⋮

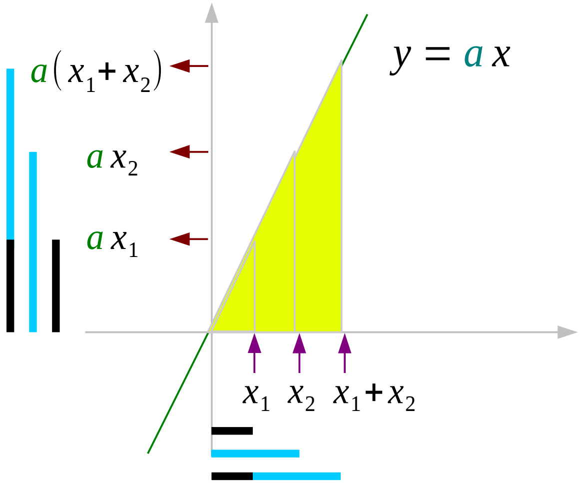


$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$f(kx) = k f(x)$$



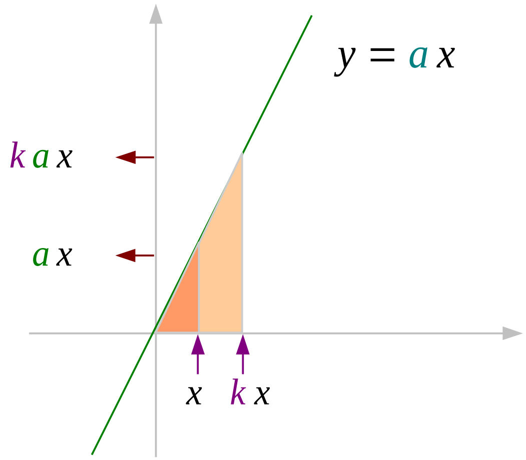
Linearity Property



Additivity

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$a \cdot (x_1 + x_2) = a \cdot (x_1) + a \cdot (x_2)$$



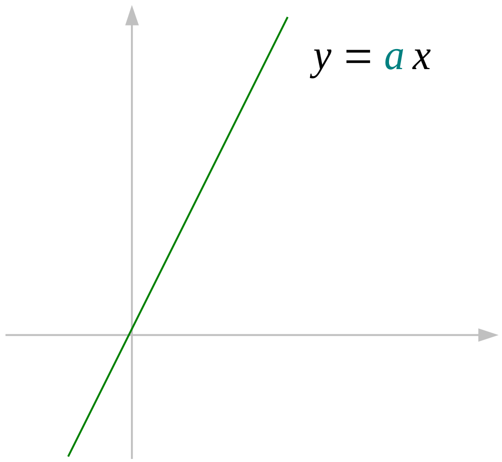
Homogeneity

$$f(k\,x) = k\,f(x)$$

$$a \cdot (k\,x) = k(a \cdot x)$$

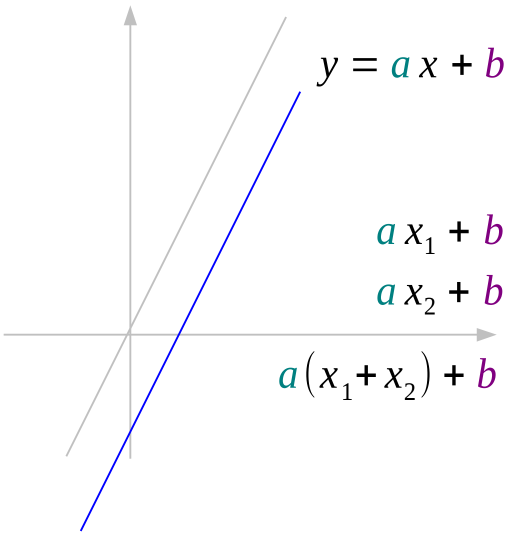
Linearity & Affinity

Linearity



Translation

Affinity



$a\,x_1 + b$

$a\,x_2 + b$

$a(x_1 + x_2) + b$

$a\,x_1 + b$

$a\,k\,x_1 + b$

$k(a\,x + b)$

from linear + -ity.

from the Latin, affinis, "connected with"

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$

$$f(k\,x) = k\,f(x)$$

Additivity

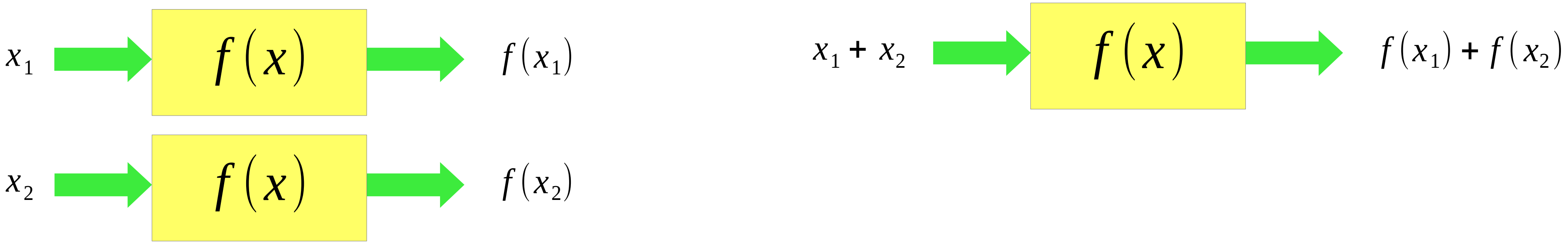
Homogeneity

~~***Additivity***~~

~~***Homogeneity***~~

Linear Map

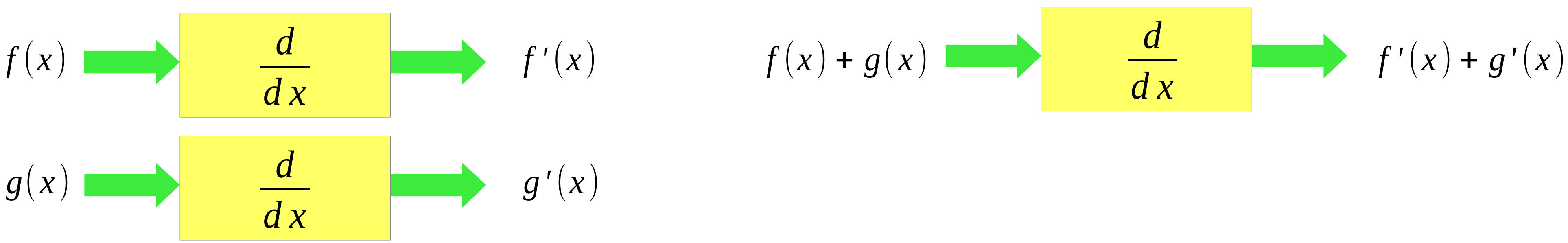
Additivity $f(x_1 + x_2) = f(x_1) + f(x_2)$



Homogeneity $f(kx) = kf(x)$

Linear Operators

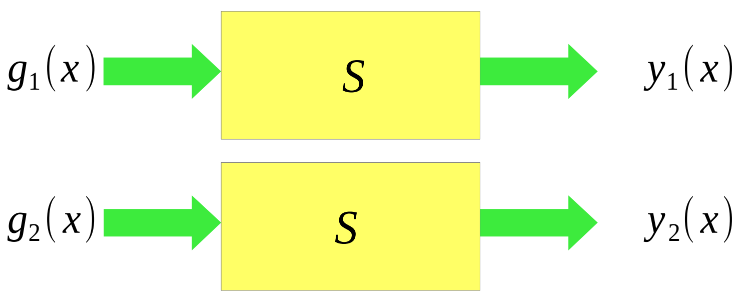
Additivity $$\frac{d}{dx}[f + g] = \frac{df}{dx} + \frac{dg}{dx}$$

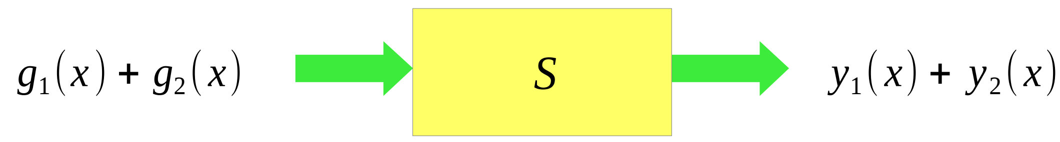


Homogeneity $$\frac{d}{dx}[k f] = k\frac{df}{dx}$$

Linear Systems

Additivity $\mathbf{S}\{g_1(x) + g_2(x)\} = \mathbf{S}\{g_1(x)\} + \mathbf{S}\{g_2(x)\}$



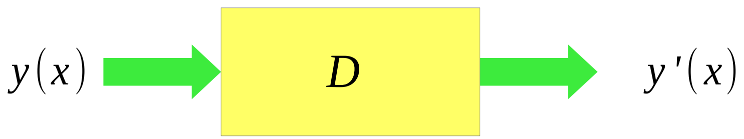


$$\frac{d^2 y}{d x^2} + a_1 \frac{d y}{d x} + a_2 y = g(x)$$

Homogeneity $\mathbf{S}\{k f(x)\} = k \mathbf{S}\{f(x)\}$

Differential Operator

Differential Operator



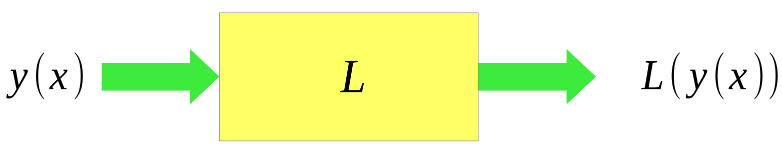
$$D = \frac{d}{dx}$$

$$D\,y = \frac{d\,y}{dx}$$

$$D(y) = \frac{d\,y}{dx}$$

$$D(y(x)) = \frac{d\,y}{dx}$$

N-th Order Differential Operator



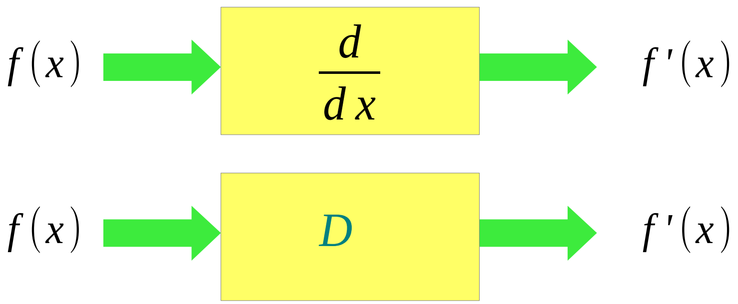
$$L \;=\; a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$

$$L(y) \;=\; \{a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\}(y)$$

$$L(y) \;=\; a_n(x)D^n(y) + a_{n-1}(x)D^{n-1}(y) + \cdots + a_1(x)D(y) + a_0(x)(y)$$

$$L(y) \;=\; a_n(x)\frac{d^n y}{d x^n} + a_{n-1}(x)\frac{d^{n-1} y}{d x^{n-1}} + \cdots + a_1(x)\frac{d\,y}{d\,x} + a_0(x)\,y$$

Examples



$$\frac{df}{dx} = f'(x)$$

$$Df = f'(x)$$

$$D(Df) = \frac{d}{dx}\left(\frac{df}{dx}\right) = f''(x)$$

$$D^2 f = \frac{d^2 f}{dx^2} = f''(x)$$

Differential Operator : Linear

$$D(c f(x)) = c D f(x)$$

$$D(f(x) + g(x)) = D f(x) + D g(x)$$

$$D(\alpha f(x) + \beta g(x)) = \alpha D f(x) + \beta D g(x)$$

n-th order Differential Operator

$$L = a_n(x) D^n + a_{n-1}(x) D^{n-1} + \cdots + a_1(x) D + a_0(x)$$

$$(D^2 + 2D + 1) f(x) = D^2 f(x) + 2 D f(x) + f(x) = f''(x) + 2f'(x) + f(x)$$

n-th order Differential Equations using the Differential Operator

$$y'' + 5y' + 2y = 3x$$

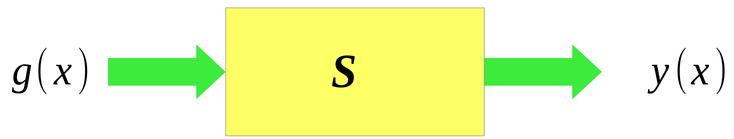
$$D^2 + 5D + 2 = L$$

$L(y) = 0$ ➡ $y'' + 5y' + 2y = 0$

$L(y) = g(x)$ ➡ $y'' + 5y' + 2y = 3x$

Linear System

Linear System



$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y \;=\; g(x)$$

$$\left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\right)(y(x)) \;=\; g(x)$$

$$L(y(x)) \;=\; g(x)$$

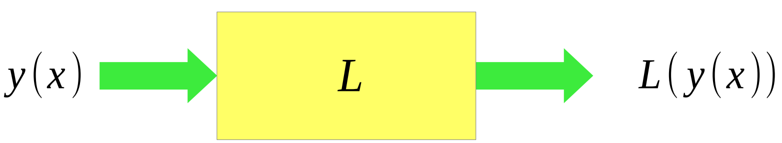
Linear Differential Equations

$$\begin{cases} a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = g(x) \\ a_n(x)\dfrac{d^n y}{dx^n} + a_{n-1}(x)\dfrac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\dfrac{dy}{dx} + a_0(x)y = 0 \end{cases}$$

Non-homogeneous Equation

Homogeneous Equation

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)$$



$$L(y) = a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y$$

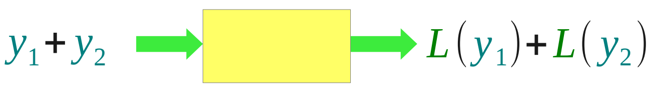
Linear Differential Equations

Linear Equation - Additivity

$$a_n(x)\frac{d^n y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{d y_1}{dx} + a_0(x) y_1 = L(y_1)$$

$$a_n(x)\frac{d^n y_2}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1(x)\frac{d y_2}{dx} + a_0(x) y_2 = L(y_2)$$

$$a_n(x)\frac{d^n (y_1+y_2)}{dx^n} + a_{n-1}(x)\frac{d^{n-1} (y_1+y_2)}{dx^{n-1}} + \cdots + a_1(x)\frac{d (y_1+y_2)}{dx} + a_0(x) (y_1+y_2) = L(y_1+y_2)$$



Linear Equation - Homogeneity

$$a_n(x)\frac{d^n y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{d y_1}{dx} + a_0(x) y_1 = L(y_1)$$

$$a_n(x)\frac{d^n k y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} k y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{d k y_1}{dx} + a_0(x) k y_1 = L(k y)$$

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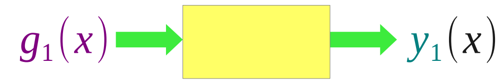
12/27/15

Linear Differential Equation Solutions

Linear Differential Equation Solution – Additivity

$$a_n(x)\frac{d^n y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{dy_1}{dx} + a_0(x)y_1 = g_1(x)$$



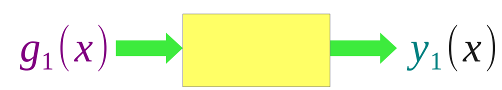
$$a_n(x)\frac{d^n y_2}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1(x)\frac{dy_2}{dx} + a_0(x)y_2 = g_2(x)$$

$$a_n(x)\frac{d^n (y_1+y_2)}{dx^n} + a_{n-1}(x)\frac{d^{n-1} (y_1+y_2)}{dx^{n-1}} + \cdots + a_1(x)\frac{d(y_1+y_2)}{dx} + a_0(x)(y_1+y_2) = g_1(x) + g_2(x)$$

Superposition

Linear Differential Equation Solution – Homogeneity

$$a_n(x)\frac{d^n y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{dy_1}{dx} + a_0(x)y_1 = g_1(x)$$



$$a_n(x)\frac{d^n k y_1}{dx^n} + a_{n-1}(x)\frac{d^{n-1} k y_1}{dx^{n-1}} + \cdots + a_1(x)\frac{dk y_1}{dx} + a_0(x)k y_1 = k g_1(x)$$

Homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Associated Homogeneous Equation

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = g(x)$$

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0$$

Associated Homogeneous Equation with constant coefficients

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Auxiliary Equation

$$m = m_1, m_2, \cdots, m_n$$

n solutions of the Auxiliary Equation

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

General Solutions of the Homogeneous Equation

Non-homogeneous Equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

Associated Homogeneous Equation

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = g(x)$$

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0$$

Associated Homogeneous Equation with constant coefficients

Particular Solution

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x} + y_p(x)$$

General Solutions of the Non-homogeneous Equation

Particular Solutions : Variation of Parameters

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$y_h(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

k-th column

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1^{(1)} & y_2^{(1)} & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = W \qquad \begin{vmatrix} y_1 & 0 & \cdots & y_n \\ y_1^{(1)} & 0 & \cdots & y_n^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & g(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = W_k \qquad u_k'(x) = \frac{W_k}{W}$$

$$y_p(x) = u_1 e^{m_1 x} + u_2 e^{m_2 x} + \cdots + u_n e^{m_n x}$$

References

[1] http://en.wikipedia.org/
[2] M.L. Boas, "Mathematical Methods in the Physical Sciences"
[3] E. Kreyszig, "Advanced Engineering Mathematics"
[4] D. G. Zill, W. S. Wright, "Advanced Engineering Mathematics"
[5] www.chem.arizona.edu/~salzmanr/480a