First Order ODEs

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$y' = xy$

- separable equation
- $\frac{dy}{dx} = xy$
- $\frac{dy}{y} = xdx$
- $ln|y| = \frac{1}{2}x^2 + c$
- $|y| = e^{x^2/2+c}$
- $y = \pm e^c e^{x^2/2}$
- $y = Ce^{x^2/2}$

$y' = \frac{xy}{y^2+1}$

- separable equation
- $\frac{y^2+1}{y} dy = x dx$
- $(y + \frac{1}{y}) dy = x dx$
- $\frac{1}{2} y^2 + ln|y| = \frac{1}{2} x^2 + c$
- $y^2 + 2ln|y| = x^2 + C$

$x^2y' = 1 - x + y^2 - x^2y^2$

- separable equation
- $x^2y' = (1-x^2)(1+y)$
- $\frac{y'}{1+y^2} = \frac{1-x^2}{x^2}$
- $arctan(y) = -\frac{1}{x} - x + c$
- $y = tan(-\frac{1}{x} - x + c)$

$$\frac{dy}{dx} = \frac{xy^2 - cos(x)sin(x)}{y(1-x^{2)}}$$

- exact equation
- $y(1-x^2)dy = (xy^2 - cos(x)sin(x))dx$
- $(xy^2 - cos(x)sin(x))dx - y(1-x^2)dy = 0$
- $\frac{\partial f}{\partial x} = xy^2 - cos(x)sin(x)$, $\frac{\partial f}{\partial y} = -y(1-x^2)$
- $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = 2xy$: exact equation !
- $f(x,y) = \int \frac{\partial f}{\partial y} dy = \int -y(1-x^2)dx = -\frac{1}{2}y^2(1-x^2) + g(x)$
- $\frac{\partial f}{\partial x} = xy^2 - cos(x)sin(x) = xy^2 + g'(x)$
- $g'(x) = -cos(x)sin(x)$

$$\frac{dy}{dx} = \frac{xy^2 - cos(x)sin(x)}{y(1-x^2)} \Rightarrow g'(x) = -cos(x)sin(x)$$

- method 1
- $g(x) = \int -cos(x)sin(x)dx$
- $h = cos(x), \;\; dh = -sin(x)dx$
- $g(x) = \int hdy = \frac{1}{2}h^2 + c = \frac{1}{2}cos^2(x) + c_1$
- method 2
- $g(x) = \int -\frac{1}{2}sin(2x)dx$ $\qquad 2cos(x)sin(x) = sin(x+x)$
- $g(x) = \frac{1}{4}cos(2x) + c_2$ $\qquad cos^2(x) - sin^2(x) = cos(x+x)$
- $g(x) = \frac{1}{4}(2cos^2(x) - 1) + c_2 = \frac{1}{2}cos^2(x) + (-\frac{1}{4} + c_2)$
- $g(x) = \frac{1}{2}cos^2(x) + c_3$

$\frac{dy}{dx} = \frac{xy^2 - cos(x)sin(x)}{y(1-x^2)} \Rightarrow f(x,y) = c$

- $g(x) = \frac{1}{2}cos^2(x) + c_3$
- $f(x,y) = \int \frac{\partial f}{\partial y} dy = \int -y(1-x^2)dx = -\frac{1}{2}y^2(1-x^2) + g(x)$
- $f(x,y) = -\frac{1}{2}y^2(1-x^2) + \frac{1}{2}cos^2(x) + c_3 = C$
- $\frac{1}{2}y^2(1-x^2) - \frac{1}{2}cos^2(x) = c$

$\frac{dy}{dx} = \frac{y-x}{y+x}$

- substitution $x = r\cos\theta, \qquad y = r\sin\theta$
- then $r = \sqrt{x^2+y^2}, \qquad \tan\theta = \frac{y}{x}, \qquad \theta = \tan^{-1}(\frac{y}{x})$
- $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta, \qquad = \cos\theta dr - r\sin\theta d\theta$
- $dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta, \qquad = \sin\theta dr + r\cos\theta d\theta$
- $\frac{dy}{dx} = \frac{y-x}{y+x} \qquad \Longrightarrow (y+x)dy = (y-x)dx$
- $r(\sin\theta + \cos\theta)dy = r(\sin\theta - \cos\theta)dx$
- $(\sin\theta + \cos\theta)(\sin\theta dr + r\cos\theta d\theta) = (\sin\theta - \cos\theta)(\cos\theta dr - r\sin\theta d\theta)$
- $(\sin^2\theta + \cos\theta\sin\theta)dr + r(\sin\theta\cos\theta + \cos^2\theta)d\theta =$
- $(\sin\theta\cos\theta - \cos^2\theta)dr + r(-\sin^2\theta + \cos\theta\sin\theta)d\theta$
- $(\sin^2\theta + \cos^2\theta)dr + r(\cos^2\theta + \sin^2\theta)d\theta = 0$
- $dr/r + d\theta = 0$

$\frac{dy}{dx} = \frac{y-x}{y+x} \Rightarrow dr/r + d\theta = 0$

- separable equation $\frac{dr}{r} + d\theta = 0$
- back substitute $r = \sqrt{x^2 + y^2}, \qquad \theta = tan^{-1}(\frac{y}{x})$
- into $ln|r| + \theta = 0$
- $ln|x^2 + y^2|^{1/2} + tan^{-1}(\frac{y}{x}) = c$
- $ln(x^2 + y^2) + 2tan^{-1}(\frac{y}{x}) = c$

Reference

[1] D. G. Zill and W. S. Wright , “Advanced Engineering Mathematics”, 4th ed.
[2] Jiˇ í Lebl, “ Notes on Diffy Qs“
[3]B E Shapiro, “Lecture Notes in Differential Equations “