Impedance

Young Won Lim
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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Sinusoidal voltage and current

$$i_C = C \cdot \frac{d\,v_C}{dt}$$

$$v_L = L \cdot \frac{d\,i_L}{dt}$$

$$\boxed{\begin{aligned} v_C &= A\cos(\omega t) \\ i_C &= -\omega C A \sin(\omega t) \\ &= \omega C A \cos(\omega t + \pi/2) \end{aligned}}$$

$$\boxed{\begin{aligned} i_L &= A\sin(\omega t) \\ &= A\cos(\omega t - \pi/2) \\ v_L &= \omega L A \cos(\omega t) \end{aligned}}$$

$$\frac{v_C}{i_C} = \xcancel{\frac{1}{\omega C} \frac{\cos(\omega t)}{\cos(\omega t + \pi/2)}}$$

$$\frac{v_L}{i_L} = \xcancel{\omega L \frac{\cos(\omega t)}{\cos(\omega t - \pi/2)}}$$

$$[i_C, \quad v_C]$$

↑ ***leads by 90***

$$[v_L, \quad i_L]$$

↑ ***leads by 90***

Leading and Lagging Current

$$i_C = C \cdot \frac{d\,v_C}{dt}$$

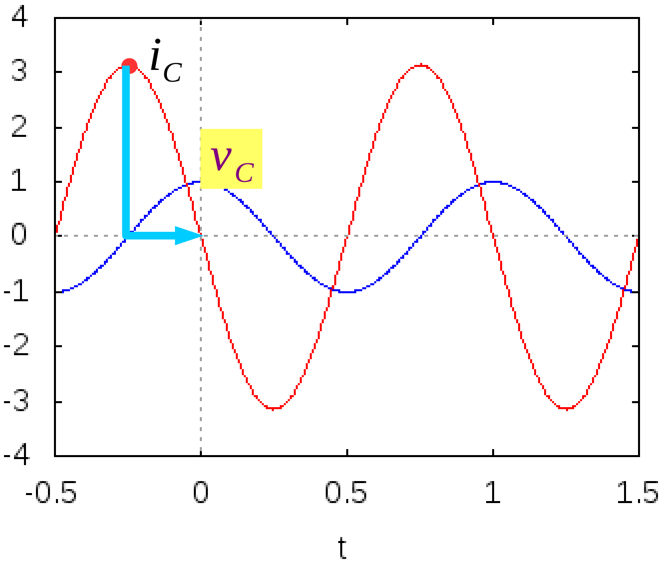
$$v_L = L \cdot \frac{d\,i_L}{dt}$$

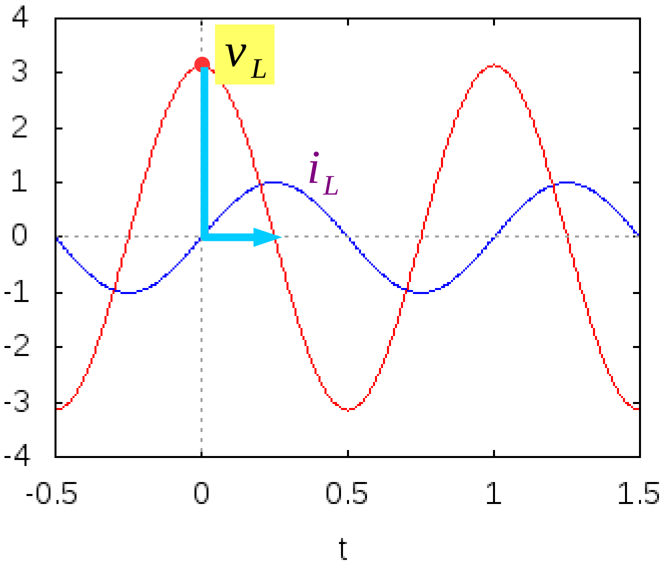
$$v_C = \cos(2\pi t)$$
$$i_C = -\pi \sin(2\pi t)$$
$$= \pi \cos(2\pi t + \pi/2)$$

$C = 0.5$

$$i_L = \sin(2\pi t)$$
$$= \cos(2\pi t - \pi/2)$$
$$v_L = \pi \cos(2\pi t)$$

$L = 0.5$

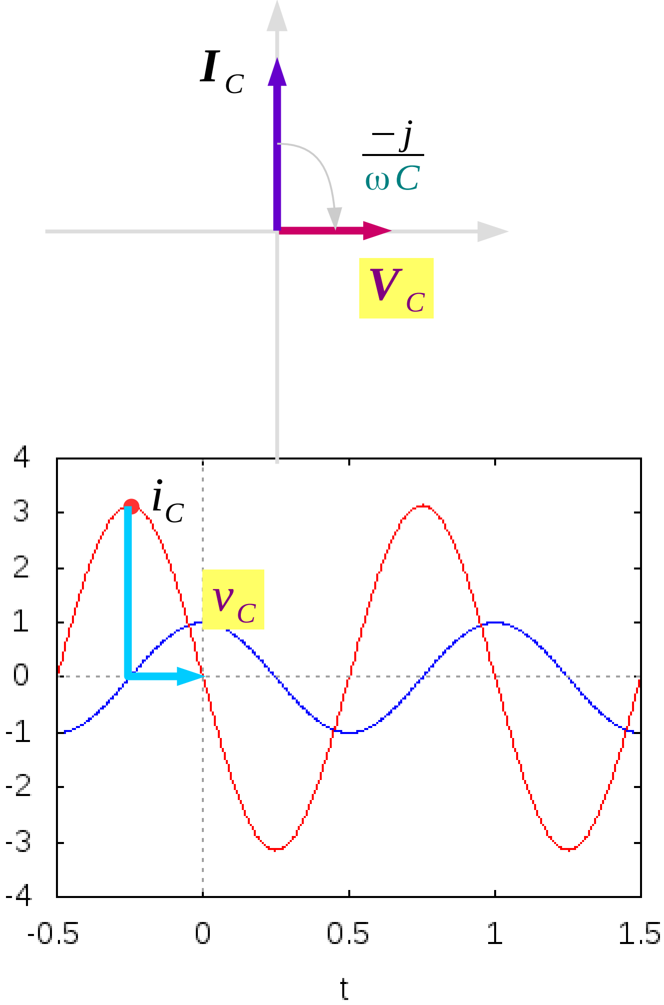


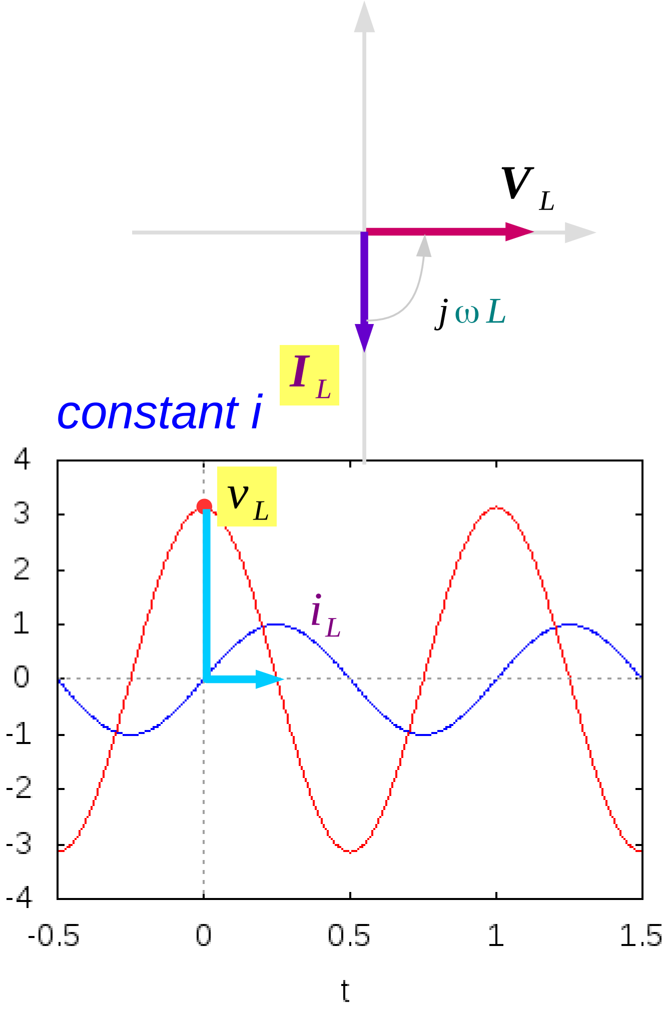


Leading and Lagging Current

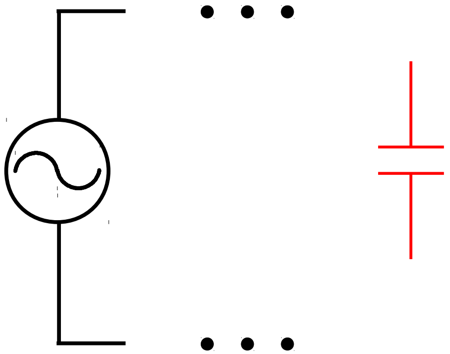
$$i_C = C \cdot \frac{d v_C}{dt}$$

$$v_L = L \cdot \frac{d i_L}{dt}$$

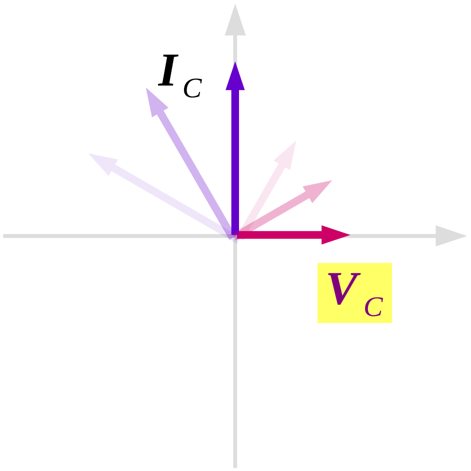


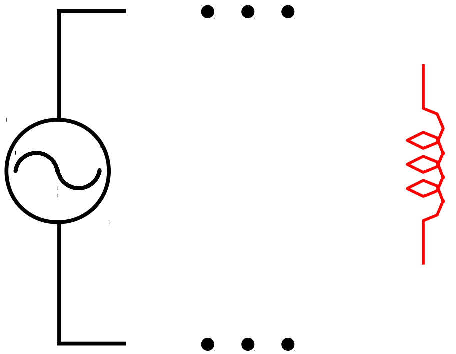


Phasor and Ohm's Law

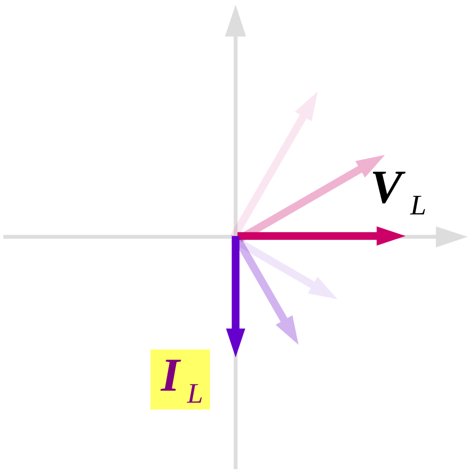


sinusoidal i, v source → **sinusoidal i, v**





sinusoidal i, v source → **sinusoidal i, v**



Impedance : a complex scaling factor

$$\boldsymbol{V}_C = A\angle\ 0$$
$$\boldsymbol{I}_C = \omega C\, A\angle\ \pi/2$$

$$\boldsymbol{I}_L = A\angle -\pi/2$$
$$\boldsymbol{V}_L = \omega L\, A\,\angle 0$$

$$\boldsymbol{Z}_C = \frac{\boldsymbol{V}_C}{\boldsymbol{I}_C}$$

⇐ *phasor*

⇐ *phasor*

⇓

a complex scaling factor

$$\boldsymbol{Z}_L = \frac{\boldsymbol{V}_L}{\boldsymbol{I}_L}$$

⇐ *phasor*

⇐ *phasor*

⇓

a complex scaling factor

$$\boldsymbol{V}_C = \boldsymbol{I}_C \cdot \boldsymbol{Z}_C = \boldsymbol{I}_C \left(\frac{1}{\omega C} \angle -\pi/2 \right)$$

$$\boldsymbol{Z}_C = \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

$$\boldsymbol{V}_L = \boldsymbol{I}_L \cdot \boldsymbol{Z}_L = \boldsymbol{I}_L \left(\omega L \angle +\pi/2 \right)$$

$$\boldsymbol{Z}_L = j\omega L$$

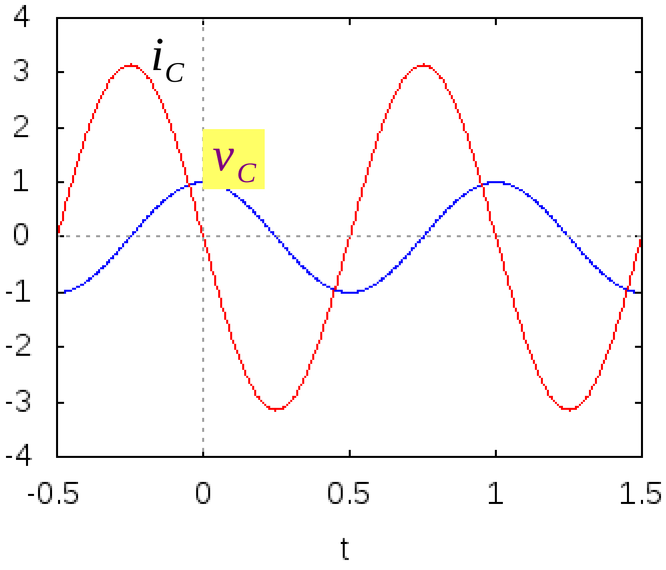
Ohm's Law

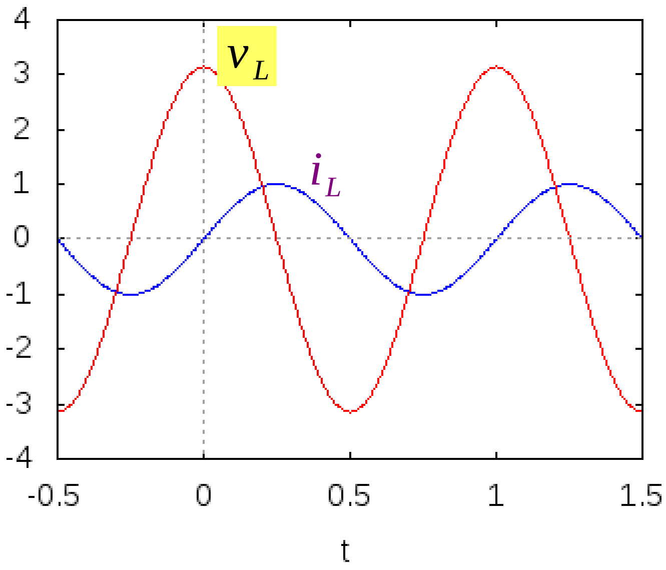
$$\boldsymbol{V}_C = A\angle 0$$
$$\boldsymbol{I}_C = \omega C A\angle \pi/2$$

$$\boldsymbol{Z}_C = \frac{\boldsymbol{V}_C}{\boldsymbol{I}_C} = \frac{1}{\omega C}\angle -\pi/2$$
$$= \frac{-j}{\omega C} = \frac{1}{j\omega C}$$

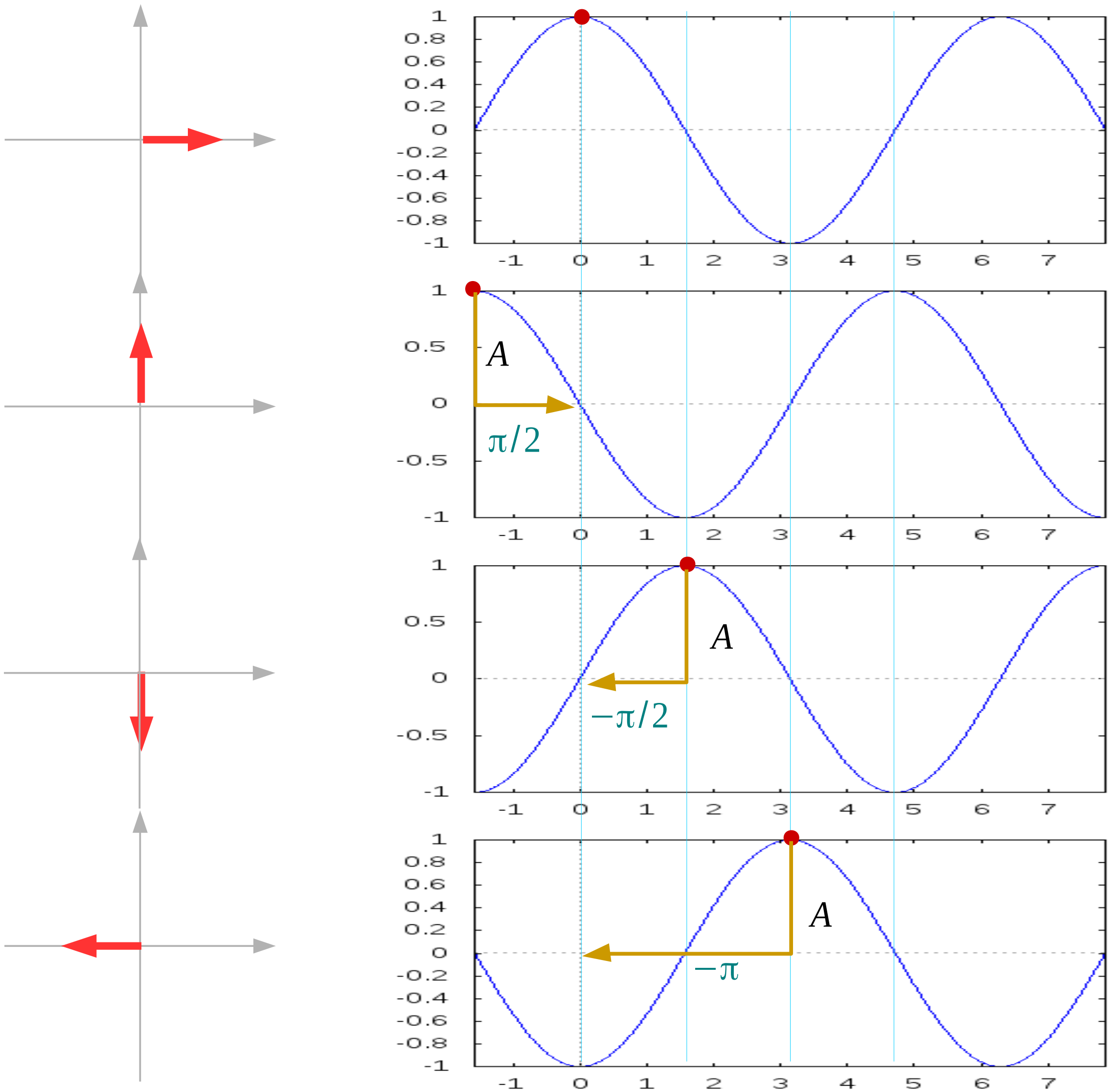
$$\boldsymbol{I}_L = A\angle -\pi/2$$
$$\boldsymbol{V}_L = \omega L A\angle 0$$

$$\boldsymbol{Z}_L = \frac{\boldsymbol{V}_L}{\boldsymbol{I}_L} = \omega L\angle +\pi/2$$
$$= j\omega L$$





Phasor Example (1)



$A\angle 0$ $\qquad A\cos(\omega t + 0)$

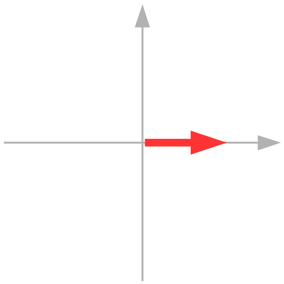
$A\angle +\pi/2$ $\qquad A\cos(\omega t + \pi/2)$

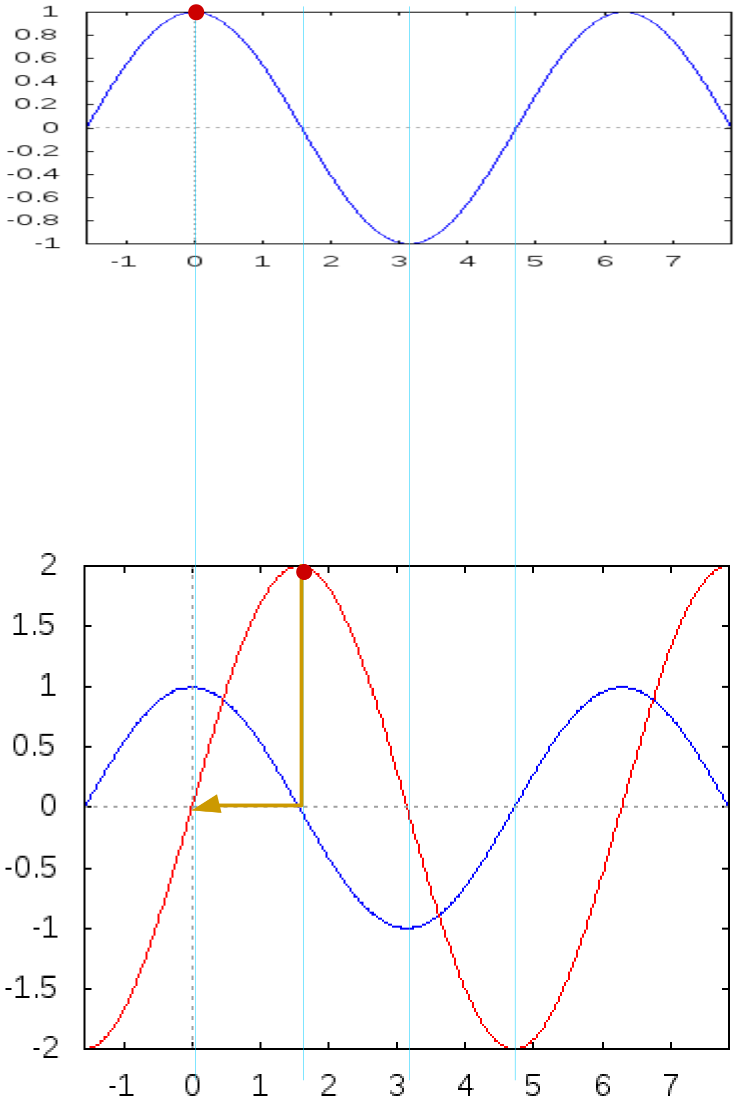
$A\angle -\pi/2$ $\qquad A\cos(\omega t - \pi/2)$

$A\angle -\pi$ $\qquad A\cos(\omega t - \pi)$

Phasor Example (2)

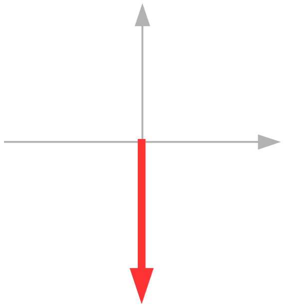
$A\angle 0$





$A\cos(\omega t + 0)$

$2A\angle -\pi/2$



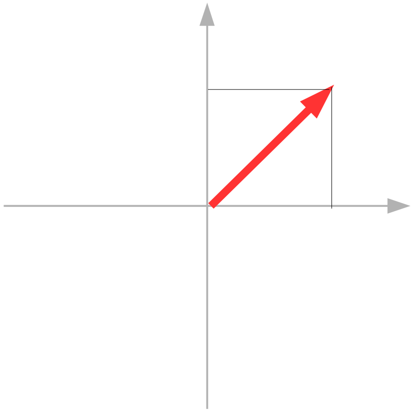
$2A\cos(\omega t - \pi/2)$

Phasor

$$A\cos(\omega t + \theta)$$

$$A\angle\theta$$

$$
\begin{aligned}
&= A\cos(\omega t)\cos(\theta) \;-\; A\sin(\omega t)\sin(\theta)\\
&= A\cos(\theta)\cos(\omega t) \;-\; A\sin(\theta)\sin(\omega t)\\
&= X\cos(\omega t) \;-\; Y\sin(\omega t)
\end{aligned}
$$



$$A\cos(\theta) = X$$
$$A\sin(\theta) = Y$$

$$(X, Y) = (A\cos\theta, A\sin\theta)$$

$$X + jY = A\cos\theta + jA\sin\theta$$

$$A = \sqrt{X^2 + Y^2}$$
$$\tan\theta = \frac{Y}{X}$$

$\theta > 0$ ***leading*** $\theta < 0$ ***lagging***

Linear Combination of cos(ωt) & sin(ωt)

$X\cos(\omega t) + Y\sin(\omega t)$

$$
\begin{aligned}
& X\cos(\omega t) + Y\sin(\omega t) \\
&= \sqrt{X^2+Y^2}\left[\frac{X}{\sqrt{X^2+Y^2}}\cos(\omega t) + \frac{Y}{\sqrt{X^2+Y^2}}\sin(\omega t)\right] \\
&= \sqrt{A^2+B^2}\left[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)\right] \\
&= \sqrt{X^2+Y^2}\cos(\theta - \omega t) \\
&= \sqrt{X^2+Y^2}\cos(\omega t - \theta)
\end{aligned}
$$

$\sqrt{X^2+Y^2}\cos(\omega t - \theta)$

$$
\begin{aligned}
& X\cos(\omega t) + Y\sin(\omega t) \\
&= \sqrt{X^2+Y^2}\cos(\omega t - \theta)
\end{aligned}
$$

$$\cos(\theta) = \frac{X}{\sqrt{X^2+Y^2}}$$

$$\sin(\theta) = \frac{Y}{\sqrt{X^2+Y^2}}$$

$X\cos(\omega t) - Y\sin(\omega t)$

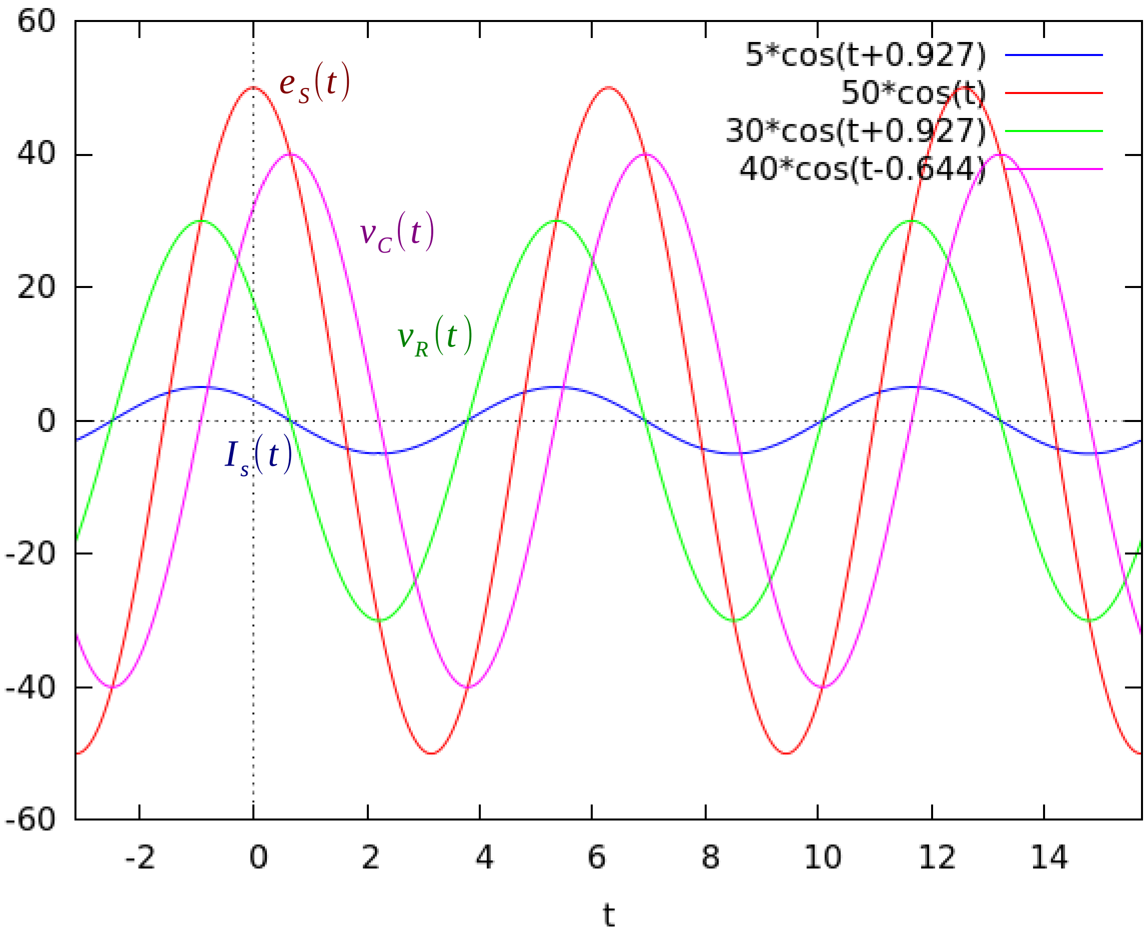
$$\frac{X}{A}\cos(\omega t) - \frac{Y}{A}\sin(\omega t)$$

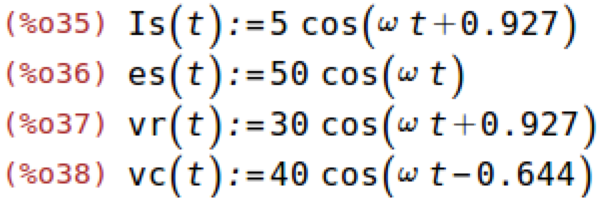
$$\cos(\theta)\cos(\omega t) - \sin(\theta)\sin(\omega t)$$

$\sqrt{X^2+Y^2}\cos(\omega t + \theta)$

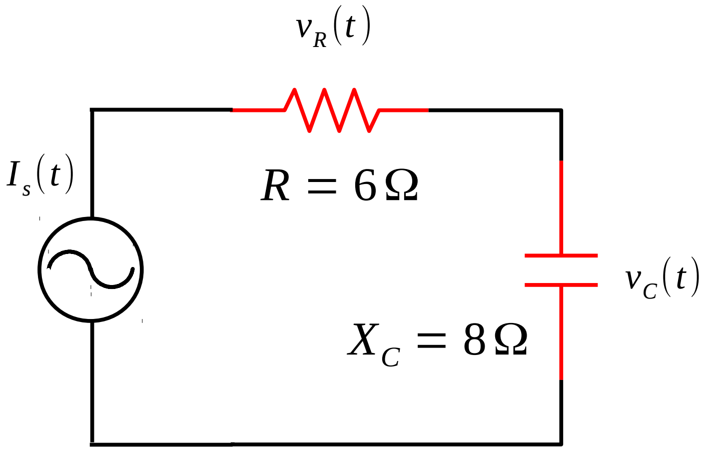
$$A\cos(\omega t + \theta)$$

RC Circuit

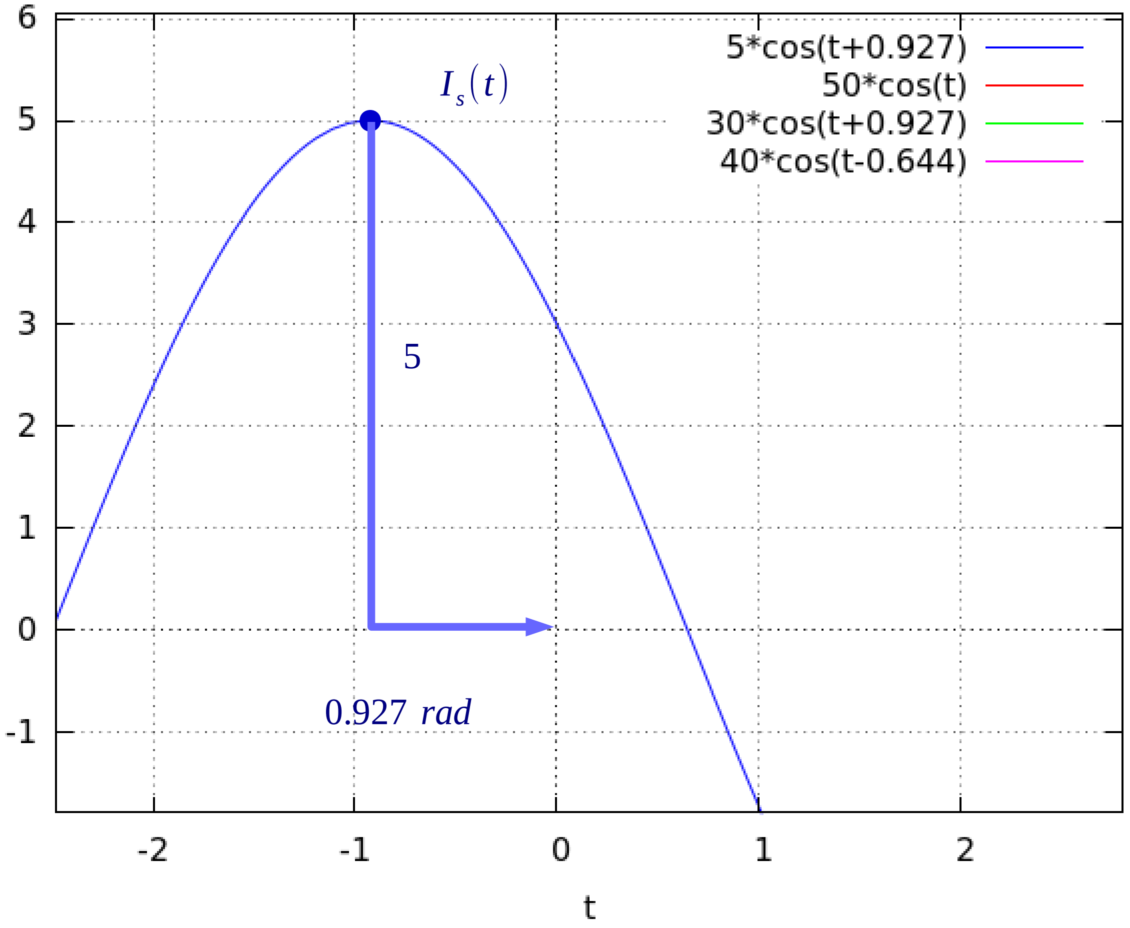




```
(%o35) Is(t):=5 cos(ω t+0.927)
(%o36) es(t):=50 cos(ω t)
(%o37) vr(t):=30 cos(ω t+0.927)
(%o38) vc(t):=40 cos(ω t-0.644)
```



Current Source



(%o35) $Is(t):=5\cos(\omega t+0.927)$

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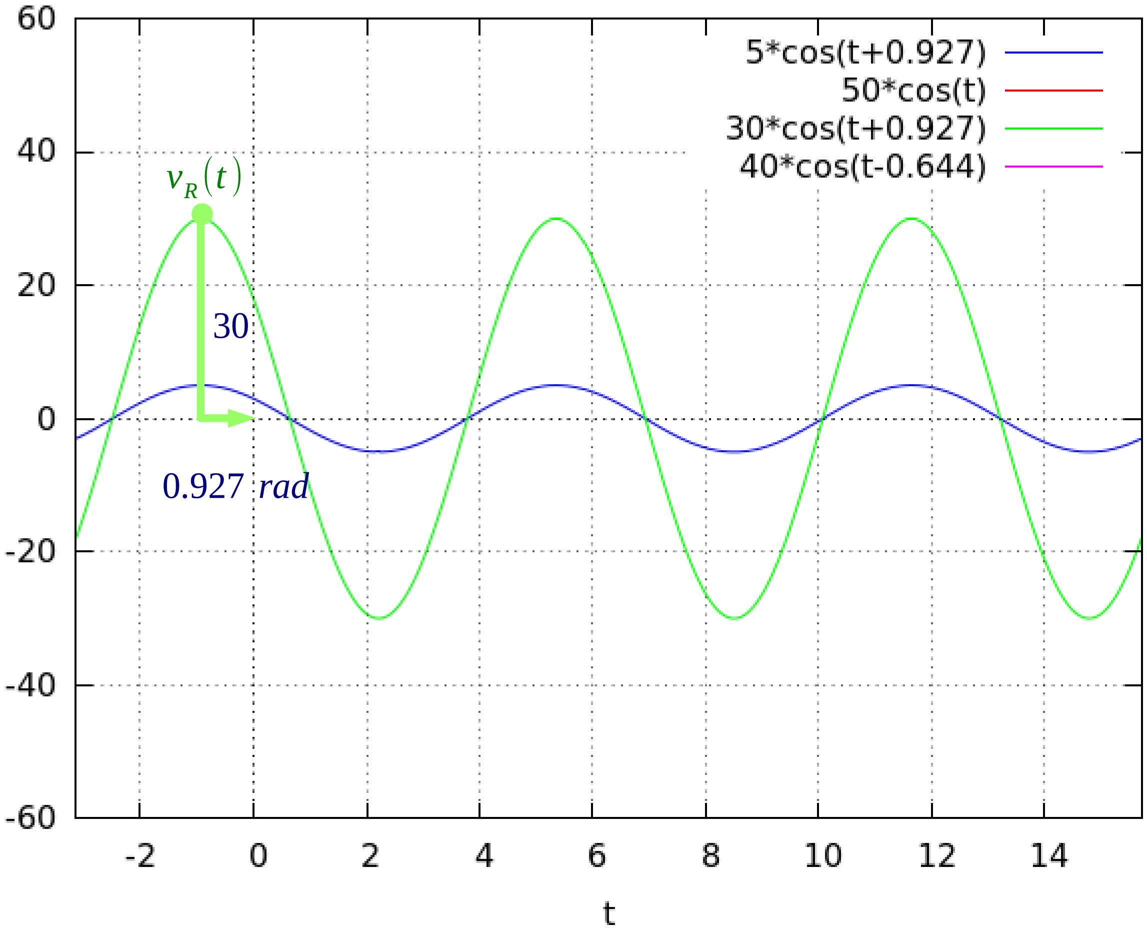
(%o37) $vr(t):=30\cos(\omega t+0.927)$

(%o38) $vc(t):=40\cos(\omega t-0.644)$

Resistor



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(%o38) vc(t):=40 cos(ω t-0.644)
```

$$v_R = i_R \cdot R$$

$$[i_R, \quad v_R]$$

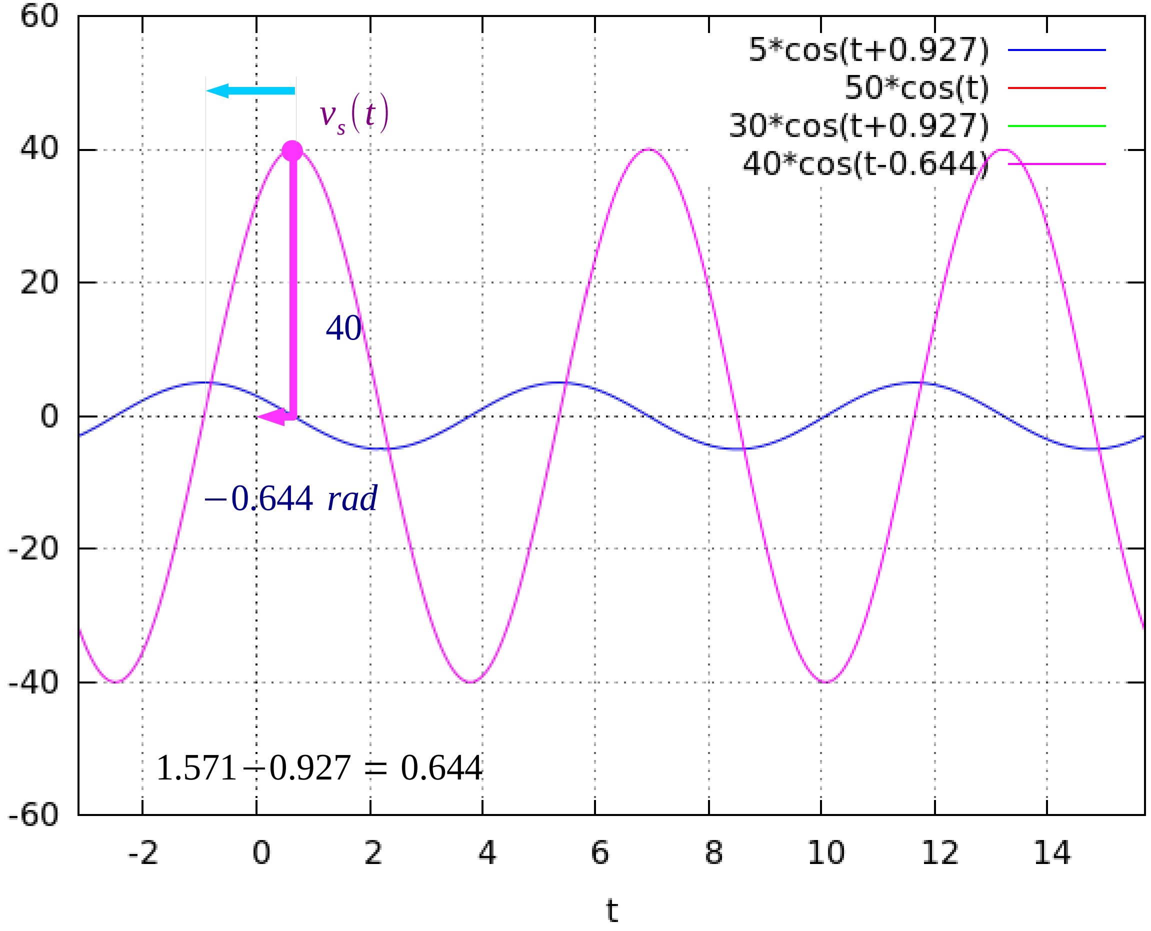
same phase

$R = 6\Omega \quad Is(t) \rightarrow 5$

$Is(t) \cdot 6\Omega \rightarrow 30$

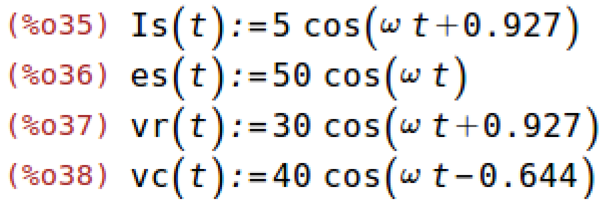
Capacitor

$\pi/2 = 1.571$



$X_C = 8\,\Omega \qquad Is(t) \rightarrow 5$

$Is(t)\cdot 8\,\Omega \rightarrow 40$



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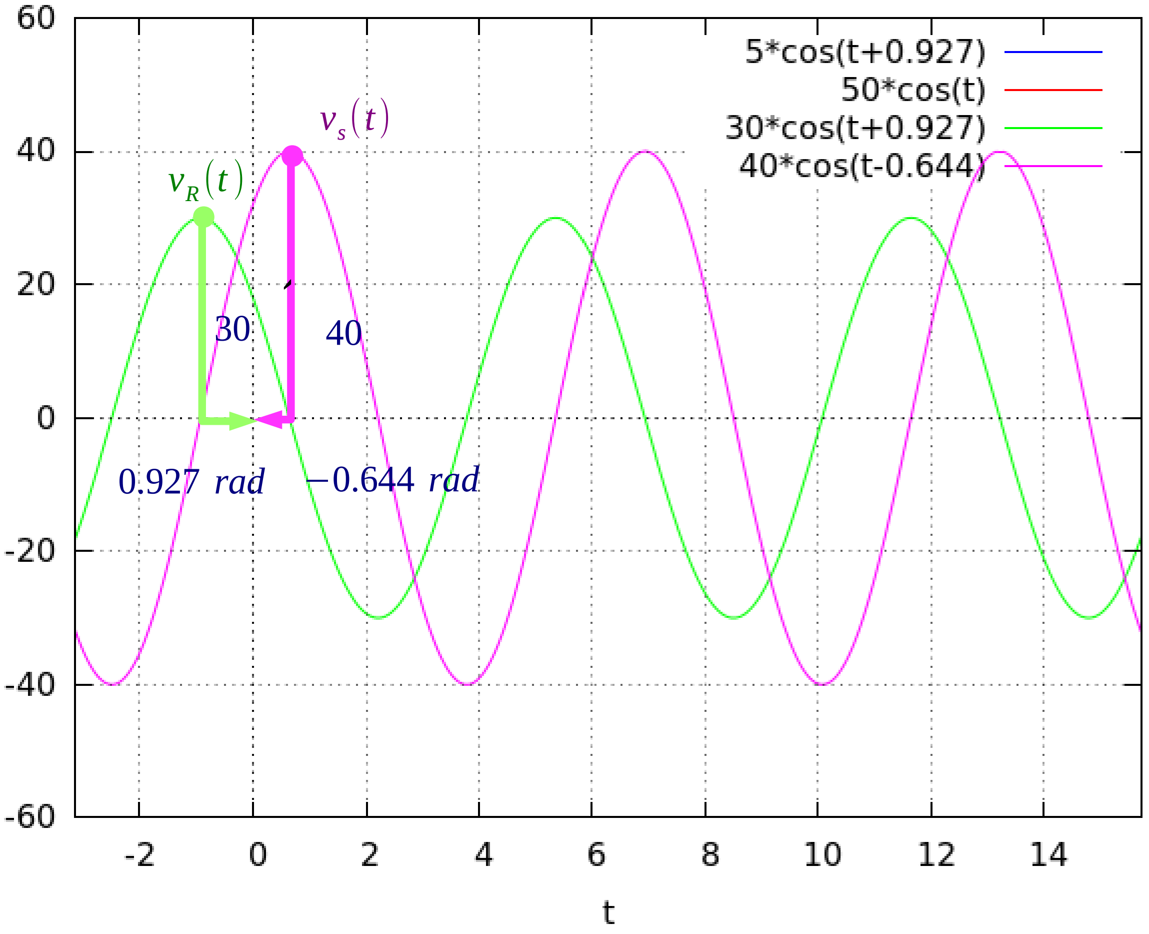
(%o37) vr(t):=30 cos(ω t+0.927)
(%o38) vc(t):=40 cos(ω t−0.644)
```

$$i_C = C\cdot\frac{d\,v_C}{dt}$$

$$[\,i_C\,,\quad v_C\,]$$

leads by 90

$v_R(t) + v_C(t)$



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(%o36) $es(t):=50\cos(\omega t)$

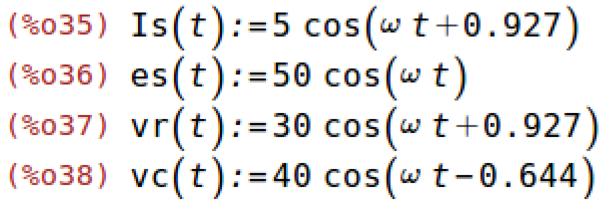
(%o37) $vr(t):=30\cos(\omega t+0.927)$

(%o38) $vc(t):=40\cos(\omega t-0.644)$

Phasor Addition

$$A_1 \cos(\omega t + \theta_1) \;=\; A_1 \cos(\omega t) \cos(\theta_1) - A_1 \sin(\omega t) \sin(\theta_1)$$

$$A_2 \cos(\omega t + \theta_2) \;=\; A_2 \cos(\omega t) \cos(\theta_2) - A_2 \sin(\omega t) \sin(\theta_2)$$



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```

$$A_1 \cos(\omega t + \theta_1) + A_2 \cos(\omega t + \theta_2) = \; X \cos(\omega t) - Y \sin(\omega t) \;=\; A \cos(\omega + \theta)$$

$$X \;=\; A_1 \cos(\theta_1) + A_2 \cos(\theta_2)$$

$$Y \;=\; A_1 \sin(\theta_1) + A_2 \sin(\theta_2)$$

$$X \;=\; 50.0$$

$$Y \;=\; 0.0$$

$$A \;=\; \sqrt{50.0^2 + 0.0^2} \;=\; 50.0$$

$$\theta \;=\; \tan^{-1} \frac{0.0}{50.81} \;=\; 0$$

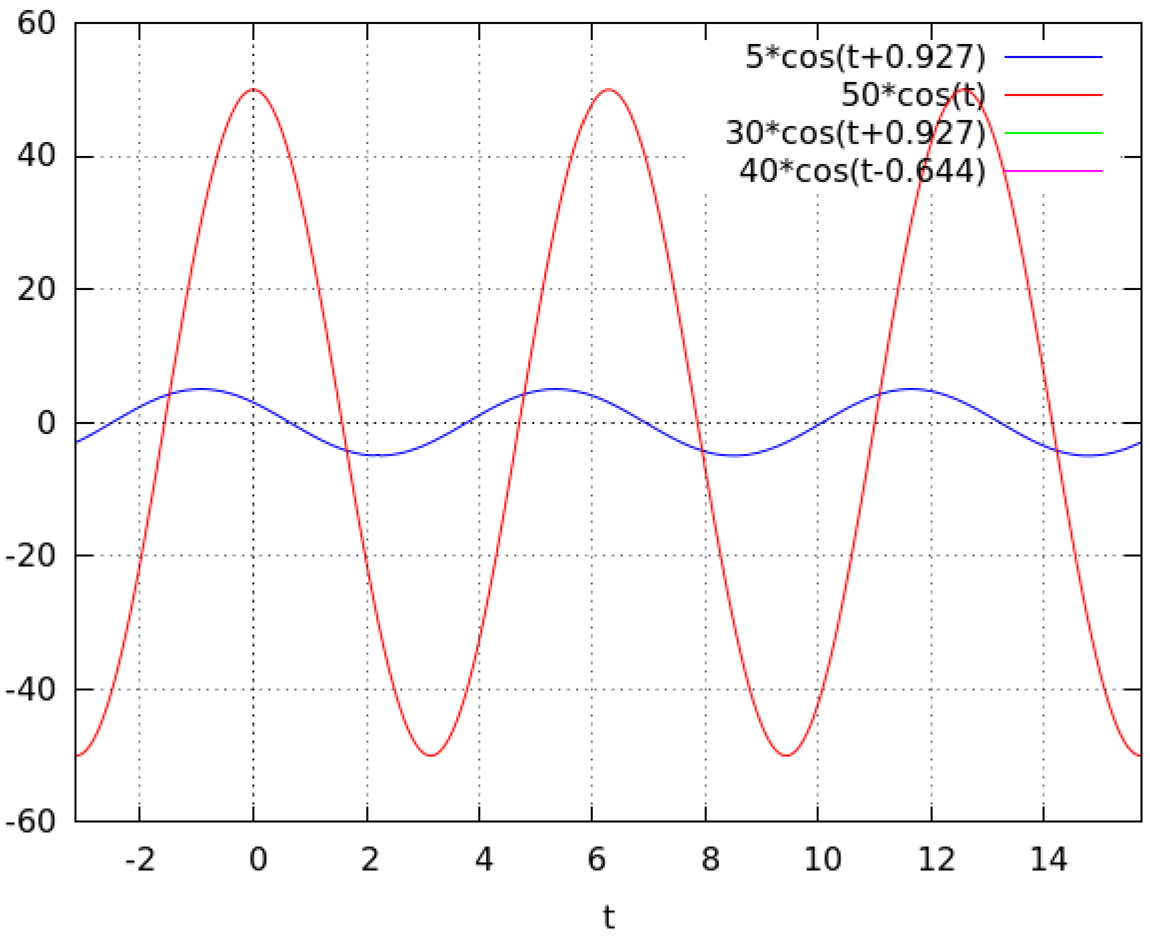
$$A_1 \cos(\theta_1) = 30 \cos(0.927) = 18.0$$

$$A_1 \sin(\theta_1) = 30 \sin(0.927) = 24.0$$

$$A_2 \cos(\theta_1) = 40 \cos(-0.644) = 32.0$$

$$A_2 \sin(\theta_1) = 40 \sin(-0.644) = -24.0$$

Is(t) and Es(t)



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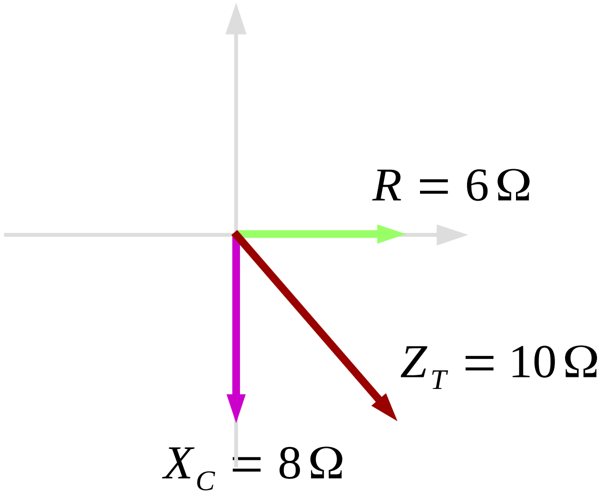
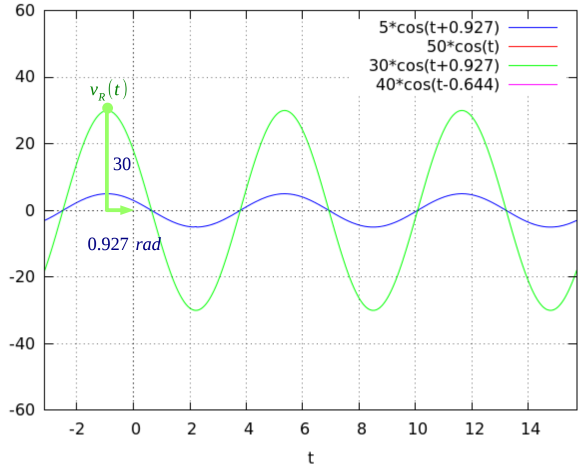


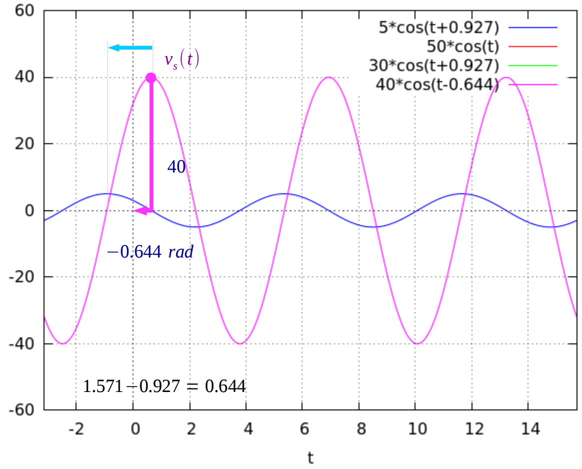
$$\boldsymbol{Z}_T = 10 \angle -0.927$$

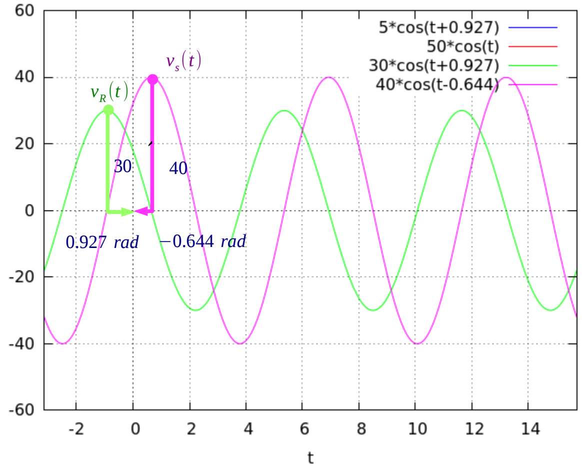
$$\boldsymbol{I}_s \boldsymbol{Z}_T = (5 \angle 0.927) \cdot (10 \angle -0.927) \quad = \; 50 \angle 0$$

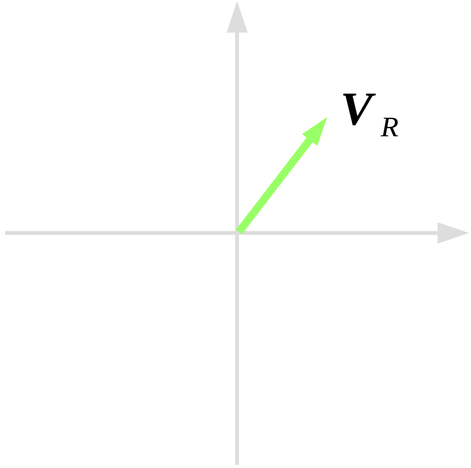
$$\boldsymbol{E}_s = \boldsymbol{I}_s \boldsymbol{Z}_T$$

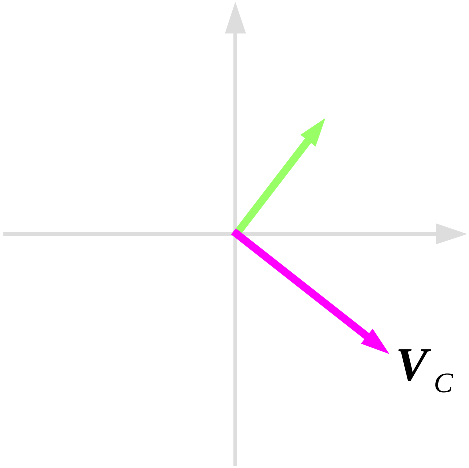
Phasor

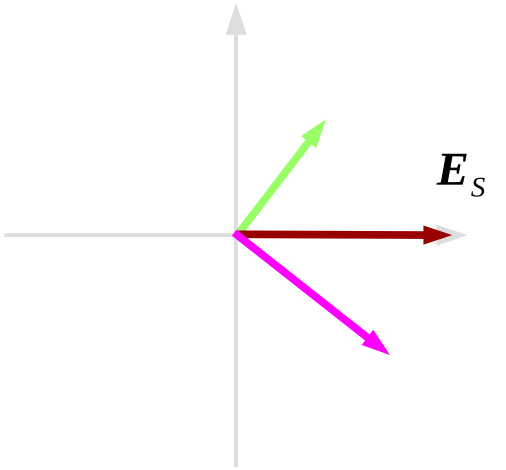

5*cos(t+0.927)
50*cos(t)
30*cos(t+0.927)
40*cos(t-0.644)
$v_R(t)$
30
0.927 rad
60
40
20
0
-20
-40
-60
-2
0
2
4
6
8
10
12
14
t


5*cos(t+0.927)
50*cos(t)
30*cos(t+0.927)
40*cos(t-0.644)
$v_s(t)$
40
-0.644 rad
$1.571 - 0.927 = 0.644$
60
40
20
0
-20
-40
-60
-2
0
2
4
6
8
10
12
14
t

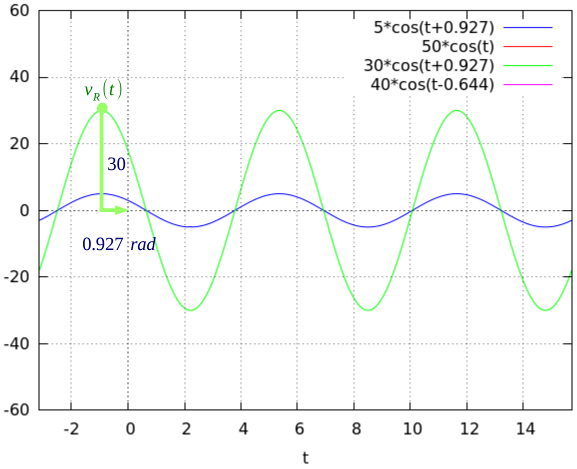

5*cos(t+0.927)
50*cos(t)
30*cos(t+0.927)
40*cos(t-0.644)
$v_R(t)$
$v_s(t)$
30
40
0.927 rad
-0.644 rad
60
40
20
0
-20
-40
-60
-2
0
2
4
6
8
10
12
14
t


$\boldsymbol{V}_R$


$\boldsymbol{V}_C$


$\boldsymbol{E}_S$

Phasor


5*cos(t+0.927)
50*cos(t)
30*cos(t+0.927)
40*cos(t-0.644)
$v_R(t)$
30
0.927 rad
60
40
20
0
-20
-40
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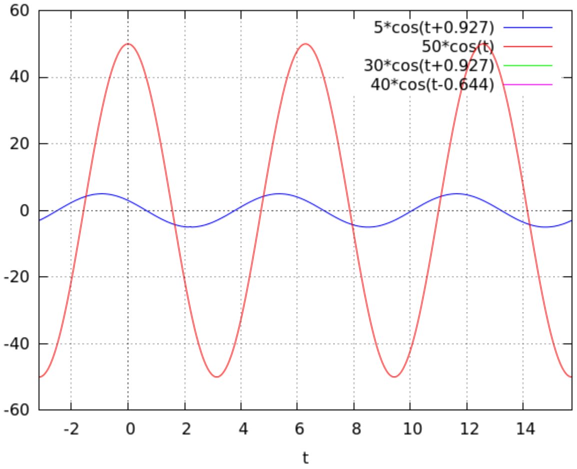
4
6
8
10
12
14
t


5*cos(t+0.927)
50*cos(t)
30*cos(t+0.927)
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60
40
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-20
-40
-60
-2
0
2
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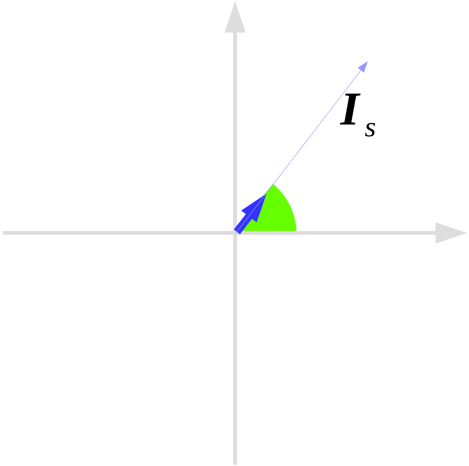
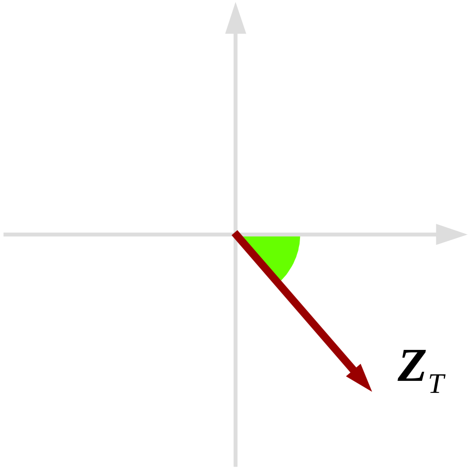
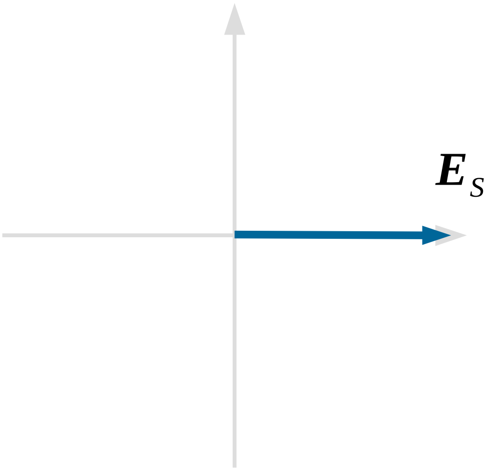
10
12
14
t


I_s


Z_T


E_S

Sinusoidal Functions

- **everlasting** sinusoid function

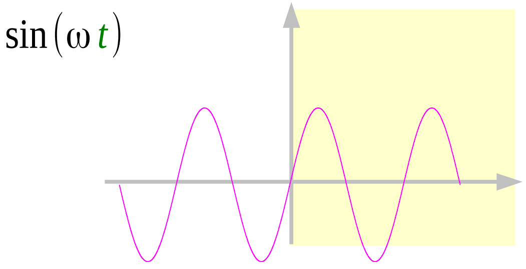
state at t = 0– = state at t = 0+

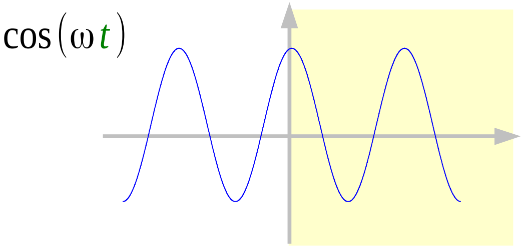
continuous state between t = 0– & 0+

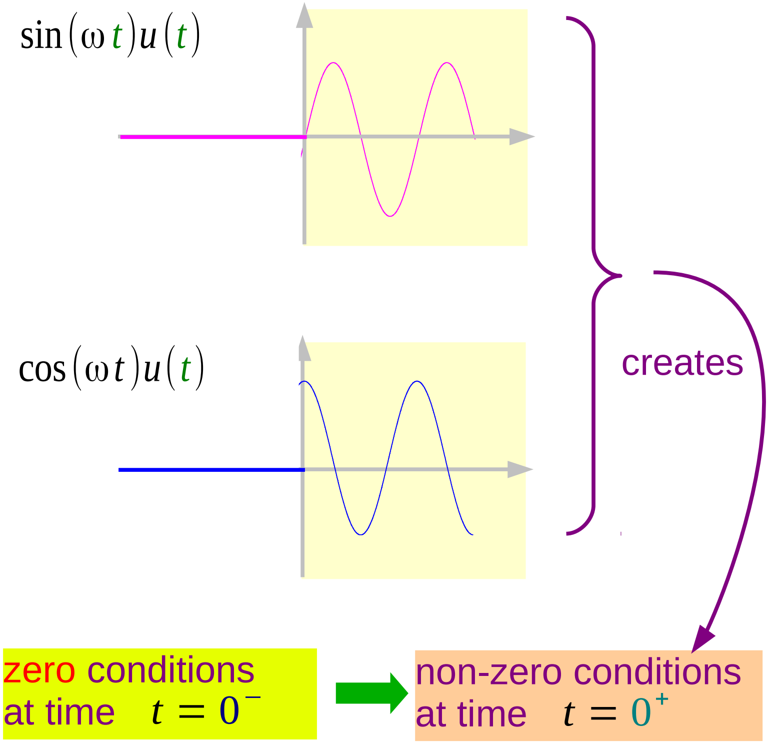
- **causal** sinusoid function

zero state at t = 0–

non-zero state at t = 0+





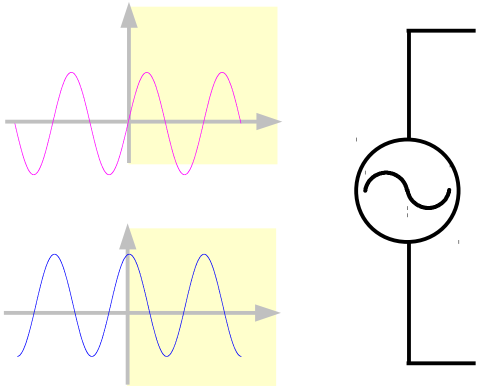


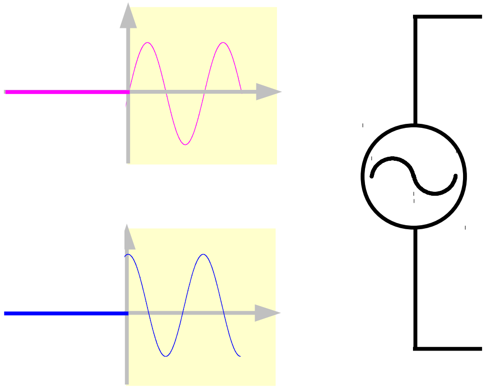
creates

zero conditions at time $t = 0^-$

non-zero conditions at time $t = 0^+$

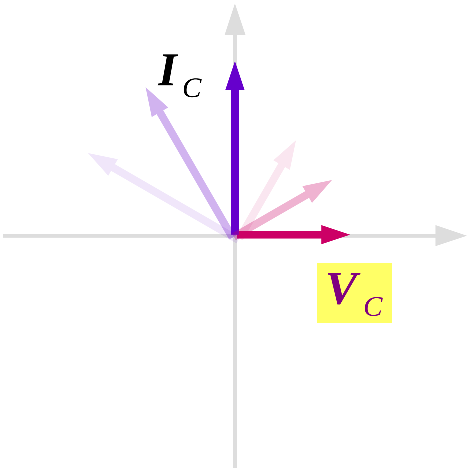
Steady State Response

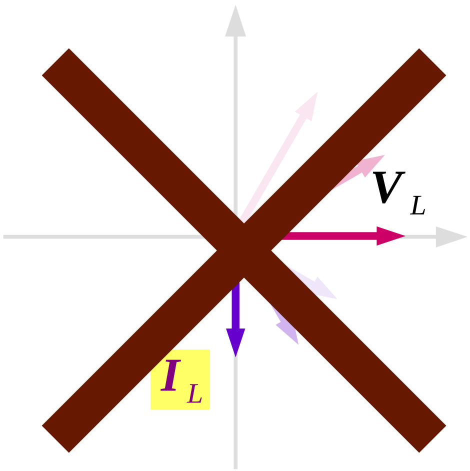




Can apply impedance method

<u>Cannot</u> apply impedance method





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003