

Laplace Transform (4B)

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Laplace Transform

Laplace Transform

$$\begin{aligned} F(s) &= \int_0^\infty f(t) e^{-st} dt & f(t) = 0 & t < 0 \\ &= \int_0^\infty \{f(t)e^{-xt}\} e^{-iyt} dt & s = x + iy & \{f(t)e^{-xt}\} = g(t) \quad \leftrightarrow \quad F(x, y) \\ &= F(x, y) & & \text{Fourier Transform} \end{aligned}$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} F(s) e^{st} ds$$

Inverse Laplace Transform (1)

Laplace Transform

$$\begin{aligned} F(s) &= \int_0^\infty f(t) e^{-st} dt \\ &= \int_0^\infty \{f(t) e^{-xt}\} e^{-yt} dt \end{aligned}$$

$\left\{ \begin{array}{l} f(t) \text{ continuous on } [0, \infty) \\ f(t) = 0 \text{ for } t < 0 \\ f(t) \text{ has exponential order } \alpha \\ f(t) \text{ piecewise continuous on } [0, \infty) \end{array} \right.$

→ $F(s)$ converges absolutely
for $\operatorname{Re}(s) = x > \alpha$

$$\int_0^\infty |f(t) e^{-st}| dt < \infty$$

$$(|e^{-st}| = |e^{-xt}| |e^{-yt}| = e^{-xt})$$

$$\int_0^\infty |f(t) e^{-st}| dt = \int_0^\infty |f(t)| e^{-xt} dt < \infty$$

$$\{f(t) e^{-xt}\} = g(\alpha)$$

absolutely integrable for $x > \alpha$

→ Use Fourier Inversion

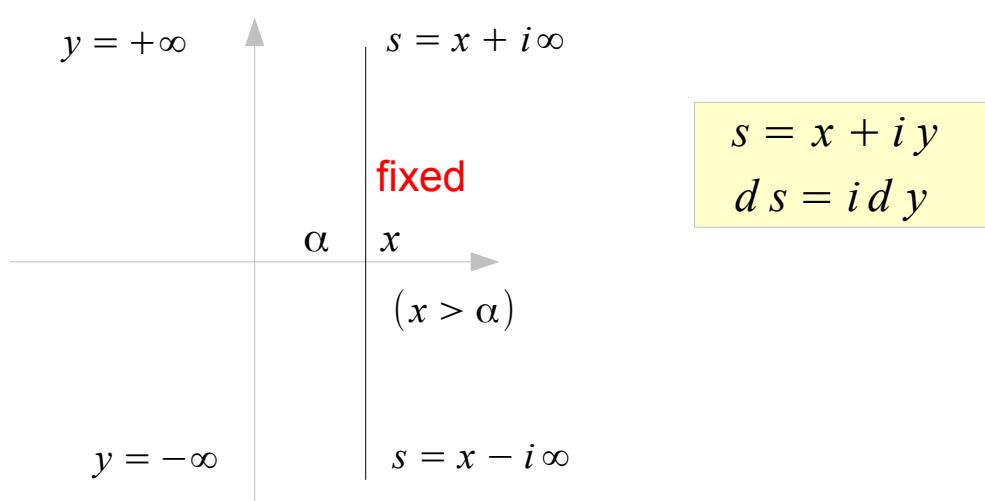
Inverse Laplace Transform (2)

$$g(t) = f(t)e^{-xt} \quad \text{absolutely integrable for } x > \alpha$$

$$F(x, y) = \int_0^{\infty} \{f(t)e^{-xt}\} e^{-iyt} dt$$

$$F(x, y) = \int_0^{\infty} g(t) e^{-iyt} dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$



$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{iyt} dy$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{xt} e^{iyt} dy$$

Inverse Fourier Transform

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{(x+iy)t} dy$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(s) e^{st} ds$$

$$= \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

Fourier-Mellin Inversion Formula

$$F(x, y) = \int_0^\infty \{f(t)e^{-xt}\} e^{-yt} dt$$

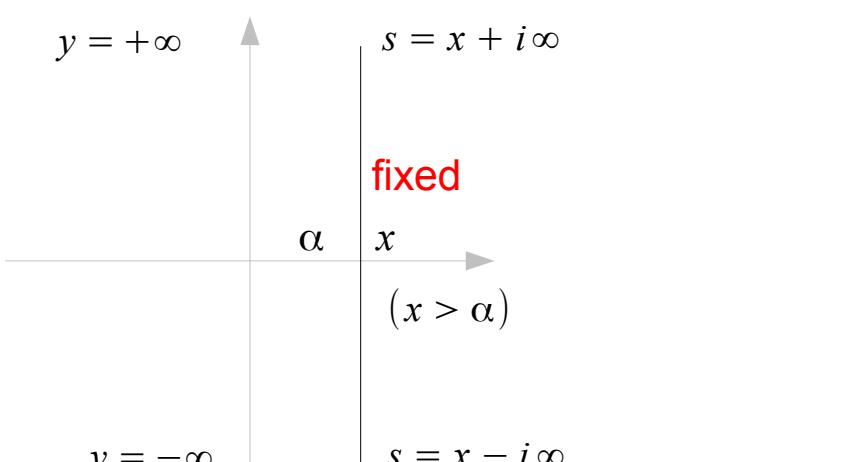
$$F(x, y) = \int_0^\infty g(t) e^{-yt} dt$$

Fourier Transform $g(t) = f(t)e^{-xt}$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{iyt} dy$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(x, y) e^{xt} e^{iyt} dy$$

Inverse Fourier Transform



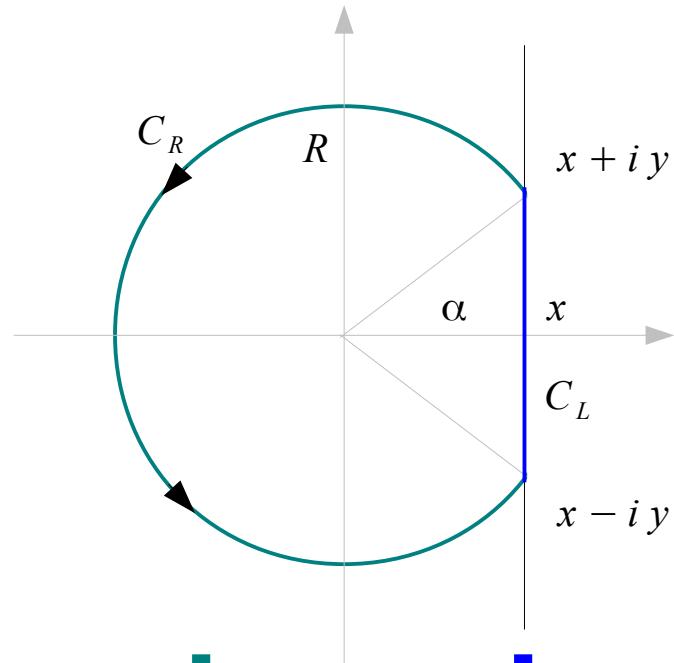
Vertical line at x : Bromwich line

$$\begin{aligned}s &= x + iy \\ ds &= idy\end{aligned}$$

$$\begin{aligned}f(t) &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(s) e^{st} ds \\ &= \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds\end{aligned}$$

Complex Inversion Formula
(Fourier-Mellin Inversion Formula)

Contour Integration (1)



$$\frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds$$

$R \rightarrow \infty$



0

$$\frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

$$\sum_{k=1}^n \text{Res}(z_k)$$

$$\frac{1}{2\pi i} \int_C F(s) e^{st} ds$$

$$= \frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{C_L} F(s) e^{st} ds$$

F(s) is analytic for $\text{Re}(s) = x > \alpha$

→ F(s) all singularities must lie to the left of Bromwich line

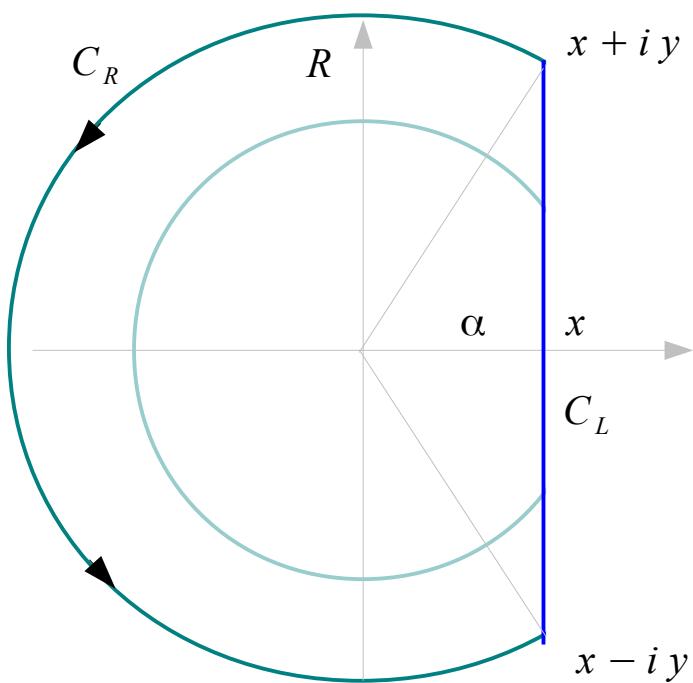
Assume F(s) is analytic for $\text{Re}(s) = x < \alpha$ except for having finitely many poles

z_1, z_2, \dots, z_n

$$\frac{1}{2\pi i} \int_C F(s) e^{st} ds = \sum_{k=1}^n \text{Res}(z_k)$$

$$= \frac{1}{2\pi i} \int_{C_R} F(s) e^{st} ds + \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds$$

Contour Integration (2)



$$\begin{aligned} \frac{1}{2\pi i} \int_C F(s) e^{st} ds &= \sum_{k=1}^n \text{Res}(z_k) \\ &= \frac{1}{2\pi i} \underbrace{\int_{C_R} F(s) e^{st} ds}_{\text{---}} + \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds \end{aligned}$$

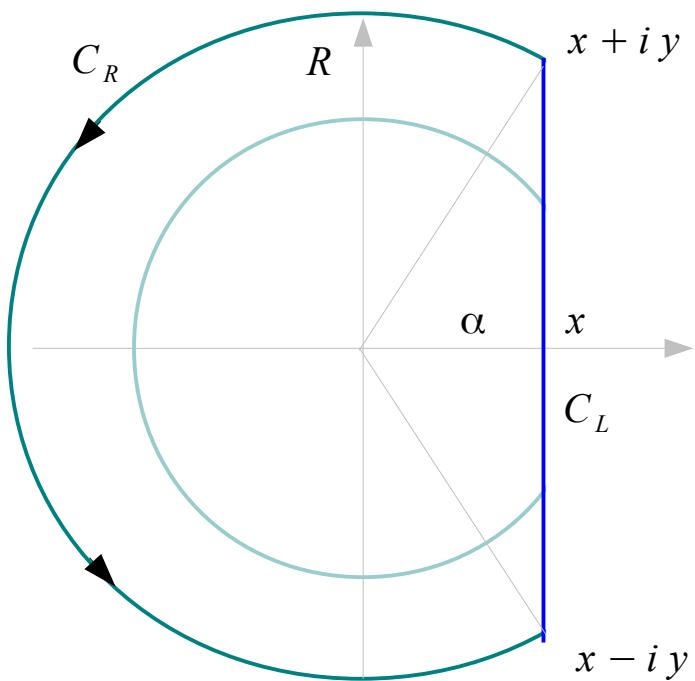
For s on C_R , some $p > 0$ all $R > R_0$

$$|F(s)| \leq \frac{M}{|s|^p}$$

→ $\lim_{R \rightarrow \infty} \underbrace{\int_{C_R} F(s) e^{st} ds}_{\text{---}} = 0 \quad (t > 0)$

→ $f(t) = \lim_{y \rightarrow \infty} \frac{1}{2\pi i} \int_{x-iy}^{x+iy} F(s) e^{st} ds = \sum_{k=1}^n \text{Res}(z_k)$

Contour Integration (3)



$$s = Re^{i\theta} \quad \theta_1 \leq \theta \leq \theta_2$$

$$ds = iRe^{i\theta} d\theta$$

$$|ds| = R d\theta$$

$$|F(s)| \leq \frac{M}{|s|^p} \quad \text{Growth Restriction}$$

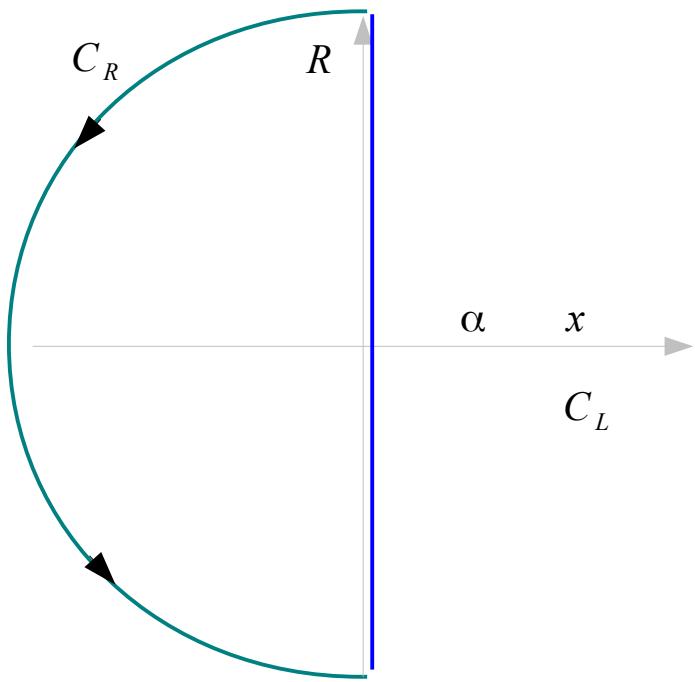
$$\rightarrow |F(s)| \rightarrow 0 \quad \text{as} \quad |s| \rightarrow \infty$$

for s on C_R

$$|F(s)| \leq \frac{M}{|s|^p} \quad \text{some } p > 0, \text{ all } R > R_0$$

$$\rightarrow \lim_{R \rightarrow \infty} \underbrace{\int_{C_R} F(s) e^{st} ds}_{=} = 0 \quad (t > 0)$$

Contour Integration (2)



$$\lim_{R \rightarrow \infty} \int_{C_R} F(s) e^{st} ds = 0 \quad (t > 0)$$

$$s = Re^{i\theta} = R(\cos\theta + i\sin\theta)$$

$$e^{st} = e^{Rt(\cos\theta + i\sin\theta)} = e^{Rt\cos\theta} e^{iRt\sin\theta}$$

$$|e^{st}| = e^{Rt\cos\theta}$$

$$\begin{aligned} \int_{C_R} F(s) e^{st} ds &\leq \int_{C_R} |F(s)| |e^{st}| |ds| \\ &\leq \frac{M}{R^{p-1}} \int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta} d\theta \end{aligned}$$

$$s = Re^{i\theta} \quad \theta_1 \leq \theta \leq \theta_2$$

$$ds = iRe^{i\theta} d\theta$$

$$|ds| = R d\theta$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann