

Determinant (5A)

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Determinant

Determinant of order 2

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \quad \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ b_2 & b_3 & \\ c_2 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & & b_3 \\ c_1 & & c_3 \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ & b_2 & \\ b_1 & b_2 & \\ c_1 & c_2 & \end{bmatrix}$$

Determinant

Determinant of order 3

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \quad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} a_1 & & \\ b_2 & \cancel{b_3} & \\ c_2 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & a_2 & \\ b_1 & \cancel{a_3} & \\ c_1 & c_3 & \end{bmatrix} \quad \begin{bmatrix} & & a_3 \\ b_1 & \cancel{b_2} & \\ c_1 & c_2 & \end{bmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = + a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Rule of Sarrus (1)

Determinant of order 3 only

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

Copy and concatenate

Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{c} a_{11} \quad a_{12} \\ a_{21} \quad a_{22} \\ a_{31} \quad a_{32} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{c} a_{11} \quad a_{12} \\ a_{21} \quad a_{22} \\ a_{31} \quad a_{32} \end{array}$$

$$+ a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32}$$

$$- a_{13} a_{22} a_{31} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33}$$

Determinant – Rule of Sarrus (2)

Determinant of order 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Recursive Method

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\ &\quad - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

Determinant of order 3 only

$$\begin{bmatrix} + & a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}$$

$$\begin{bmatrix} a_{11} & + & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33}$$

Rule of Sarrus

$$\begin{bmatrix} a_{11} & a_{12} & + & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$+ a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}$$

Minor

The **minor** of entry a_{ij}

M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Sub-matrix

A diagram illustrating the formation of a submatrix. A 3x3 matrix is shown with its first row and first column highlighted in red. A 2x2 submatrix is formed by removing these highlighted rows and columns, resulting in the matrix $\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \rightarrow \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

The **cofactor** of entry a_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

The **minor** of entry a_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$(n-1) \times (n-1)$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Cofactor

$$C_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$\begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} (-1)^2 & (-1)^3 & (-1)^4 \\ (-1)^3 & (-1)^4 & (-1)^5 \\ (-1)^4 & (-1)^5 & (-1)^6 \end{bmatrix}$$

Minor Example (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{11} = +M_{11}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{12} = -M_{12}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{13} = +M_{13}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$$

Minor

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

$$C_{21} = -M_{21}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix}$$

$$M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{22} = +M_{22}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{23} = -M_{23}$$

Minor Example (2)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} & a_{12} & a_{13} \\ & a_{22} & a_{23} \end{bmatrix}$$

Sub-matrix

$$\begin{bmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{bmatrix}$$

Minor / Cofactor

$$M_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

$$C_{31} = +M_{31}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & & a_{13} \\ a_{21} & & a_{23} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$C_{32} = -M_{32}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$C_{33} = +M_{33}$$

Determinant

The determinant of an $n \times n$ matrix \mathbf{A} $det(\mathbf{A})$

Cofactor expansion along the i-th row
(elements of the i-th row) · (cofactors at the i-th row)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\mathbf{A}) &= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} \\ det(\mathbf{A}) &= a_{21} C_{21} + a_{22} C_{22} + a_{23} C_{23} \\ det(\mathbf{A}) &= a_{31} C_{31} + a_{32} C_{32} + a_{33} C_{33} \end{aligned}$$

Cofactor expansion along the j-th column
(elements of the j-th column) · (cofactors at the j-th column)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{aligned} det(\mathbf{A}) &= a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} \\ det(\mathbf{A}) &= a_{12} C_{12} + a_{22} C_{22} + a_{32} C_{32} \\ det(\mathbf{A}) &= a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33} \end{aligned}$$

Adjoint

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

$$\begin{array}{lll} a_{11} \Leftrightarrow C_{11} & a_{12} \Leftrightarrow C_{12} & a_{13} \Leftrightarrow C_{13} \\ a_{21} \Leftrightarrow C_{21} & a_{22} \Leftrightarrow C_{22} & a_{23} \Leftrightarrow C_{23} \\ a_{31} \Leftrightarrow C_{31} & a_{32} \Leftrightarrow C_{32} & a_{33} \Leftrightarrow C_{33} \end{array}$$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

Inverse Matrix

The **cofactor** of entry a_{ij} $C_{ij} = (-1)^{i+j} M_{ij}$

The **minor** of entry a_{ij} M_{ij}

The determinant of the submatrix
that remains after **deleting** i-th row and j-th column

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^{-1}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \text{adj}(\mathbf{A}) = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

Inverse Matrix

Given matrix

$$\begin{bmatrix} +2 & +2 & 0 \\ -2 & +1 & +1 \\ +3 & 0 & +1 \end{bmatrix}$$

$$\left. \begin{array}{c} +\begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} -\begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} +\begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} \\ -\begin{vmatrix} +2 & 0 \\ 0 & +1 \end{vmatrix} +\begin{vmatrix} +2 & 0 \\ +3 & +1 \end{vmatrix} -\begin{vmatrix} +2 & +2 \\ +3 & 0 \end{vmatrix} \\ +\begin{vmatrix} +2 & 0 \\ +1 & +1 \end{vmatrix} -\begin{vmatrix} +2 & 0 \\ -2 & +1 \end{vmatrix} +\begin{vmatrix} +2 & +2 \\ -2 & +1 \end{vmatrix} \end{array} \right\}$$

$$+2 \cdot \begin{vmatrix} +1 & +1 \\ 0 & +1 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & +1 \\ +3 & +1 \end{vmatrix} + 0 \cdot \begin{vmatrix} -2 & +1 \\ +3 & 0 \end{vmatrix} = 12$$

Matrix of Cofactors

$$\begin{bmatrix} +1 & +5 & -3 \\ -2 & +2 & +6 \\ +2 & -2 & +6 \end{bmatrix}$$

Adjoint $adj(\mathbf{A})$

$$\begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} adj(\mathbf{A})$$

a number a matrix

$$\mathbf{A}^{-1} = \frac{1}{12} \begin{bmatrix} +1 & -2 & +2 \\ +5 & +2 & -2 \\ -3 & +6 & +6 \end{bmatrix}$$

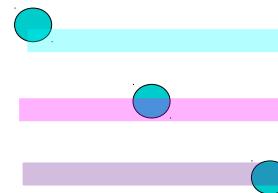
Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

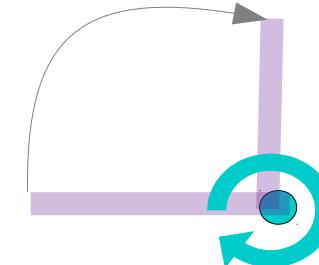
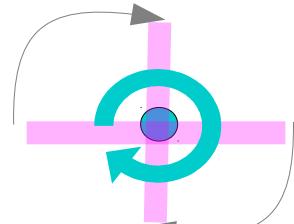
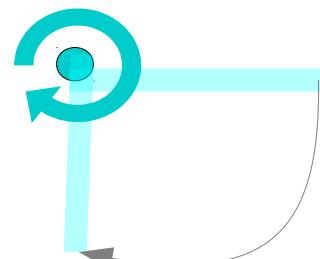
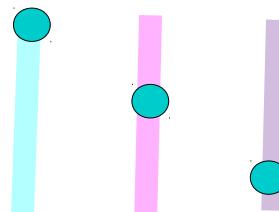
$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$\det(A) = \det(A^T)$$

$$[a_{ij}]$$



$$[a_{ji}]$$



Cofactor Expansion and Determinant

$A \quad n \times n$ zero row zero col
has $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\Rightarrow \det(A) = 0$

$$\begin{aligned}\det(A) &= a_{i_1} C_{i_1} + a_{i_2} C_{i_2} + a_{i_3} C_{i_3} && \text{i-th row cofactor expansion} \\ &= a_{1j} C_{1j} + a_{2j} C_{2j} + a_{3j} C_{3j} && \text{j-th column cofactor expansion} \\ &= 0\end{aligned}$$

$A \quad n \times n$ $A^T \quad n \times n$
 $\begin{bmatrix} * & * & * \end{bmatrix}$ i-th row $\begin{bmatrix} * \\ * \\ * \end{bmatrix}$ i-th col $\Rightarrow \det(A^T) = \det(A)$

$$\begin{aligned}\det(A) &= a_{i_1} C_{i_1} + a_{i_2} C_{i_2} + a_{i_3} C_{i_3} && \text{i-th row cofactor expansion of } A \\ &= a_{1i} C_{1i} + a_{2i} C_{2i} + a_{3i} C_{3i} && \text{i-th column cofactor expansion of } A^T\end{aligned}$$

Elementary Matrix and Determinant (1)

Interchange two rows

$$\begin{bmatrix} \text{green} \\ \text{green} \\ \text{cyan} \end{bmatrix} \leftrightarrow \begin{bmatrix} \text{cyan} \\ \text{green} \\ \text{green} \end{bmatrix}$$

B **A**

$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

Multiply a row by a nonzero constant

$$\begin{bmatrix} \text{purple} \end{bmatrix} \leftarrow \times c \begin{bmatrix} \text{cyan} \end{bmatrix}$$

B **A**

$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

Add a multiple of one row to another

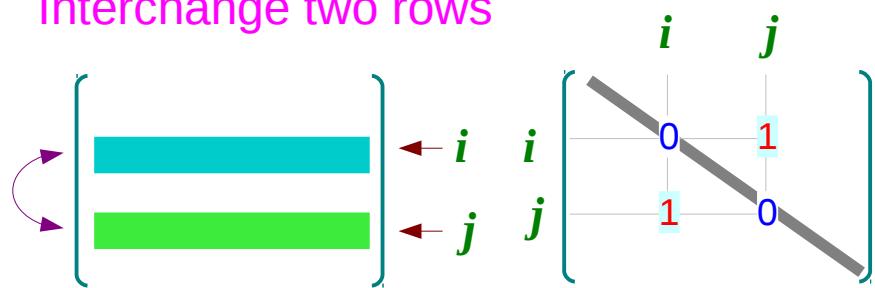
$$\begin{bmatrix} \text{cyan} \\ \text{orange} \end{bmatrix} \leftarrow \times c \begin{bmatrix} \text{cyan} \\ \text{cyan} \\ \text{yellow} \end{bmatrix}$$

B **A**

$$\det(\mathbf{B}) = \det(\mathbf{A})$$
$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Elementary Matrix and Determinant (2)

Interchange two rows



$$\det(\mathbf{B}) = -\det(\mathbf{A})$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B})$$

$$\begin{vmatrix} & a_{22} & a_{23} \\ & a_{32} & a_{33} \\ a_{11} & & \end{vmatrix}$$

$$\det(\mathbf{A})$$

$$\begin{vmatrix} & a_{22} & a_{23} \\ a_{32} & & a_{33} \\ a_{11} & & \end{vmatrix}$$

$$\det(\mathbf{B})$$

$$\begin{aligned} \det(\mathbf{B}) &= b_{21}C_{21} + b_{22}C_{22} + b_{23}C_{23} \\ &= -a_{11}M_{21} + a_{12}M_{22} - a_{13}M_{23} \end{aligned}$$

$$\det(\mathbf{A})$$

$$\begin{aligned} \det(\mathbf{A}) &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \end{aligned}$$

$$\begin{vmatrix} a_{21} & & a_{23} \\ & a_{32} & a_{33} \\ a_{11} & a_{22} & \end{vmatrix}$$

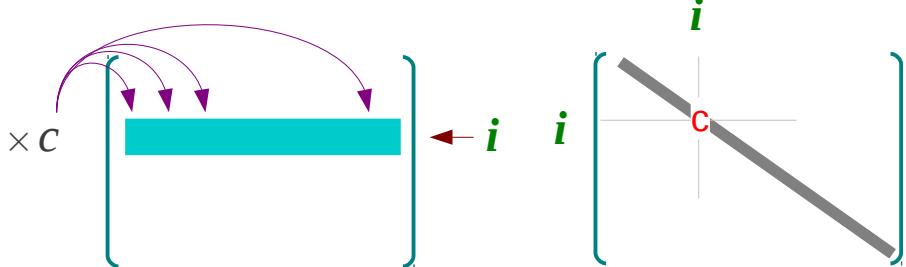
$$\begin{vmatrix} a_{21} & & a_{23} \\ a_{31} & & a_{33} \\ a_{11} & a_{22} & \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & \\ & a_{32} & a_{33} \\ a_{11} & a_{22} & \end{vmatrix}$$

$$\begin{vmatrix} a_{21} & a_{22} & \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{22} & \end{vmatrix}$$

Elementary Matrix and Determinant (3)

Multiply a row by a nonzero constant



$$\det(\mathbf{B}) = c \det(\mathbf{A})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = c \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

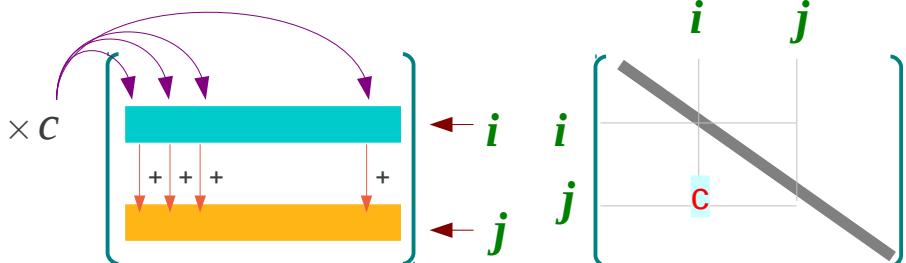
$$\det(\mathbf{B})$$

$$\begin{vmatrix} c a_{11} & c a_{12} & c a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \det(\mathbf{B}) &= b_{11} C_{11} + b_{12} C_{12} + b_{13} C_{13} \\ &= c \cdot a_{11} C_{11} + c \cdot a_{12} C_{12} + c \cdot a_{13} C_{13} \\ &= c (a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}) \\ &= c \cdot \det(\mathbf{A}) \end{aligned}$$

Elementary Matrix and Determinant (4)

Add a multiple of one row to another



$$\det(\mathbf{B}) = \det(\mathbf{A})$$

$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B})$$

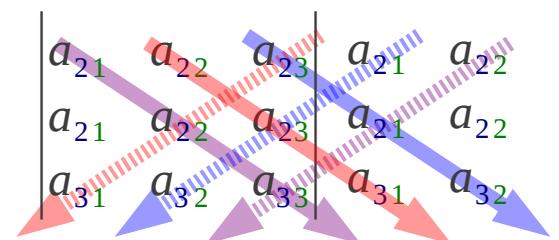
$$\begin{vmatrix} a_{11} + c a_{21} & a_{12} + c a_{22} & a_{13} + c a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det(\mathbf{B}) = b_{11} C_{11} + b_{12} C_{12} + b_{13} C_{13}$$

$$\begin{aligned} &= (a_{11} + c a_{21}) C_{11} + (a_{12} + c a_{22}) C_{12} + (a_{13} + c a_{23}) C_{13} \\ &= (a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}) \xrightarrow{\text{det}(\mathbf{A})} \\ &\quad + c(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) \xrightarrow{0} 0 \end{aligned}$$

$$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$(a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}) = 0$$



Determinant of Diagonal Matrix

Lower Triangular Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & 0 \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Upper Triangular Matrix

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Diagonal Matrix

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \quad \begin{matrix} a_{11} & a_{12} \\ 0 & a_{22} \\ 0 & 0 \end{matrix} \quad \det(A) = a_{11} a_{22} a_{33}$$

Determinant of an Elementary Matrix

Interchange two rows

A diagram illustrating the interchange of two rows in a 2x2 matrix. On the left, a teal bracket labeled i and a green bracket labeled j enclose two horizontal bars: a teal bar at the top and a green bar at the bottom. A curved arrow above the bars indicates a swap. To the right, a 2x2 matrix is shown with columns labeled i and j . The first column has entries 0 and 1. The second column has entries 1 and 0. A diagonal line connects the top-left entry (0) to the bottom-right entry (0), and another diagonal line connects the top-right entry (1) to the bottom-left entry (1).

$$\det(E_k) = -1$$

Multiply a row by a nonzero constant

A diagram illustrating the multiplication of a row by a nonzero constant c . On the left, a teal bracket labeled i encloses a teal bar. A curved arrow labeled $\times c$ points to the bar. To the right, a 2x2 matrix is shown with columns labeled i and j . The first column has entries 0 and c . The second column has entries 1 and 0. A diagonal line connects the top-left entry (0) to the bottom-right entry (0), and another diagonal line connects the top-right entry (c) to the bottom-left entry (1).

$$\det(E_k) = c$$

Add a multiple of one row to another

A diagram illustrating the addition of a multiple of one row to another. On the left, a teal bracket labeled i encloses a teal bar, and a green bracket labeled j encloses an orange bar. A curved arrow labeled $\times c$ points to the teal bar. Below the teal bar, red arrows labeled with '+' signs point to the orange bar. To the right, a 2x2 matrix is shown with columns labeled i and j . The first column has entries 0 and c . The second column has entries 1 and c . A diagonal line connects the top-left entry (0) to the bottom-right entry (0), and another diagonal line connects the top-right entry (c) to the bottom-left entry (1).

$$\det(E_k) = 1$$

Properties of Determinants

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A)+\det(B)$$

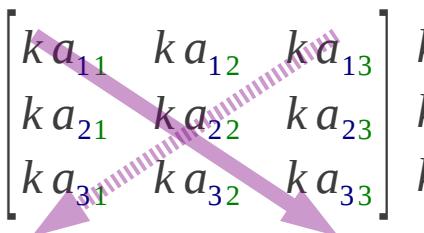
$$\det(AB) = \det(A)\det(B)$$

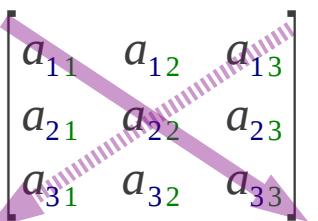
Proof of $\det(kA) = k^n \det(A)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A)+\det(B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix} \begin{array}{ll} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \\ ka_{31} & ka_{32} \end{array}$$


$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{array}{ll} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{array}$$


Proof of $\det(A+B) \neq \det(A) + \det(B)$

$$\det(kA) = k^n \det(A)$$

$$\det(A+B) \neq \det(A) + \det(B)$$

$$\det(AB) = \det(A)\det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} \begin{array}{l} a_{11}+b_{11} \quad a_{12}+b_{12} \\ a_{21}+b_{21} \quad a_{22}+b_{22} \\ a_{31}+b_{31} \quad a_{32}+b_{32} \end{array}$$

$$A \quad n \times n \quad B \quad n \times n$$

$$\begin{bmatrix} \$ \$ \$ \\ \$ \$ \$ \end{bmatrix} + \begin{bmatrix} \# \# \# \\ \# \# \# \end{bmatrix}$$

$$C = A + B$$



$$\det(C) = \det(A) + \det(B)$$

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{bmatrix} \begin{array}{l} a_{11}+b_{11} \quad a_{12}+b_{12} \\ 2a_{21} \quad 2a_{22} \\ 2a_{31} \quad 2a_{32} \end{array}$$

Proof of $\det(AB) = \det(A) \det(B)$ (1)

E_k (Elementary Matrices)

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$

$$E_k \begin{bmatrix} i & j \\ i & j \end{bmatrix}$$



$$\det(E_k B) = \det(E_k) \det(B)$$

$A \quad n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

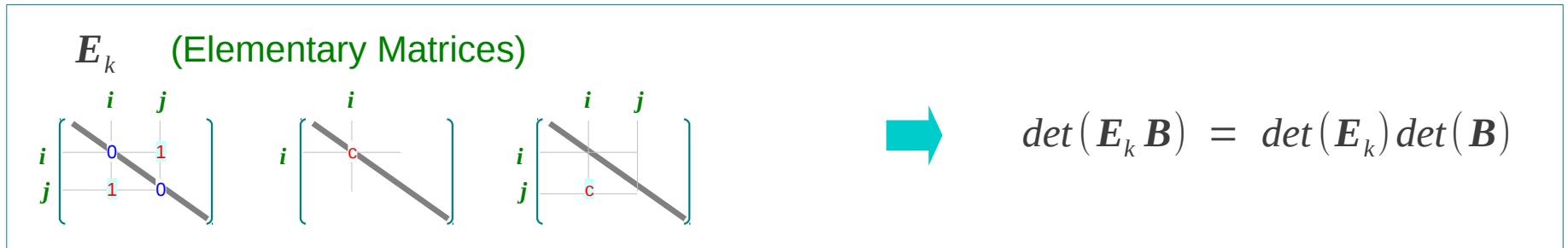
$A \quad n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof of $\det(AB) = \det(A) \det(B)$ (2)



$$i \begin{bmatrix} i & j \\ j & i \end{bmatrix}$$

$$i \begin{bmatrix} i & j \\ j & i \end{bmatrix}$$

$$i \begin{bmatrix} i & j \\ j & i \end{bmatrix}$$

$$\det(E_k B) = -\det(B)$$

$$\det(E_k) = -1$$

$$\det(E_k B) = \det(E_k) \det(B)$$

$$\det(E_k B) = c \cdot \det(B)$$

$$\det(E_k) = c$$

$$\det(E_k B) = \det(E_k) \det(B)$$

$$\det(E_k B) = \det(B)$$

$$\det(E_k) = 1$$

$$\det(E_k B) = \det(E_k) \det(B)$$

Proof of $\det(AB) = \det(A) \det(B)$ (3)

A $n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A) \neq 0$$

$$E_r \cdots E_2 E_1 A = R$$

Reduced Row Echelon Form

$$\underbrace{\det(E_r) \cdots \det(E_2) \det(E_1)}_{\text{non-zero}} \det(A) = \det(R)$$

A $n \times n$: invertible



$$R = I \quad \det(R) = 1 (\neq 0)$$

$$\det(A) \neq 0$$



$$\det(R) \neq 0$$

No zero row $R = I$

A $n \times n$: invertible

Proof of $\det(AB) = \det(A) \det(B)$ (4)

$$\det(AB) = \det(A)\det(B)$$

A $n \times n$: not invertible \rightarrow AB $n \times n$: not invertible

$$\det(A) = 0$$

$$\det(AB) = 0$$

A $n \times n$: invertible \rightarrow $A = E_r \cdots E_2 E_1$
 $AB = E_r \cdots E_2 E_1 B$

$$\det(AB) = \det(E_r) \cdots \det(E_2) \det(E_1) \det(B)$$

$$\det(AB) = \boxed{\det(E_r \cdots E_2 E_1) \det(B)}$$

$$\det(AB) = \boxed{\det(A) \det(B)}$$

Proof of $\det(AB) = \det(A) \det(B)$ (5)

A $n \times n$: invertible

$$AA^{-1} = A^{-1}A = I$$



$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$AA^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = 1$$

Computing $A \cdot \text{adj}(A)$ – diagonal elements

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

Computing $A \cdot \text{adj}(A)$ – off-diagonal elements (1)

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

transpose 

\mathbf{A}

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$\text{adj}(\mathbf{A})$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$\det(\mathbf{A}) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$0 = a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$0 = a_{31}C_{11} + a_{32}C_{12} + a_{33}C_{13}$$

Computing $A \cdot \text{adj}(A)$ – off-diagonal elements (2)

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix}$$

$$B = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \begin{bmatrix} C'_{11} \\ C'_{12} \\ C'_{13} \end{bmatrix}$$

- the redundant row of B
- linearly dependent
- $\det(B) = 0$
- turned into a zero row
- $\det(R) = 0$

$$E_r \cdots E_2 E_1 B = R$$

$$C_{11} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the same cofactor

$$C'_{11} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$C_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

the same cofactor

$$C'_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$C_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the same cofactor

$$C'_{13} = + \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

the cofactor along the 1st row of B

$$B \rightarrow a_{21} C'_{11} + a_{22} C'_{12} + a_{23} C'_{13} = 0$$

$$A \rightarrow = a_{21} C_{11} + a_{22} C_{12} + a_{23} C_{13}$$

an off-diagonal element of $A \cdot \text{adj}(A)$

Result of $\mathbf{A} \cdot \text{adj}(\mathbf{A})$

Given matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$n \times n$

Matrix of Cofactors

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$n \times n$

transpose

Adjoint

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$n \times n$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \det(\mathbf{A}) & 0 & 0 \\ 0 & \det(\mathbf{A}) & 0 \\ 0 & 0 & \det(\mathbf{A}) \end{bmatrix}$$

$$= \det(\mathbf{A}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

cofactor expansion
along the 1st, 2nd, 3rd rows

$$\mathbf{A} \text{adj}(\mathbf{A}) = \det(\mathbf{A}) \mathbf{I}$$

matrix

$$\det(\mathbf{A})$$

value

$$\mathbf{A} \left[\frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})} \right] = \mathbf{I}$$

$$\mathbf{A}[\mathbf{A}^{-1}] = \mathbf{I}$$

Linear Equations

$$(\text{Eq 1}) \rightarrow a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$$

$$(\text{Eq 2}) \rightarrow a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$$

 \vdots \vdots \vdots \vdots

$$(\text{Eq 3}) \rightarrow a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$$

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

$$\mathbf{A} \mathbf{x} = \mathbf{b}$$

Cramer's Rule (1) – solutions

$$\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \right] = \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array} \right]$$

$$\left[\begin{array}{ccccc} b_1 & a_{12} & \cdots & a_{1n} & \\ b_2 & a_{22} & \cdots & a_{2n} & \\ \vdots & \vdots & & \vdots & \\ b_n & a_{n2} & \cdots & a_{nn} & \end{array} \right] \left[\begin{array}{ccccc} a_{11} & b_1 & \cdots & a_{1n} & \\ a_{21} & b_2 & \cdots & a_{2n} & \\ \vdots & \vdots & & \vdots & \\ a_{n1} & b_n & \cdots & a_{nn} & \end{array} \right]$$

$$\left[\begin{array}{ccccc} a_{11} & a_{12} & \cdots & b_1 & \\ a_{21} & a_{22} & \cdots & b_2 & \\ \vdots & \vdots & & \vdots & \\ a_{n1} & a_{n2} & \cdots & b_n & \end{array} \right]$$

$$x_1 = \frac{\det(\mathbf{A}_1)}{\det(\mathbf{A})}$$

$$x_2 = \frac{\det(\mathbf{A}_2)}{\det(\mathbf{A})}$$

$$x_n = \frac{\det(\mathbf{A}_n)}{\det(\mathbf{A})}$$

Cramer's Rule (2) - determinants

$$\mathbf{A}_1 = \begin{bmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$det(\mathbf{A}_1) = b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1}$$

cofactor expansion along
the first column

$$x_1 = \frac{det(\mathbf{A}_1)}{det(\mathbf{A})}$$

$$\mathbf{A}_2 = \begin{bmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{bmatrix}$$

$$det(\mathbf{A}_2) = b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2}$$

cofactor expansion along
the second column

$$x_2 = \frac{det(\mathbf{A}_2)}{det(\mathbf{A})}$$

$$\mathbf{A}_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{bmatrix}$$

$$det(\mathbf{A}_n) = b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn}$$

cofactor expansion along
the last column

$$x_n = \frac{det(\mathbf{A}_n)}{det(\mathbf{A})}$$

Cramer's Rule (3) – inverse matrix

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{\text{adj}(\mathbf{A})}{\det(\mathbf{A})}\mathbf{b}$$

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

the transposed matrix
note reverse order index

$$\mathbf{x} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} b_1 C_{11} + b_2 C_{21} + \cdots + b_n C_{n1} \\ b_1 C_{12} + b_2 C_{22} + \cdots + b_n C_{n2} \\ \vdots & \vdots & \vdots \\ b_1 C_{1n} + b_2 C_{2n} + \cdots + b_n C_{nn} \end{pmatrix} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} \det(\mathbf{A}_1) \\ \det(\mathbf{A}_2) \\ \vdots \\ \det(\mathbf{A}_n) \end{pmatrix}$$

Equivalent Statements

A : invertible

$$\begin{matrix} A & A^{-1} \\ \left[\begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \right] & \left[\begin{array}{|c|} \hline \text{blue} \\ \hline \end{array} \right] \end{matrix} = \begin{matrix} A^{-1} & A \\ \left[\begin{array}{|c|} \hline \text{blue} \\ \hline \end{array} \right] & \left[\begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \right] \end{matrix} = \begin{matrix} I_n \\ \left[\begin{array}{|cc|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} \right] \end{matrix}$$

$Ax = 0$
only the trivial solution

$$\begin{matrix} A & x \\ \left[\begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \right] & \left[\begin{array}{|c|} \hline \text{orange} \\ \hline \end{array} \right] \end{matrix} = \begin{matrix} 0 \\ \left[\begin{array}{|c|} \hline 0 \\ 0 \\ 0 \\ \hline \end{array} \right] \end{matrix}$$

A the RREF is I_n
(Reduced Row Echelon Form)

$$\begin{matrix} A \\ \left[\begin{array}{|c|} \hline \text{red} \\ \hline \end{array} \right] \end{matrix} \xrightarrow{\text{Elem Row Op}} \begin{matrix} I_n \\ \left[\begin{array}{|cc|} \hline 1 & 0 \\ 0 & 1 \\ \hline \end{array} \right] \end{matrix}$$

A can be written as a product of E_k
(Elementary Matrices)

$$\begin{matrix} i & j \\ \left[\begin{array}{|cc|} \hline & 0 \\ 0 & 1 \\ \hline \end{array} \right] \end{matrix}, \quad \begin{matrix} i & j \\ \left[\begin{array}{|cc|} \hline & c \\ 0 & 1 \\ \hline \end{array} \right] \end{matrix}, \quad \begin{matrix} i & j \\ \left[\begin{array}{|cc|} \hline 1 & 0 \\ & c \\ \hline \end{array} \right] \end{matrix}$$

References

- [1] <http://en.wikipedia.org/>
- [2] Anton & Busby, "Contemporary Linear Algebra"
- [3] Anton & Rorres, "Elementary Linear Algebra"