

Hilbert Inner Product Space (2B)

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Trigonometric Identities

$$\cos \theta \cos \phi = \frac{1}{2} (\cos(\theta - \phi) + \cos(\theta + \phi))$$

$$\sin \theta \sin \phi = \frac{1}{2} (\cos(\theta - \phi) - \cos(\theta + \phi))$$

$$\sin \theta \cos \phi = \frac{1}{2} (\sin(\theta + \phi) + \sin(\theta - \phi))$$

$$\cos \theta \sin \phi = \frac{1}{2} (\sin(\theta + \phi) - \sin(\theta - \phi))$$

$$\frac{1}{2} (1 + \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (1 - \cos(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\frac{1}{2} (\sin(\theta + \phi)) \quad \text{when } \theta = \phi$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = \pi \quad (n = m)$$

n, m : integer

Trigonometric Orthogonality

$$f(x) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x) dx$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin kx dx$$

$$k = 1, 2, 3, \dots$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = 0 \quad (n \neq m)$$

$$\int_{-\pi}^{+\pi} \sin nx \cos mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \sin mx dx = 0$$

$$\int_{-\pi}^{+\pi} \cos nx \cos mx dx = \pi \quad (n = m)$$

$$\int_{-\pi}^{+\pi} \sin nx \sin mx dx = \pi \quad (n = m)$$

n, m : integer

$$a_k \leftarrow \underline{f(x) \cdot \cos kx} = a_0 \cdot \cos kx + \sum_{m=1}^{\infty} (a_m \underline{\cos mx \cdot \cos kx} + b_m \sin mx \cdot \cos kx)$$

$$b_k \leftarrow \underline{f(x) \cdot \sin kx} = a_0 \cdot \sin kx + \sum_{m=1}^{\infty} (a_m \cos mx \cdot \sin nx + b_m \underline{\sin mx \cdot \sin kx})$$

Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$$

complex conjugate

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

linear

$$\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$$

positive semidefinite

$$\langle x, x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Orthogonal Functions (1)

$$e^{j \textcolor{violet}{n} \omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+j k \omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-j k \omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

fundamental frequency $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{j \textcolor{red}{m} \omega_0 t}, e^{j \textcolor{violet}{n} \omega_0 t} \rangle = \int_0^T e^{+j(\textcolor{red}{m}-\textcolor{violet}{n})\omega_0 t} dt = \begin{cases} 0 & (\textcolor{red}{m} \neq \textcolor{violet}{n}) \\ T & (\textcolor{red}{m} = \textcolor{violet}{n}) \end{cases} \quad \textcolor{blue}{m, n : integer}$$

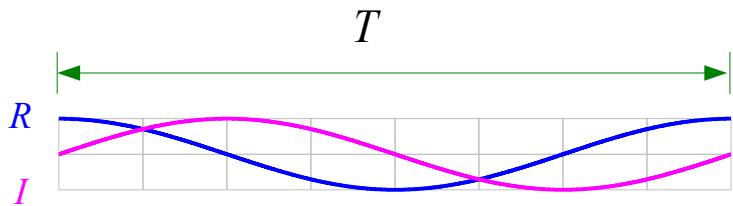
Orthogonal Functions (2)

$$e^{j \textcolor{green}{n} \omega_0 t}$$

$$\langle e^{j \textcolor{red}{m} \omega_0 t}, e^{j \textcolor{green}{n} \omega_0 t} \rangle = \int_0^T e^{+j(\textcolor{red}{m}-\textcolor{green}{n})\omega_0 t} dt = \begin{cases} 0 & (\textcolor{red}{m} \neq \textcolor{green}{n}) \\ T & (\textcolor{red}{m} = \textcolor{green}{n}) \end{cases} \quad \textcolor{blue}{m, n : \text{integer}}$$

$$\begin{aligned}
 e^{+j \textcolor{red}{m} \omega_0 t} \cdot e^{-j \textcolor{green}{n} \omega_0 t} &= (\cos \textcolor{red}{m} \omega_0 t + j \sin \textcolor{red}{m} \omega_0 t) \cdot (\cos \textcolor{green}{n} \omega_0 t - j \sin \textcolor{green}{n} \omega_0 t) \\
 &= \{\cos \textcolor{red}{m} \omega_0 t \cdot \cos \textcolor{green}{n} \omega_0 t + \sin \textcolor{red}{m} \omega_0 t \cdot \sin \textcolor{green}{n} \omega_0 t\} \\
 &\quad + j \{\sin \textcolor{red}{m} \omega_0 t \cdot \cos \textcolor{green}{n} \omega_0 t - \cos \textcolor{red}{m} \omega_0 t \sin \textcolor{green}{n} \omega_0 t\} \\
 &= \cos \{\textcolor{red}{m} \omega_0 t - \textcolor{green}{n} \omega_0 t\} + j \sin \{\textcolor{red}{m} \omega_0 t - \sin \textcolor{green}{n} \omega_0 t\} \\
 &= \frac{\cos \{(\textcolor{red}{m} - \textcolor{green}{n}) \omega_0 t\}}{1} + j \frac{\sin \{(\textcolor{red}{m} - \textcolor{green}{n}) \omega_0 t\}}{0} \quad (\textcolor{red}{m} = \textcolor{green}{n})
 \end{aligned}$$

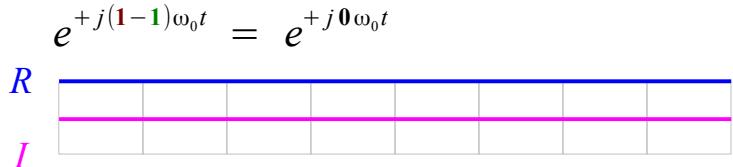
Inner Product Examples (1)



$$f_0 = 1/T$$

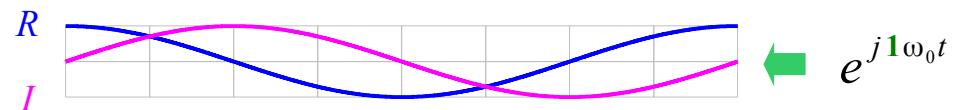
$$\omega_0 = 2\pi/T$$

$$\leftarrow e^{j \mathbf{1} \omega_0 t}$$



$$e^{+j(\mathbf{1}-\mathbf{1})\omega_0 t} = e^{+j\mathbf{0}\omega_0 t}$$

$$\leftarrow$$

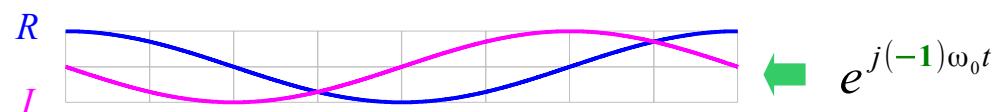


$$\leftarrow e^{j \mathbf{1} \omega_0 t}$$

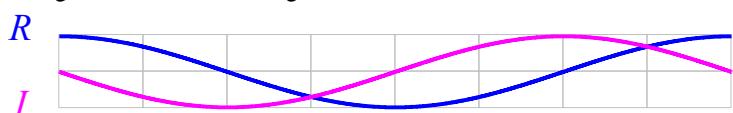


$$e^{+j(\mathbf{1}+\mathbf{1})\omega_0 t} = e^{+j\mathbf{2}\omega_0 t}$$

$$\leftarrow$$

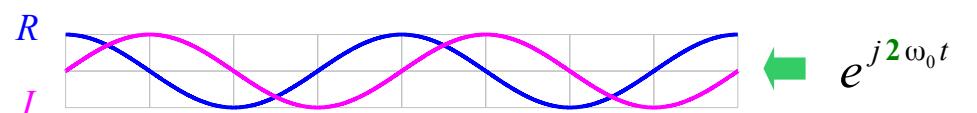


$$\leftarrow e^{j(-\mathbf{1})\omega_0 t}$$

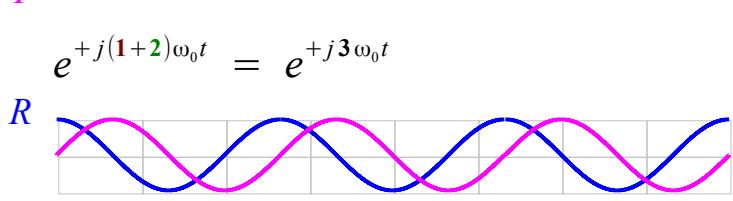


$$e^{+j(\mathbf{1}-\mathbf{2})\omega_0 t} = e^{+j(-\mathbf{1})\omega_0 t}$$

$$\leftarrow$$

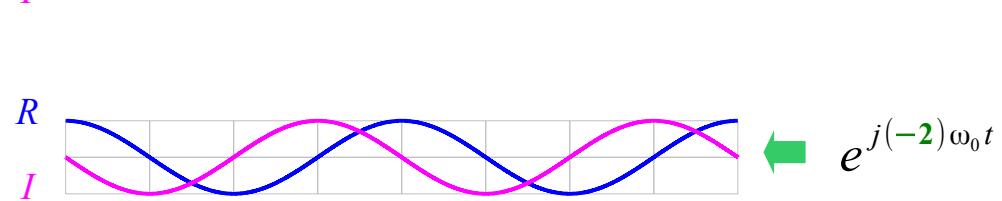


$$\leftarrow e^{j\mathbf{2}\omega_0 t}$$



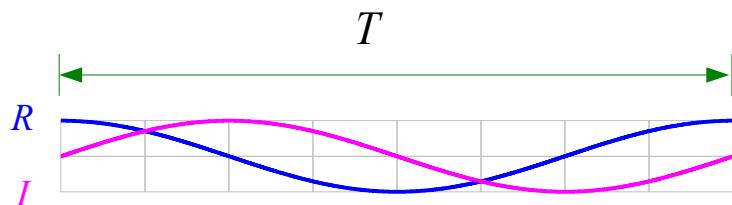
$$e^{+j(\mathbf{1}+\mathbf{2})\omega_0 t} = e^{+j\mathbf{3}\omega_0 t}$$

$$\leftarrow$$



$$\leftarrow e^{j(-\mathbf{2})\omega_0 t}$$

Inner Product Examples (2)



$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

$$\leftarrow e^{j \mathbf{1} \omega_0 t}$$

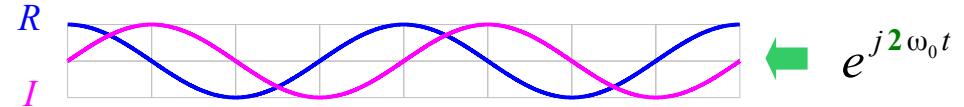
$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j \mathbf{1} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{1})\omega_0 t} dt = T \quad \leftarrow$$



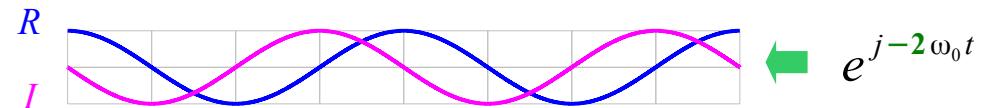
$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j -\mathbf{1} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{-1})\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j \mathbf{2} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{2})\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j -\mathbf{2} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{-2})\omega_0 t} dt = 0 \quad \leftarrow$$



Orthogonal Functions

$\cos \textcolor{violet}{n} \omega_0 t$

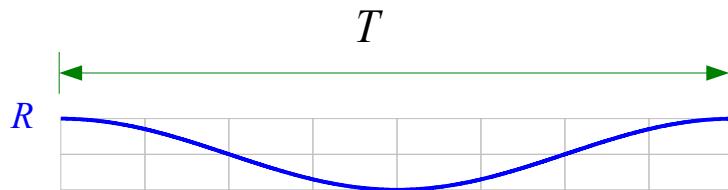
$$\langle \cos \textcolor{red}{m} \omega_0 t, \cos \textcolor{violet}{n} \omega_0 t \rangle = \int_0^T \cos \textcolor{red}{m} \omega_0 t \cdot \cos \textcolor{violet}{n} \omega_0 t \, dt = \begin{cases} 0 & (\textcolor{red}{m} \neq \textcolor{violet}{n}) \\ T/2 & (\textcolor{red}{m} = \textcolor{violet}{n}) \end{cases}$$

$m, n : \text{integer}$

$$\cos \textcolor{red}{m} \omega_0 t \cdot \cos \textcolor{violet}{n} \omega_0 t = \frac{1}{2} \left\{ \underbrace{\cos(\textcolor{red}{m}-\textcolor{violet}{n}) \omega_0 t + \cos(\textcolor{red}{m}+\textcolor{violet}{n}) \omega_0 t}_1 \right\}$$
$$(\textcolor{red}{m} = \pm \textcolor{violet}{n})$$

Inner Product Examples (1)

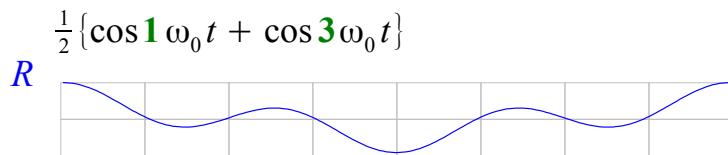
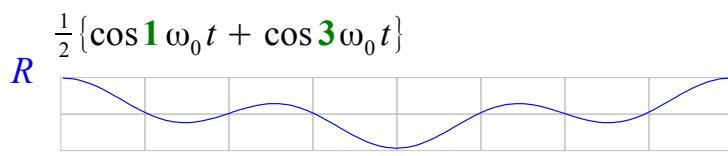
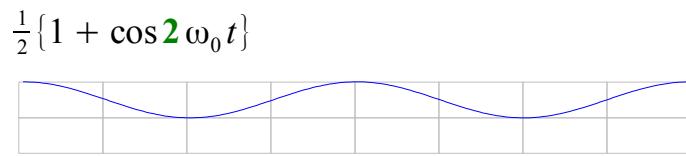
$\cos \textcolor{green}{n} \omega_0 t$



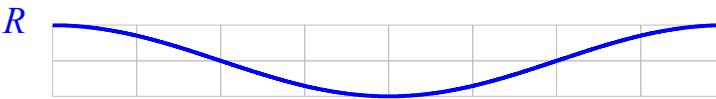
$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

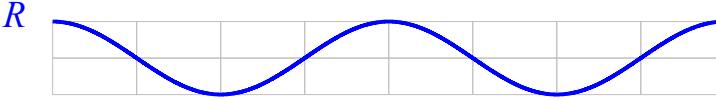
$\leftarrow \cos \textcolor{green}{1} \omega_0 t$



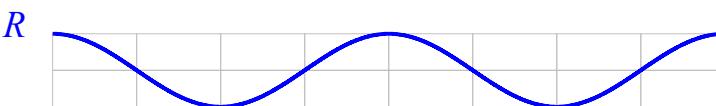
$\leftarrow \cos \textcolor{green}{1} \omega_0 t$



$\leftarrow \cos \textcolor{green}{1} \omega_0 t$



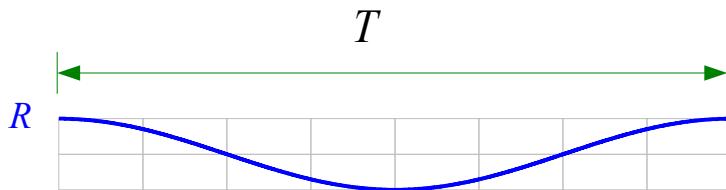
$\leftarrow \cos \textcolor{green}{2} \omega_0 t$



$\leftarrow \cos(-2) \omega_0 t$

Inner Product Examples (2)

$\cos \textcolor{violet}{n} \omega_0 t$



$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

$$\leftarrow \cos \textcolor{violet}{1} \omega_0 t$$

$$\langle \cos \textcolor{violet}{1} \omega_0 t, \cos \textcolor{violet}{1} \omega_0 t \rangle$$

$$\begin{aligned} &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{violet}{1}-\textcolor{violet}{1}) \omega_0 t + \cos(\textcolor{violet}{1}+\textcolor{violet}{1}) \omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ 1 + \cos 2\omega_0 t \} dt = \frac{T}{2} \end{aligned}$$



$$\langle \cos \textcolor{violet}{1} \omega_0 t, \cos(-\textcolor{violet}{1}) \omega_0 t \rangle$$

$$\begin{aligned} &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{violet}{1}+\textcolor{violet}{1}) \omega_0 t + \cos(\textcolor{violet}{1}-\textcolor{violet}{1}) \omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ 1 + \cos 2\omega_0 t \} dt = \frac{T}{2} \end{aligned}$$



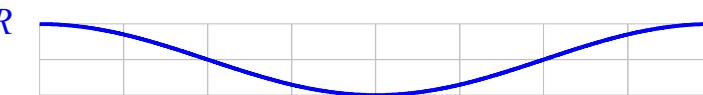
$$\langle \cos \textcolor{violet}{1} \omega_0 t, \cos \textcolor{violet}{2} \omega_0 t \rangle$$

$$\begin{aligned} &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{violet}{1}-\textcolor{violet}{2}) \omega_0 t + \cos(\textcolor{violet}{1}+\textcolor{violet}{2}) \omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ \cos 1\omega_0 t + \cos 3\omega_0 t \} dt = 0 \end{aligned}$$

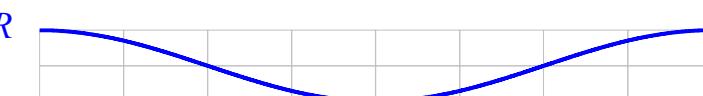


$$\langle \cos \textcolor{violet}{1} \omega_0 t, \cos(-\textcolor{violet}{2}) \omega_0 t \rangle$$

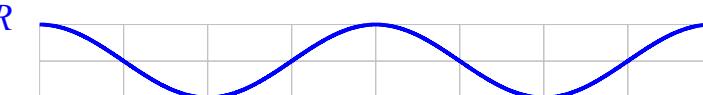
$$\begin{aligned} &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{violet}{1}+\textcolor{violet}{2}) \omega_0 t + \cos(\textcolor{violet}{1}-\textcolor{violet}{2}) \omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ \cos 1\omega_0 t + \cos 3\omega_0 t \} dt = 0 \end{aligned}$$



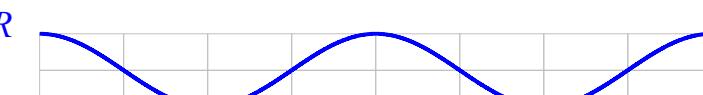
$$\leftarrow \cos \textcolor{violet}{1} \omega_0 t$$



$$\leftarrow \cos(-\textcolor{violet}{1}) \omega_0 t$$



$$\leftarrow \cos \textcolor{violet}{2} \omega_0 t$$



$$\leftarrow \cos(-\textcolor{violet}{2}) \omega_0 t$$

Orthogonal Functions

$\sin \textcolor{blue}{n} \omega_0 t$

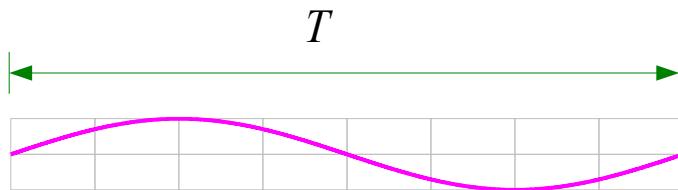
$$\langle \sin \textcolor{red}{m} \omega_0 t, \sin \textcolor{blue}{n} \omega_0 t \rangle = \int_0^T \sin \textcolor{red}{m} \omega_0 t \cdot \sin \textcolor{blue}{n} \omega_0 t \, dt = \begin{cases} 0 & (\textcolor{red}{m} \neq \textcolor{blue}{n}) \\ T/2 & (\textcolor{red}{m} = \textcolor{blue}{n}) \end{cases}$$

$\textcolor{blue}{m}, \textcolor{blue}{n} : \text{integer}$

$$\begin{aligned} \sin \textcolor{red}{m} \omega_0 t \cdot \sin \textcolor{blue}{n} \omega_0 t &= \frac{1}{2} \left\{ \underbrace{\cos(\textcolor{red}{m}-\textcolor{blue}{n}) \omega_0 t - \cos(\textcolor{red}{m}+\textcolor{blue}{n}) \omega_0 t}_{1} \right\} \\ &\quad (\textcolor{red}{m} = \pm \textcolor{blue}{n}) \end{aligned}$$

Inner Product Examples (1)

$\sin \textcolor{green}{n} \omega_0 t$



$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

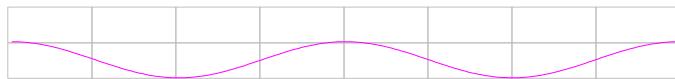
$\leftarrow \sin \textcolor{green}{1} \omega_0 t$

$$\frac{1}{2} \{ 1 - \cos 2\omega_0 t \}$$



$\leftarrow \sin \textcolor{green}{1} \omega_0 t$

$$\frac{1}{2} \{ \cos 2\omega_0 t - 1 \}$$



$\leftarrow \sin(-\textcolor{green}{1}) \omega_0 t$

$$\frac{1}{2} \{ \cos \textcolor{green}{1} \omega_0 t - \cos 3\omega_0 t \}$$



$\leftarrow \sin \textcolor{green}{2} \omega_0 t$

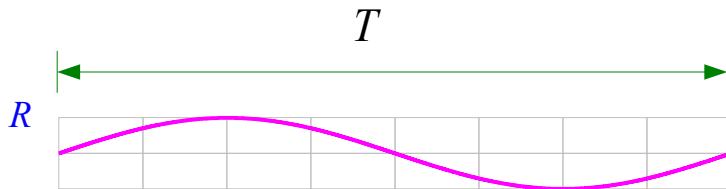
$$\frac{1}{2} \{ \cos 3\omega_0 t - \cos \textcolor{red}{1} \omega_0 t \}$$



$\leftarrow \sin(-\textcolor{green}{2}) \omega_0 t$

Inner Product Examples (2)

$\sin \textcolor{green}{n} \omega_0 t$



$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

← $\sin \textcolor{green}{1} \omega_0 t$

$$\begin{aligned} & \langle \sin \textcolor{red}{1} \omega_0 t, \sin \textcolor{green}{1} \omega_0 t \rangle \\ &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{red}{1}-\textcolor{green}{1})\omega_0 t - \cos(\textcolor{red}{1}+\textcolor{green}{1})\omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ 1 - \cos 2\omega_0 t \} dt = \frac{T}{2} \end{aligned}$$

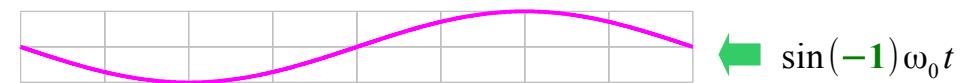
$$\begin{aligned} & \langle \sin \textcolor{red}{1} \omega_0 t, \sin(-\textcolor{green}{1})\omega_0 t \rangle \\ &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{red}{1}+\textcolor{green}{1})\omega_0 t - \cos(\textcolor{red}{1}-\textcolor{green}{1})\omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ \cos 2\omega_0 t - 1 \} dt = -\frac{T}{2} \end{aligned}$$

$$\begin{aligned} & \langle \sin \textcolor{red}{1} \omega_0 t, \sin \textcolor{green}{2} \omega_0 t \rangle \\ &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{red}{1}-\textcolor{green}{2})\omega_0 t - \cos(\textcolor{red}{1}+\textcolor{green}{2})\omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ \cos \textcolor{red}{1} \omega_0 t - \cos 3\omega_0 t \} dt = 0 \end{aligned}$$

$$\begin{aligned} & \langle \sin \textcolor{red}{1} \omega_0 t, \sin(-\textcolor{green}{2})\omega_0 t \rangle \\ &= \int_0^T \frac{1}{2} \{ \cos(\textcolor{red}{1}+\textcolor{green}{2})\omega_0 t - \cos(\textcolor{red}{1}-\textcolor{green}{2})\omega_0 t \} dt \\ &= \int_0^T \frac{1}{2} \{ \cos 3\omega_0 t - \cos \textcolor{red}{1} \omega_0 t \} dt = 0 \end{aligned}$$



← $\sin \textcolor{green}{1} \omega_0 t$



← $\sin(-\textcolor{green}{1})\omega_0 t$



← $\sin \textcolor{green}{2} \omega_0 t$



← $\sin(-\textcolor{green}{2})\omega_0 t$

Infinite Set of Orthogonal Functions

$$\{1, \cos \omega_0 t, \cos 2\omega_0 t, \dots, \cos n\omega_0 t, \dots\} \rightarrow g(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega_0 t$$

Linear Combination: even function only

$$\{\sin \omega_0 t, \sin 2\omega_0 t, \dots, \sin n\omega_0 t, \dots\} \rightarrow h(t) = \sum_{n=1}^{+\infty} b_n \sin n\omega_0 t$$

Linear Combination: odd function only

$$\langle \cos m\omega_0 t, \cos n\omega_0 t \rangle = \int_0^T \cos m\omega_0 t \cdot \cos n\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0, n > 0$: integer

$$\langle \sin m\omega_0 t, \sin n\omega_0 t \rangle = \int_0^T \sin m\omega_0 t \cdot \sin n\omega_0 t \, dt = \begin{cases} 0 & (m \neq n) \\ T/2 & (m = n) \end{cases}$$

$m > 0, n > 0$: integer

Completeness (1)

$$\{1, \cos \omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \dots, \sin \omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \dots\}$$

Linear Combination: 

$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t + \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

Can be even / odd function

When the collection of orthogonal functions are **complete**,
every function can be expanded via a linear combination of these infinite set of functions

$$\langle \cos \mathbf{m} \omega_0 t, \cos \mathbf{n} \omega_0 t \rangle = \int_0^T \cos \mathbf{m} \omega_0 t \cdot \cos \mathbf{n} \omega_0 t \, dt = \begin{cases} 0 & (\mathbf{m} \neq \mathbf{n}) \\ T/2 & (\mathbf{m} = \mathbf{n}) \end{cases}$$

$\mathbf{m}, \mathbf{n} > \mathbf{0}$: integer

$$\langle \sin \mathbf{m} \omega_0 t, \sin \mathbf{n} \omega_0 t \rangle = \int_0^T \sin \mathbf{m} \omega_0 t \cdot \sin \mathbf{n} \omega_0 t \, dt = \begin{cases} 0 & (\mathbf{m} \neq \mathbf{n}) \\ T/2 & (\mathbf{m} = \mathbf{n}) \end{cases}$$

$\mathbf{m}, \mathbf{n} > \mathbf{0}$: integer

Completeness (2)

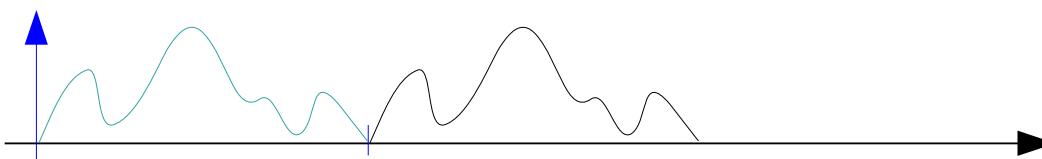
$$\{1, \cos \omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \dots, \sin \omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \dots\}$$

Linear Combination: 

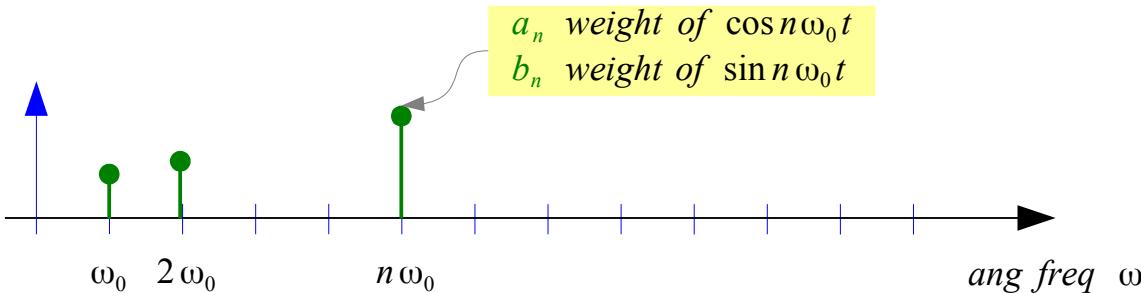
$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{+\infty} b_n \sin n\omega_0 t$$

*even / odd function
periodic function*

periodic continuous time function $f(t)$



$$T = \frac{2\pi}{\omega_0}$$

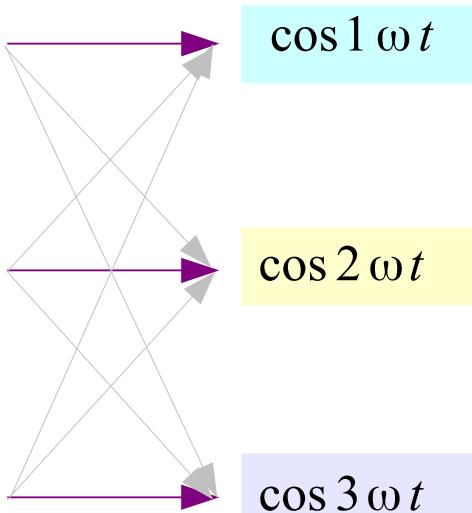


infinite but discrete set of angular frequency

Completeness (3)

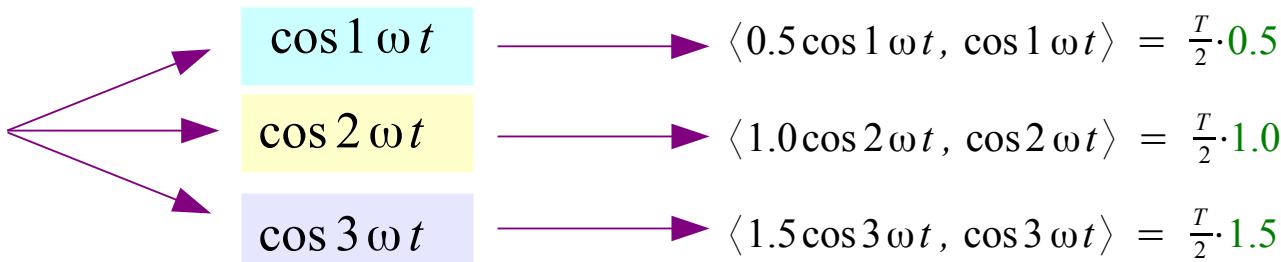
$$f(t) = \sum_{n=0}^{+\infty} a_n \cos n\omega t + \sum_{n=1}^{+\infty} b_n \sin n\omega t$$

$$0.5 \cos 1\omega t \\ + \\ 1.0 \cos 2\omega t \\ + \\ 1.5 \cos 3\omega t$$



$$||$$

$$f(t)$$



$$a_1 = 0.5, \quad a_2 = 1.0, \quad a_3 = 1.5$$

$$a_4 = a_5 = \dots = 0 \quad b_1 = b_2 = \dots = 0$$

$$\langle 0.5 \cos 1\omega t, \cos 1\omega t \rangle = \frac{T}{2} \cdot 0.5$$

$$\langle 1.0 \cos 2\omega t, \cos 1\omega t \rangle = 0$$

$$\langle 1.5 \cos 3\omega t, \cos 1\omega t \rangle = 0$$

$$\langle 0.5 \cos 1\omega t, \cos 2\omega t \rangle = 0$$

$$\langle 1.0 \cos 2\omega t, \cos 2\omega t \rangle = \frac{T}{2} \cdot 1.0$$

$$\langle 1.5 \cos 3\omega t, \cos 2\omega t \rangle = 0$$

$$\langle 0.5 \cos 1\omega t, \cos 3\omega t \rangle = 0$$

$$\langle 1.0 \cos 2\omega t, \cos 3\omega t \rangle = 0$$

$$\langle 1.5 \cos 3\omega t, \cos 3\omega t \rangle = \frac{T}{2} \cdot 1.5$$

Hilbert Space

Complete Orthogonal System

$$\{ \dots, e^{-jn\omega_0 t}, \dots, e^{-j\omega_0 t}, 1, e^{+j\omega_0 t}, \dots, e^{+jn\omega_0 t}, \dots \}$$

$$\langle e^{j\mathbf{m}\omega_0 t}, e^{j\mathbf{n}\omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{m}-\mathbf{n})\omega_0 t} dt = \begin{cases} 0 & (\mathbf{m} \neq \mathbf{n}) \\ T & (\mathbf{m} = \mathbf{n}) \end{cases} \quad \mathbf{m}, \mathbf{n} : \text{integer}$$

Complete BiOrthogonal System

$$\{1, \cos\omega_0 t, \cos 2\omega_0 t, \cos 3\omega_0 t, \dots, \sin\omega_0 t, \sin 2\omega_0 t, \sin 3\omega_0 t, \dots\}$$

$$\langle \cos \mathbf{m} \omega_0 t, \cos \mathbf{n} \omega_0 t \rangle = \int_0^T \cos \mathbf{m} \omega_0 t \cdot \cos \mathbf{n} \omega_0 t dt = \begin{cases} 0 & (\mathbf{m} \neq \mathbf{n}) \\ T/2 & (\mathbf{m} = \mathbf{n}) \end{cases} \quad \mathbf{m}, \mathbf{n} > 0 : \text{integer}$$

$$\langle \sin \mathbf{m} \omega_0 t, \sin \mathbf{n} \omega_0 t \rangle = \int_0^T \sin \mathbf{m} \omega_0 t \cdot \sin \mathbf{n} \omega_0 t dt = \begin{cases} 0 & (\mathbf{m} \neq \mathbf{n}) \\ T/2 & (\mathbf{m} = \mathbf{n}) \end{cases} \quad \mathbf{m}, \mathbf{n} > 0 : \text{integer}$$

Cauchy-Schwartz Inequality

For all vectors \mathbf{x} and \mathbf{y} of an inner product space

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent  maximum

$$\left| \int_a^b \mathbf{x}(t) \overline{\mathbf{y}(t)} dt \right| \leq \sqrt{\int_a^b \mathbf{x}(t) \overline{\mathbf{x}(t)} dt} \sqrt{\int_a^b \mathbf{y}(t) \overline{\mathbf{y}(t)} dt}$$

Inner product is maximum
when $\mathbf{y} = k \mathbf{x}$

Orthogonality

$$\begin{matrix}
 & r_0^* & r_1^* & r_2^* & r_3^* & r_4^* & r_5^* & r_6^* & r_7^*
 \end{matrix} = \left[\begin{array}{cccccccc}
 r_0 \cdot r_0^* & r_0 \cdot r_1^* & r_0 \cdot r_2^* & r_0 \cdot r_3^* & r_0 \cdot r_4^* & r_0 \cdot r_5^* & r_0 \cdot r_6^* & r_0 \cdot r_7^* \\
 r_1 \cdot r_0^* & r_1 \cdot r_1^* & r_1 \cdot r_2^* & r_1 \cdot r_3^* & r_1 \cdot r_4^* & r_1 \cdot r_5^* & r_1 \cdot r_6^* & r_1 \cdot r_7^* \\
 r_2 \cdot r_0^* & r_2 \cdot r_1^* & r_2 \cdot r_2^* & r_2 \cdot r_3^* & r_2 \cdot r_4^* & r_2 \cdot r_5^* & r_2 \cdot r_6^* & r_2 \cdot r_7^* \\
 r_3 \cdot r_0^* & r_3 \cdot r_1^* & r_3 \cdot r_2^* & r_3 \cdot r_3^* & r_3 \cdot r_4^* & r_3 \cdot r_5^* & r_3 \cdot r_6^* & r_3 \cdot r_7^* \\
 r_4 \cdot r_0^* & r_4 \cdot r_1^* & r_4 \cdot r_2^* & r_4 \cdot r_3^* & r_4 \cdot r_4^* & r_4 \cdot r_5^* & r_4 \cdot r_6^* & r_4 \cdot r_7^* \\
 r_5 \cdot r_0^* & r_5 \cdot r_1^* & r_5 \cdot r_2^* & r_5 \cdot r_3^* & r_5 \cdot r_4^* & r_5 \cdot r_5^* & r_5 \cdot r_6^* & r_5 \cdot r_7^* \\
 r_6 \cdot r_0^* & r_6 \cdot r_1^* & r_6 \cdot r_2^* & r_6 \cdot r_3^* & r_6 \cdot r_4^* & r_6 \cdot r_5^* & r_6 \cdot r_6^* & r_6 \cdot r_7^* \\
 r_7 \cdot r_0^* & r_7 \cdot r_1^* & r_7 \cdot r_2^* & r_7 \cdot r_3^* & r_7 \cdot r_4^* & r_7 \cdot r_5^* & r_7 \cdot r_6^* & r_7 \cdot r_7^*
 \end{array} \right]$$

$$\begin{aligned}
 \langle \mathbf{r}_i^H, \mathbf{r}_i^H \rangle &= \mathbf{r}_i \cdot \mathbf{r}_i^* = N \\
 \langle \mathbf{r}_i^H, \mathbf{r}_j^H \rangle &= \mathbf{r}_i \cdot \mathbf{r}_j^* = 0 \quad (i \neq j)
 \end{aligned}$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} = \sum x_i^* y_i \quad \mathbf{x}^H : \text{conjugate transpose}$$

Norm of Hermitian inner products

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^H \cdot \mathbf{x}} = \sqrt{\sum x_i^* x_i} \quad \text{the length of a vector}$$

$$\mathbf{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \quad \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \cdot \mathbf{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^n a_i^2 + b_i^2$$

The 1st Row of the DFT Matrix



$$W_8^{k n} = e^{-j(\frac{2\pi}{8})kn} \quad k = 0, \quad n = 0, 1, \dots, 7$$

$$\begin{aligned} R &\rightarrow \text{samples of } \cos(-\omega t) = \cos(\omega t) \\ I &\rightarrow \text{samples of } \sin(-\omega t) = -\sin(\omega t) \end{aligned} \quad \underbrace{\qquad}_{\text{measure}} \quad \begin{aligned} \omega t &= 2\pi f t \\ &2\pi \cdot (\frac{0}{8}) \cdot f_s \cdot t \end{aligned}$$

$X[0]$ measures how much of the $+0 \cdot \omega$ component is present in \mathbf{x} .

The 3rd Row of the DFT Matrix

R

2 cycles

I

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$$\begin{aligned} R &\rightarrow \text{samples of } \cos(-2\omega t) = \cos(2\omega t) \\ I &\rightarrow \text{samples of } \sin(-2\omega t) = -\sin(2\omega t) \end{aligned}$$

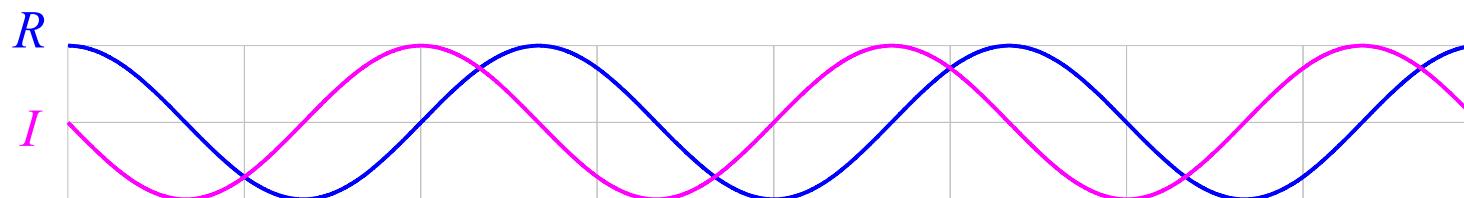
measure

$$\omega t = 2\pi f t$$

$$2\pi \cdot (\frac{2}{8}) \cdot f_s \cdot t$$

$X[2]$ measures how much of the $+2\cdot\omega$ component is present in \mathbf{x} .

The 4th Row of the DFT Matrix



3 cycles

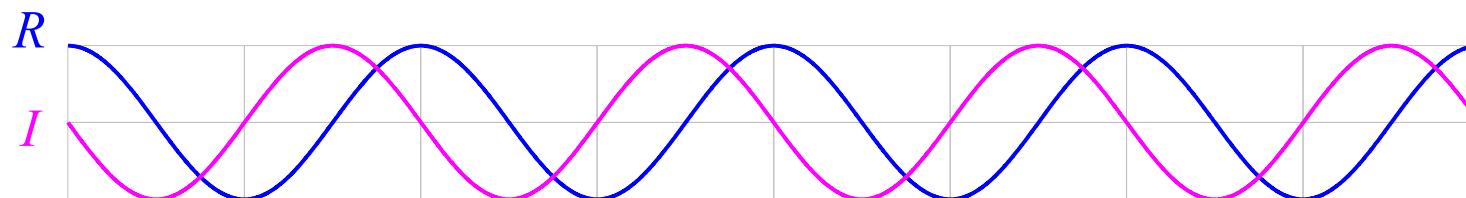
$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 3, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-3\omega t) = \cos(3\omega t)$
I \rightarrow samples of $\sin(-3\omega t) = -\sin(3\omega t)$

$$\left. \begin{aligned} \omega t &= 2\pi ft \\ &2\pi \cdot (\frac{3}{8}) \cdot f_s \cdot t \end{aligned} \right\} \text{measure}$$

X[3] measures how much of the $+3\cdot\omega$ component is present in *x*.

The 5th Row of the DFT Matrix



4 cycles

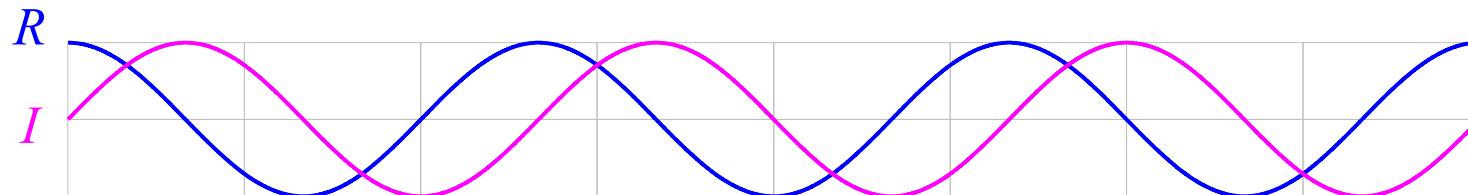
$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 4, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-4\omega t) = \cos(4\omega t)$
I \rightarrow samples of $\sin(-4\omega t) = -\sin(4\omega t)$

$$\left. \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot (\frac{4}{8}) \cdot f_s \cdot t \end{array} \right\} \text{measure}$$

$X[4]$ measures how much of the $+4 \cdot \omega$ component is present in \mathbf{x} .

The 6th Row of the DFT Matrix



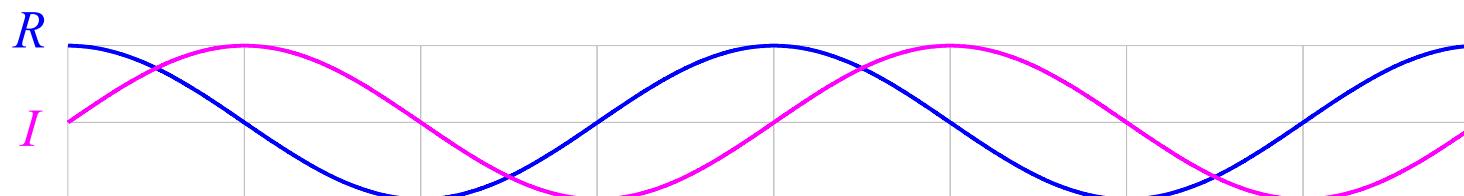
3 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 5, \quad n = 0, 1, \dots, 7$$

R → samples of $\cos(-(-3\omega)t) = \cos(3\omega t)$ } measure $-\omega t = -2\pi f t$
I → samples of $\sin(-(-3\omega)t) = \sin(3\omega t)$ } $2\pi \cdot (\frac{-3}{8}) \cdot f_s \cdot t$

$X[5]$ measures how much of the $-3\cdot\omega$ component is present in \mathbf{x} .

The 7th Row of the DFT Matrix



2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

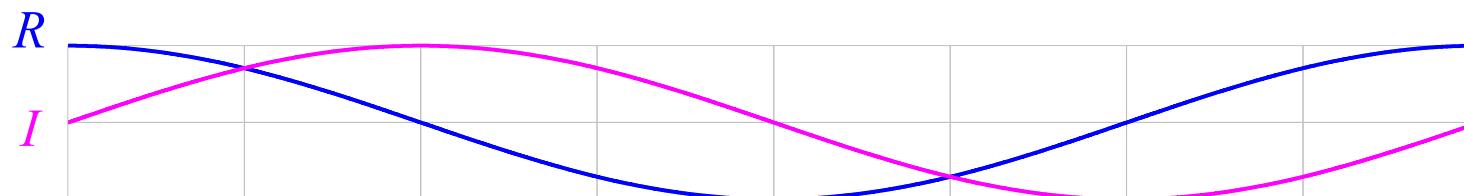
R \rightarrow samples of $\cos(-(-2\omega)t) = \cos(2\omega t)$

I \rightarrow samples of $\sin(-(-2\omega)t) = \sin(2\omega t)$

$\left. \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot (\frac{-2}{8}) \cdot f_s \cdot t \end{array} \right\} \text{measure}$

$X[6]$ measures how much of the $-2\cdot\omega$ component is present in \mathbf{x} .

The 8th Row of the DFT Matrix



$$W_8^{k n} = e^{-j(\frac{2\pi}{8})kn} \quad k = 7, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-(-\omega)t) = \cos(\omega t)$

I \rightarrow samples of $\sin(-(-\omega)t) = \sin(\omega t)$

} measure

$$\begin{aligned} -\omega t &= -2\pi f t \\ 2\pi \cdot (\frac{-1}{8}) \cdot f_s \cdot t \end{aligned}$$

X[7] measures how much of the $-1 \cdot \omega$ component is present in *x*.

Any Period $p = 2L$

$$g(v) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kv + b_k \sin kv)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} g(v) dv$$

$$a_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \cos kv dv$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{+\pi} g(v) \sin kv dv$$

$$k = 1, 2, \dots$$

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

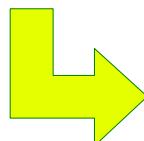
$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$$k = 1, 2, 3, \dots$$

$$v: [-\pi, +\pi]$$

$$x: [-L, +L]$$



$$v = \frac{\pi}{L} x$$

$$dv = \frac{\pi}{L} dx$$



Time and Frequency

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi}{L} x + b_k \sin \frac{k\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{+L} f(x) dx$$

$$a_k = \frac{1}{L} \int_{-L}^{+L} f(x) \cos \frac{k\pi x}{L} dx$$

$$b_k = \frac{1}{L} \int_{-L}^{+L} f(x) \sin \frac{k\pi x}{L} dx$$

$$k = 1, 2, 3, \dots$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

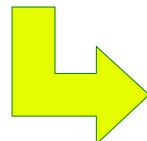
$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

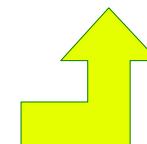
$$k = 1, 2, \dots$$

$$x: [-L, +L]$$

$$t: [0, T]$$



$$2L = T$$



Continuous Time Periodic Signal $x(t)$

Harmonic Frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2\pi k}{T} t + b_k \sin \frac{2\pi k}{T} t \right)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi k t}{T} dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin \frac{2\pi k t}{T} dt$$

$$k = 1, 2, \dots$$

$$t: [0, T]$$

resolution frequency

n-th harmonic frequency

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(2\pi k f_0 t) dt \quad k = 1, 2, \dots$$

$$t: [0, T]$$

$$f_0 = \frac{1}{T}$$

$$f_n = n f_0 = n \frac{1}{T}$$

Radial Frequency

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(k 2\pi f_0 t) + b_n \sin(k 2\pi f_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k 2\pi f_0 t) dt \quad k = 1, 2, \dots$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\mathbf{k} \omega_0 t) + b_n \sin(\mathbf{k} \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(\mathbf{k} \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(\mathbf{k} \omega_0 t) dt \\ k = 1, 2, \dots$$

$$t: [0, T]$$

$$t: [0, T]$$

linear frequency

$$f$$

angular (radial) frequency

$$\omega = 2\pi f$$

Complex Fourier Series Coefficients

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$

$t: [0, T]$

$t: [0, T]$

Real coefficients

$$a_0, a_k, b_k, k = 1, 2, \dots$$

Complex coefficients

$$A_0, A_k, B_k, k = 1, 2, \dots$$

one-sided spectrum

only positive frequencies

two-sided spectrum

Both pos and neg frequencies

Euler Equation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$k = 1, 2, \dots$

$$e^{+j\omega t} = \cos \omega t + j \sin \omega t$$

$$e^{-j\omega t} = \cos \omega t - j \sin \omega t$$

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$a_k \underline{\cos(k\omega_0 t)} + b_k \underline{\sin(k\omega_0 t)}$$

$$= a_k \frac{1}{2} (e^{jk\omega_0 t} + e^{-jk\omega_0 t}) + b_k \frac{1}{2j} (e^{jk\omega_0 t} - e^{-jk\omega_0 t})$$

$$= \frac{(a_k - jb_k)}{2} e^{jk\omega_0 t} + \frac{(a_k + jb_k)}{2} e^{-jk\omega_0 t}$$

$$= A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t}$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq \rightarrow	$A_0 = a_0$	}
pos freq \rightarrow	$A_k = \frac{1}{2} (a_k - jb_k)$	
neg freq \rightarrow	$B_k = \frac{1}{2} (a_k + jb_k)$	

only positive frequencies

Euler Equation (2)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$A_k = \frac{1}{T} \int_0^T x(t) (\cos(k\omega_0 t) - j \sin(k\omega_0 t)) dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) (\cos(k\omega_0 t) + j \sin(k\omega_0 t)) dt$$



$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

zero freq $\rightarrow A_0 = a_0$

pos freq $\rightarrow A_k = \frac{1}{2} (a_k - j b_k)$

neg freq $\rightarrow B_k = \frac{1}{2} (a_k + j b_k)$

only positive frequencies

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

Complex Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$x(t) = A_0 + \sum_{k=1}^{\infty} (A_k e^{jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_0 = a_0$$

$$A_k = \frac{1}{2} (a_k - j b_k)$$

$$B_k = \frac{1}{2} (a_k + j b_k)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=0}^{\infty} (A_k e^{+jk\omega_0 t} + B_k e^{-jk\omega_0 t})$$

$$A_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = 0, 1, 2, \dots$$

$$B_k = \frac{1}{T} \int_0^T x(t) e^{+jk\omega_0 t} dt$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} A_0 & (k = 0) \\ A_k & (k > 0) \\ B_k & (k < 0) \end{cases}$$

Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$k = +1, +2, \dots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha)} \cos(\beta) - \underline{\sin(\alpha)} \sin(\beta)$$

$$g_k \cos(k\omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k)} \cos(k\omega_0 t) - \underline{g_k \sin(\phi_k)} \sin(k\omega_0 t)$$

$$\underline{a_k \cos(k\omega_0 t)} + \underline{b_k \sin(k\omega_0 t)}$$

$$\begin{aligned} a_k &= g_k \cos(\phi_k) \\ -b_k &= g_k \sin(\phi_k) \end{aligned}$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Phasor Representation (1)

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t))$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k\omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k\omega_0 t) dt$$

$$k = 1, 2, \dots$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = 1, 2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re \{ X_k e^{+jk\omega_0 t} \}$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \Re \{ e^{+j(k\omega_0 t + \phi_k)} \}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1, 2, \dots$$

Phasor Representation (2)

$$x(t) = g_0 + \sum_{k=1}^{\infty} \frac{g_k}{2} \cdot (e^{+j(k\omega_0 t + \phi_k)} + e^{-j(k\omega_0 t + \phi_k)})$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k}{2} e^{+j\phi_k} e^{+jk\omega_0 t} + \frac{g_k}{2} e^{-j\phi_k} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} \left(\frac{g_k e^{+j\phi_k}}{2} e^{+jk\omega_0 t} + \frac{g_k e^{-j\phi_k}}{2} e^{-jk\omega_0 t} \right)$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k\omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = 1, 2, \dots$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{g_k e^{+j\phi_k}}{2} \quad (k > 0)$$

$$C_{-k} = \frac{g_k e^{-j\phi_k}}{2} \quad (k < 0)$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$x(t) = X_0 + \sum_{k=1}^{\infty} \Re \{ X_k e^{+jk\omega_0 t} \}$$

$$X_0 = g_0$$

$$X_k = g_k \cdot e^{+j\phi_k}$$

$$k = 1, 2, \dots$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}g_k e^{+jk\phi_k} & (k > 0) \\ \frac{1}{2}g_k e^{-jk\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Power Spectrum Two-Sided

$$\underline{|C_k|^2 + |C_{-k}|^2} = \frac{1}{2}|g_k|^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

Periodogram One-Sided

$$2 \cdot |C_k| = \underline{|g_k|} = \underline{\sqrt{a_k^2 + b_k^2}}$$

CTFS of Impulse Train (1)

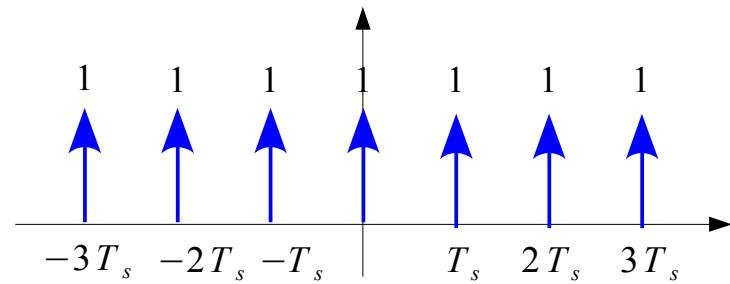
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

Fourier Series Expansion of Impulse Train

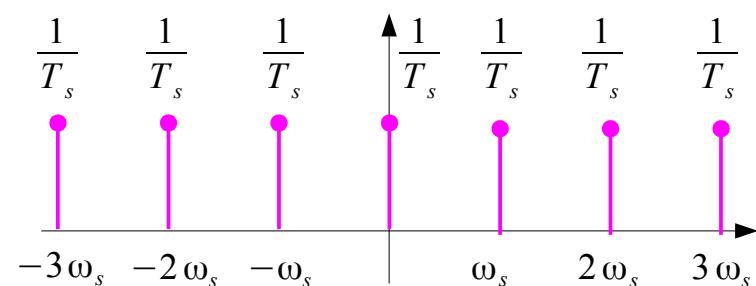
$$p(t) = \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t}$$

Fourier Series Coefficients

$$\begin{aligned} a_k &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s t} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) e^{-jk\omega_s 0} dt \\ &= \frac{1}{T_s} \int_{-T_s/2}^{+T_s/2} \delta(t) dt = \frac{1}{T_s} \end{aligned}$$



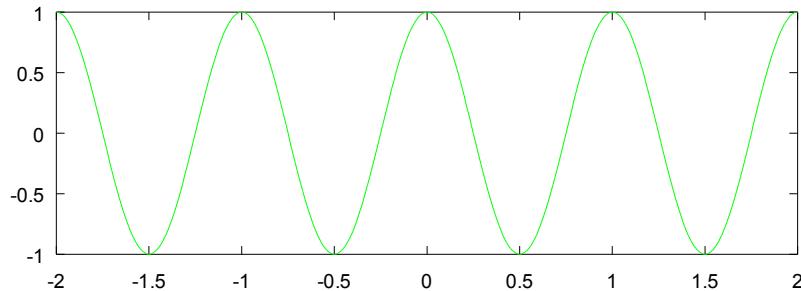
$$\omega_s = \frac{2\pi}{T_s}$$



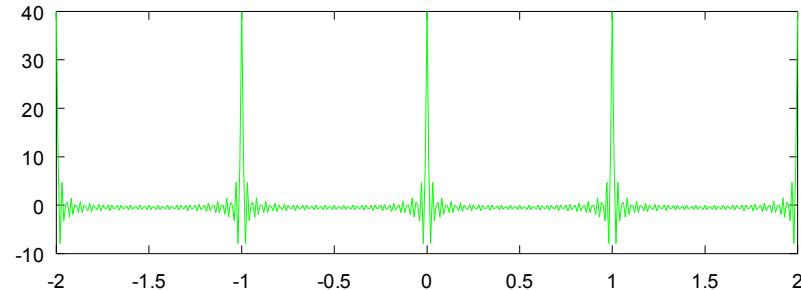
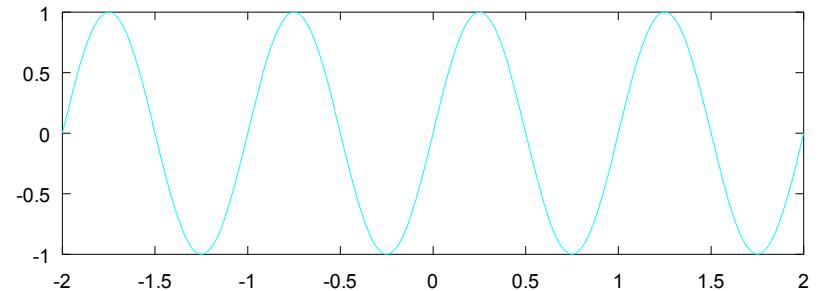
CTFS of Impulse Train (2)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

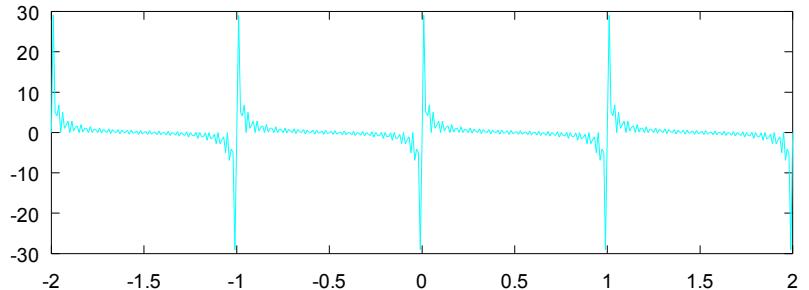
$\cos 2\pi \cdot 1 \cdot t$



$\sin 2\pi \cdot 1 \cdot t$



$$\sum_{k=1}^{40} \cos 2\pi \cdot k \cdot t$$

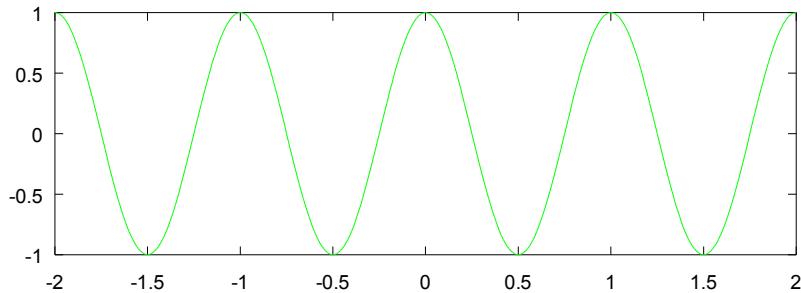


$$\sum_{k=1}^{40} \sin 2\pi \cdot k \cdot t$$

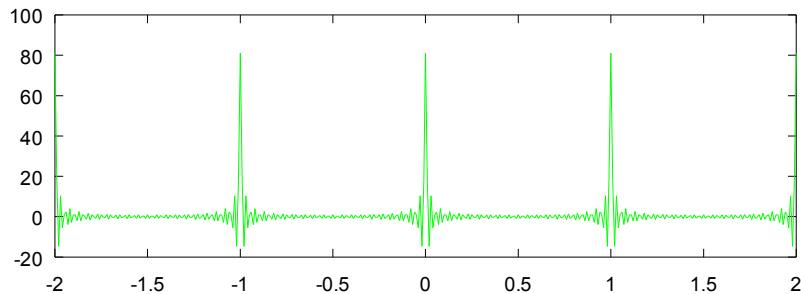
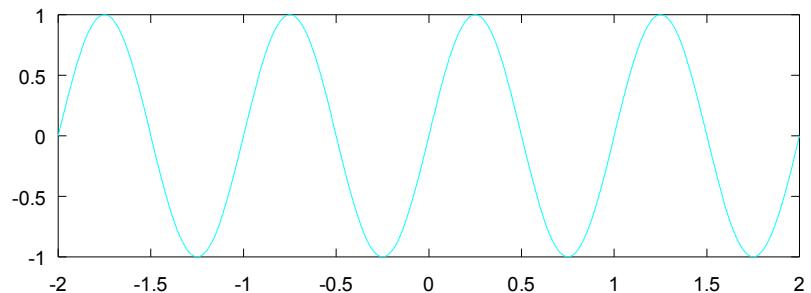
CTFS of Impulse Train (3)

$$p(t) = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} a_k e^{+jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{+\infty} (\cos k\omega_s t - j \sin k\omega_s t)$$

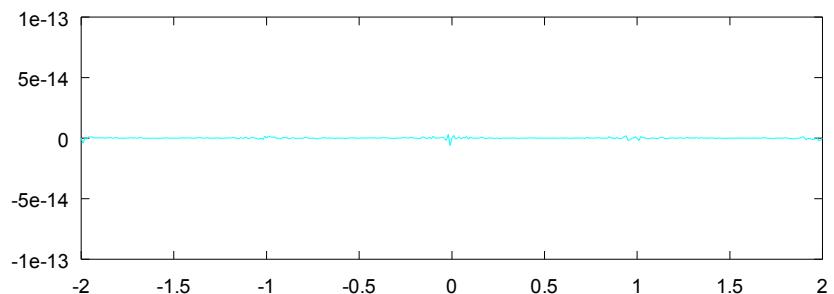
$\cos 2\pi \cdot 1 \cdot t$



$\sin 2\pi \cdot 1 \cdot t$



$$\sum_{k=-40}^{40} \cos 2\pi \cdot k \cdot t$$



$$\sum_{k=-40}^{40} \sin 2\pi \cdot k \cdot t$$

Inner Product Space

Hilbert Space real / complex inner product space

$$\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$$

complex conjugate

$$\langle y, x \rangle = \overline{\langle x, y \rangle}$$

linear

$$\langle a x_1 + b x_2, y \rangle = a \langle x_1, y \rangle + b \langle x_2, y \rangle$$

positive semidefinite

$$\langle x, x \rangle \geq 0$$

Norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

Orthogonality

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+j k \omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-j k \omega_0 t} dt$$

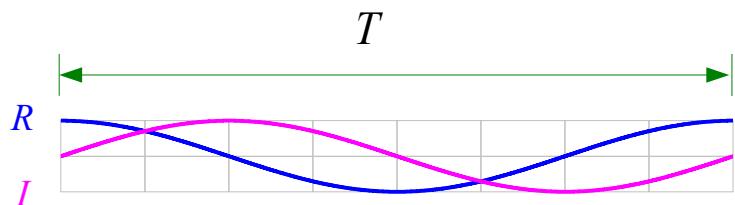
$$k = \dots, -2, -1, 0, +1, +2, \dots$$

fundamental frequency $f_0 = \frac{1}{T}$ $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

n-th harmonic frequency $f_n = n f_0$ $\omega_n = 2\pi f_n = \frac{2\pi n}{T}$

$$\langle e^{j \textcolor{red}{m} \omega_0 t}, e^{j \textcolor{green}{n} \omega_0 t} \rangle = \int_0^T e^{+j(\textcolor{red}{m}-\textcolor{green}{n})\omega_0 t} dt = \begin{cases} 0 & (\textcolor{red}{m} \neq \textcolor{green}{n}) \\ T & (\textcolor{red}{m} = \textcolor{green}{n}) \end{cases} \quad \textcolor{blue}{m, n : \text{integer}}$$

Inner Product Examples



$$f_0 = 1/T$$

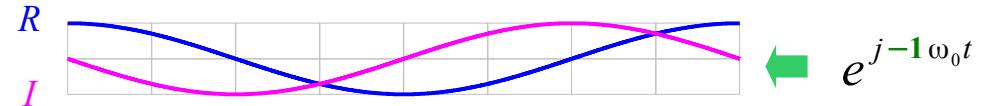
$$\omega_0 = 2\pi/T$$

$$\leftarrow e^{j \mathbf{1} \omega_0 t}$$

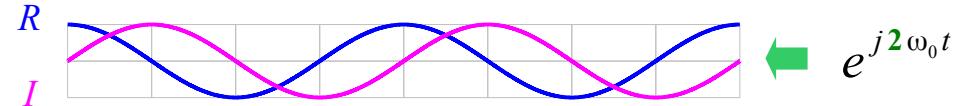
$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j \mathbf{1} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{1})\omega_0 t} dt = T \quad \leftarrow$$



$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j -\mathbf{1} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}+\mathbf{-1})\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j \mathbf{2} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}-\mathbf{2})\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j \mathbf{1} \omega_0 t}, e^{j -\mathbf{2} \omega_0 t} \rangle = \int_0^T e^{+j(\mathbf{1}+\mathbf{-2})\omega_0 t} dt = 0 \quad \leftarrow$$



Cauchy-Schwartz Inequality

For all vectors \mathbf{x} and \mathbf{y} of an inner product space

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent  maximum

$$\left| \int_a^b \mathbf{x}(t) \overline{\mathbf{y}(t)} dt \right| \leq \sqrt{\int_a^b \mathbf{x}(t) \overline{\mathbf{x}(t)} dt} \sqrt{\int_a^b \mathbf{y}(t) \overline{\mathbf{y}(t)} dt}$$

Inner product is maximum
when $\mathbf{y} = k \mathbf{x}$

Orthogonality

$$\begin{bmatrix} \mathbf{r}_0^* & \mathbf{r}_1^* & \mathbf{r}_2^* & \mathbf{r}_3^* & \mathbf{r}_4^* & \mathbf{r}_5^* & \mathbf{r}_6^* & \mathbf{r}_7^* \end{bmatrix} = \begin{bmatrix} \mathbf{r}_0 \cdot \mathbf{r}_0^* & \mathbf{r}_0 \cdot \mathbf{r}_1^* & \mathbf{r}_0 \cdot \mathbf{r}_2^* & \mathbf{r}_0 \cdot \mathbf{r}_3^* & \mathbf{r}_0 \cdot \mathbf{r}_4^* & \mathbf{r}_0 \cdot \mathbf{r}_5^* & \mathbf{r}_0 \cdot \mathbf{r}_6^* & \mathbf{r}_0 \cdot \mathbf{r}_7^* \\ \mathbf{r}_1 \cdot \mathbf{r}_0^* & \mathbf{r}_1 \cdot \mathbf{r}_1^* & \mathbf{r}_1 \cdot \mathbf{r}_2^* & \mathbf{r}_1 \cdot \mathbf{r}_3^* & \mathbf{r}_1 \cdot \mathbf{r}_4^* & \mathbf{r}_1 \cdot \mathbf{r}_5^* & \mathbf{r}_1 \cdot \mathbf{r}_6^* & \mathbf{r}_1 \cdot \mathbf{r}_7^* \\ \mathbf{r}_2 \cdot \mathbf{r}_0^* & \mathbf{r}_2 \cdot \mathbf{r}_1^* & \mathbf{r}_2 \cdot \mathbf{r}_2^* & \mathbf{r}_2 \cdot \mathbf{r}_3^* & \mathbf{r}_2 \cdot \mathbf{r}_4^* & \mathbf{r}_2 \cdot \mathbf{r}_5^* & \mathbf{r}_2 \cdot \mathbf{r}_6^* & \mathbf{r}_2 \cdot \mathbf{r}_7^* \\ \mathbf{r}_3 \cdot \mathbf{r}_0^* & \mathbf{r}_3 \cdot \mathbf{r}_1^* & \mathbf{r}_3 \cdot \mathbf{r}_2^* & \mathbf{r}_3 \cdot \mathbf{r}_3^* & \mathbf{r}_3 \cdot \mathbf{r}_4^* & \mathbf{r}_3 \cdot \mathbf{r}_5^* & \mathbf{r}_3 \cdot \mathbf{r}_6^* & \mathbf{r}_3 \cdot \mathbf{r}_7^* \\ \mathbf{r}_4 \cdot \mathbf{r}_0^* & \mathbf{r}_4 \cdot \mathbf{r}_1^* & \mathbf{r}_4 \cdot \mathbf{r}_2^* & \mathbf{r}_4 \cdot \mathbf{r}_3^* & \mathbf{r}_4 \cdot \mathbf{r}_4^* & \mathbf{r}_4 \cdot \mathbf{r}_5^* & \mathbf{r}_4 \cdot \mathbf{r}_6^* & \mathbf{r}_4 \cdot \mathbf{r}_7^* \\ \mathbf{r}_5 \cdot \mathbf{r}_0^* & \mathbf{r}_5 \cdot \mathbf{r}_1^* & \mathbf{r}_5 \cdot \mathbf{r}_2^* & \mathbf{r}_5 \cdot \mathbf{r}_3^* & \mathbf{r}_5 \cdot \mathbf{r}_4^* & \mathbf{r}_5 \cdot \mathbf{r}_5^* & \mathbf{r}_5 \cdot \mathbf{r}_6^* & \mathbf{r}_5 \cdot \mathbf{r}_7^* \\ \mathbf{r}_6 \cdot \mathbf{r}_0^* & \mathbf{r}_6 \cdot \mathbf{r}_1^* & \mathbf{r}_6 \cdot \mathbf{r}_2^* & \mathbf{r}_6 \cdot \mathbf{r}_3^* & \mathbf{r}_6 \cdot \mathbf{r}_4^* & \mathbf{r}_6 \cdot \mathbf{r}_5^* & \mathbf{r}_6 \cdot \mathbf{r}_6^* & \mathbf{r}_6 \cdot \mathbf{r}_7^* \\ \mathbf{r}_7 \cdot \mathbf{r}_0^* & \mathbf{r}_7 \cdot \mathbf{r}_1^* & \mathbf{r}_7 \cdot \mathbf{r}_2^* & \mathbf{r}_7 \cdot \mathbf{r}_3^* & \mathbf{r}_7 \cdot \mathbf{r}_4^* & \mathbf{r}_7 \cdot \mathbf{r}_5^* & \mathbf{r}_7 \cdot \mathbf{r}_6^* & \mathbf{r}_7 \cdot \mathbf{r}_7^* \end{bmatrix}$$

$\langle \mathbf{r}_i^H, \mathbf{r}_i^H \rangle = \mathbf{r}_i \cdot \mathbf{r}_i^* = N$
 $\langle \mathbf{r}_i^H, \mathbf{r}_j^H \rangle = \mathbf{r}_i \cdot \mathbf{r}_j^* = 0 \quad (i \neq j)$

Complex Vector Inner Product

Hermitian inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} = \sum x_i^* y_i \quad \mathbf{x}^H : \text{conjugate transpose}$$

Norm of Hermitian inner products

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^H \cdot \mathbf{x}} = \sqrt{\sum x_i^* x_i} \quad \text{the length of a vector}$$

$$\mathbf{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \quad \langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \cdot \mathbf{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - j b_1 & a_2 - j b_2 & \cdots & a_n - j b_n \end{pmatrix} \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} = \sum_{i=1}^n a_i^2 + b_i^2$$

The 1st Row of the DFT Matrix



$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 0, \quad n = 0, 1, \dots, 7$$

$$\begin{aligned} R &\rightarrow \text{samples of } \cos(-\omega t) = \cos(\omega t) \\ I &\rightarrow \text{samples of } \sin(-\omega t) = -\sin(\omega t) \end{aligned} \quad \underbrace{\qquad}_{\text{measure}} \quad \begin{aligned} \omega t &= 2\pi f t \\ &2\pi \cdot (\frac{0}{8}) \cdot f_s \cdot t \end{aligned}$$

$X[0]$ measures how much of the $+0 \cdot \omega$ component is present in \mathbf{x} .

The 3rd Row of the DFT Matrix

R

2 cycles

I

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

$$\begin{aligned} R &\rightarrow \text{samples of } \cos(-2\omega t) = \cos(2\omega t) \\ I &\rightarrow \text{samples of } \sin(-2\omega t) = -\sin(2\omega t) \end{aligned}$$

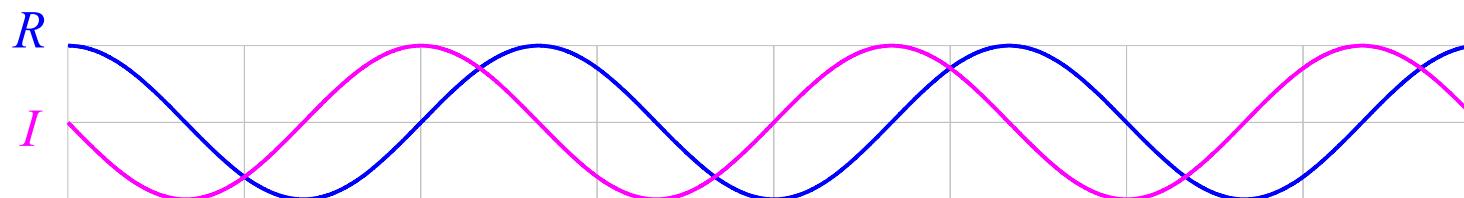
measure

$$\omega t = 2\pi f t$$

$$2\pi \cdot (\frac{2}{8}) \cdot f_s \cdot t$$

$X[2]$ measures how much of the $+2\cdot\omega$ component is present in \mathbf{x} .

The 4th Row of the DFT Matrix



3 cycles

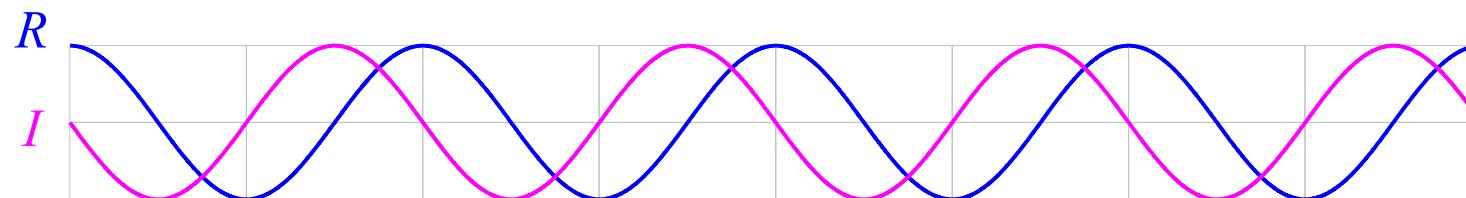
$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 3, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-3\omega t) = \cos(3\omega t)$
I \rightarrow samples of $\sin(-3\omega t) = -\sin(3\omega t)$

$$\left. \begin{array}{l} \omega t = 2\pi ft \\ 2\pi \cdot (\frac{3}{8}) \cdot f_s \cdot t \end{array} \right\} \text{measure}$$

X[3] measures how much of the $+3 \cdot \omega$ component is present in *x*.

The 5th Row of the DFT Matrix



4 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 4, \quad n = 0, 1, \dots, 7$$

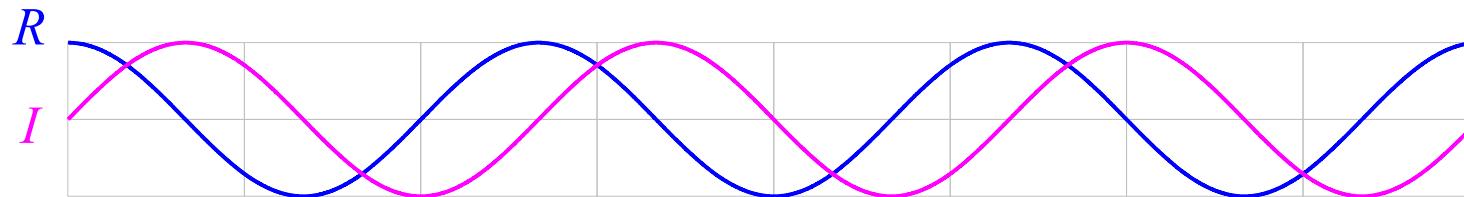
$R \rightarrow$ samples of $\cos(-4\omega t) = \cos(4\omega t)$

$I \rightarrow$ samples of $\sin(-4\omega t) = -\sin(4\omega t)$

$$\left. \begin{array}{l} \omega t = 2\pi f t \\ 2\pi \cdot (\frac{4}{8}) \cdot f_s \cdot t \end{array} \right\} \text{measure}$$

$X[4]$ measures how much of the $+4\cdot\omega$ component is present in \mathbf{x} .

The 6th Row of the DFT Matrix



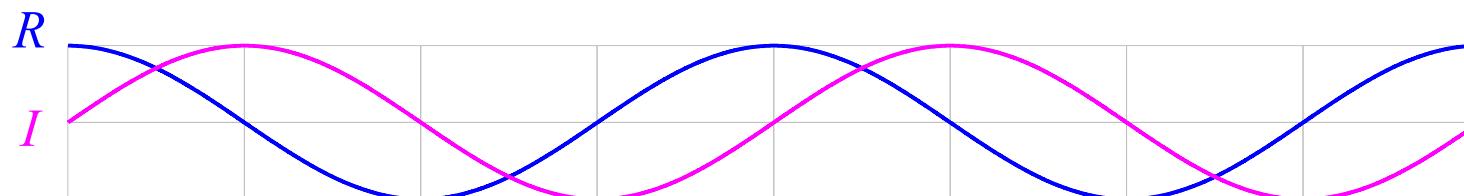
3 cycles

$$W_8^{k n} = e^{-j(\frac{2\pi}{8})kn} \quad k = 5, \quad n = 0, 1, \dots, 7$$

R → samples of $\cos(-(-3\omega)t) = \cos(3\omega t)$ } measure $-\omega t = -2\pi f t$
I → samples of $\sin(-(-3\omega)t) = \sin(3\omega t)$ } $2\pi \cdot (\frac{-3}{8}) \cdot f_s \cdot t$

$X[5]$ measures how much of the $-3\cdot\omega$ component is present in \mathbf{x} .

The 7th Row of the DFT Matrix



2 cycles

$$W_8^{kn} = e^{-j(\frac{2\pi}{8})kn} \quad k = 2, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-(-2\omega)t) = \cos(2\omega t)$

I \rightarrow samples of $\sin(-(-2\omega)t) = \sin(2\omega t)$

$\left. \begin{array}{l} -\omega t = -2\pi f t \\ 2\pi \cdot (\frac{-2}{8}) \cdot f_s \cdot t \end{array} \right\} \text{measure}$

$X[6]$ measures how much of the $-2\cdot\omega$ component is present in \mathbf{x} .

The 8th Row of the DFT Matrix



$$W_8^{k,n} = e^{-j(\frac{2\pi}{8})kn} \quad k = 7, \quad n = 0, 1, \dots, 7$$

R \rightarrow samples of $\cos(-(-\omega)t) = \cos(\omega t)$

I \rightarrow samples of $\sin(-(-\omega)t) = \sin(\omega t)$

} measure \rightarrow

$$\begin{aligned} -\omega t &= -2\pi f t \\ 2\pi \cdot (\frac{-1}{8}) \cdot f_s \cdot t &\end{aligned}$$

$X[7]$ measures how much of the $-1 \cdot \omega$ component is present in \mathbf{x} .

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
- [4] C. Langton, <http://www.complextoreal.com/chapters/fft1.pdf>