Hyperbolic Functions (1A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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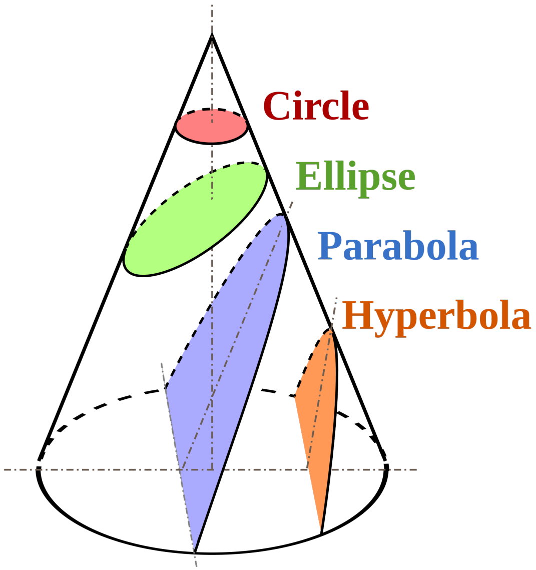
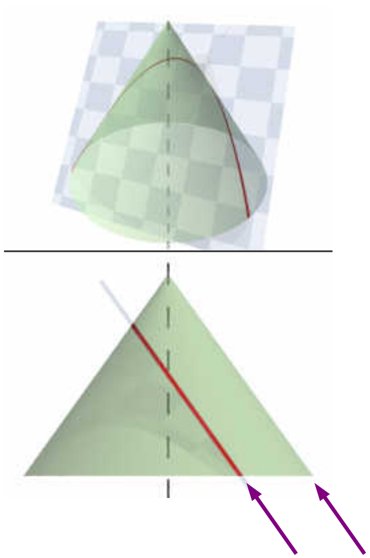
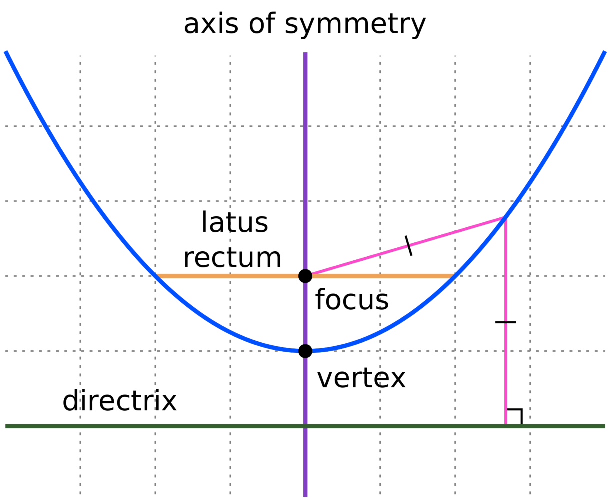
Parabola







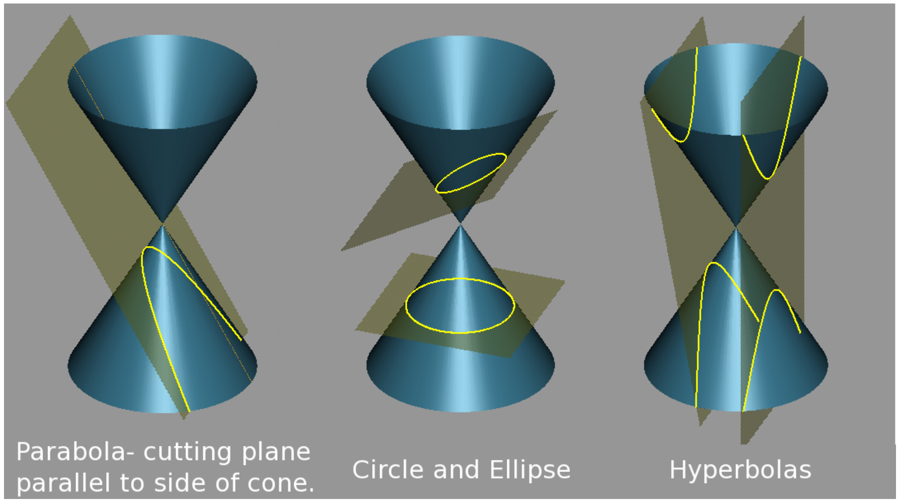
Parabola

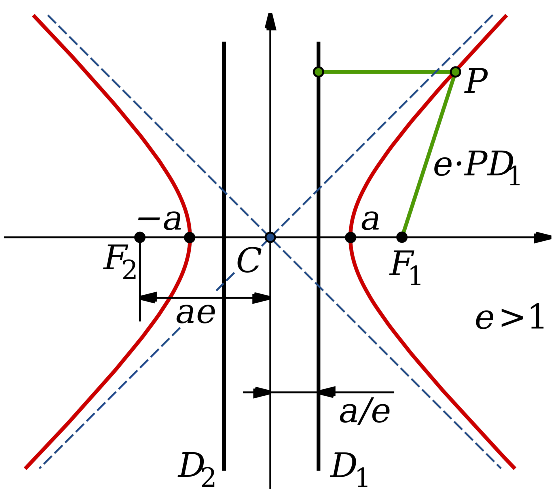
From Ancient Greek παραβολή (parabolē), from παραβάλλω (paraballō, “I set side by side”), from παρά (para, “beside”) + βάλλω (ballō, “I throw”).

The conic section formed by the intersection of a cone with a plane parallel to a tangent plane to the cone; the locus of points equidistant from a fixed point (the focus) and line (the directrix).

http://en.wikipedia.org/

Hyperbola





Hyperbola

From ὑπερβάλλω "I go beyond, exceed", from ὑπέρ "above" + βάλλω "I throw"

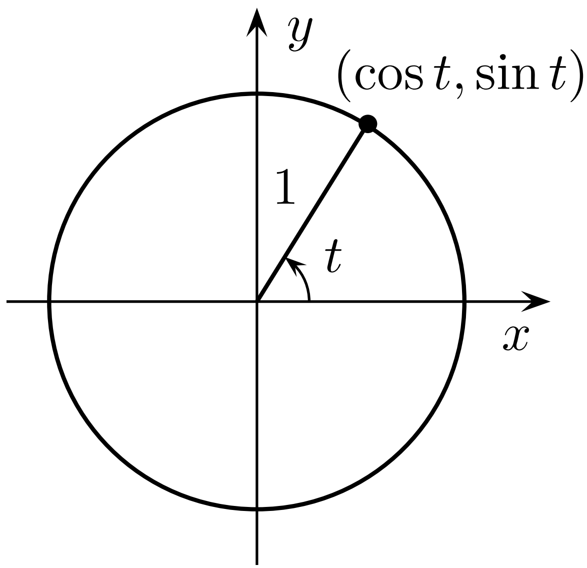
In mathematics, a hyperbola (plural hyperbolas or hyperbolae) is a type of smooth curve, lying in a plane, defined by its geometric properties or by equations for which it is the solution set. A hyperbola has two pieces, called connected components or branches, that are mirror images of each other and resemble two infinite bows.

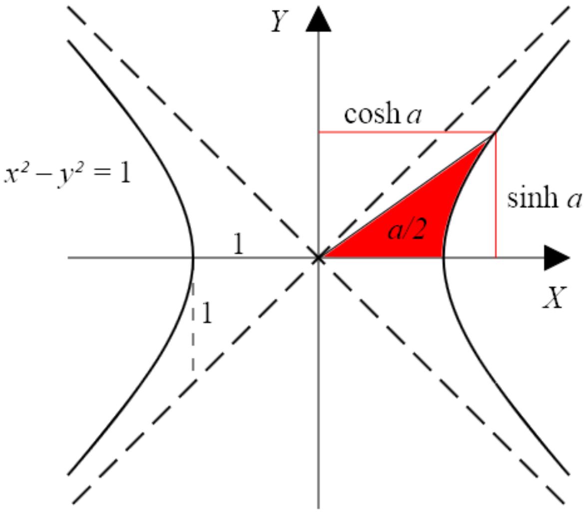
If the plane intersects both halves of the double cone but does **not** pass through the **apex** of the cones then the conic is a hyperbola.

A conic section formed by the intersection of a cone with a plane that intersects the base of the cone and is **not** tangent to the cone.

http://en.wikipedia.org/

Trigonometric & Hyperbolic Functions



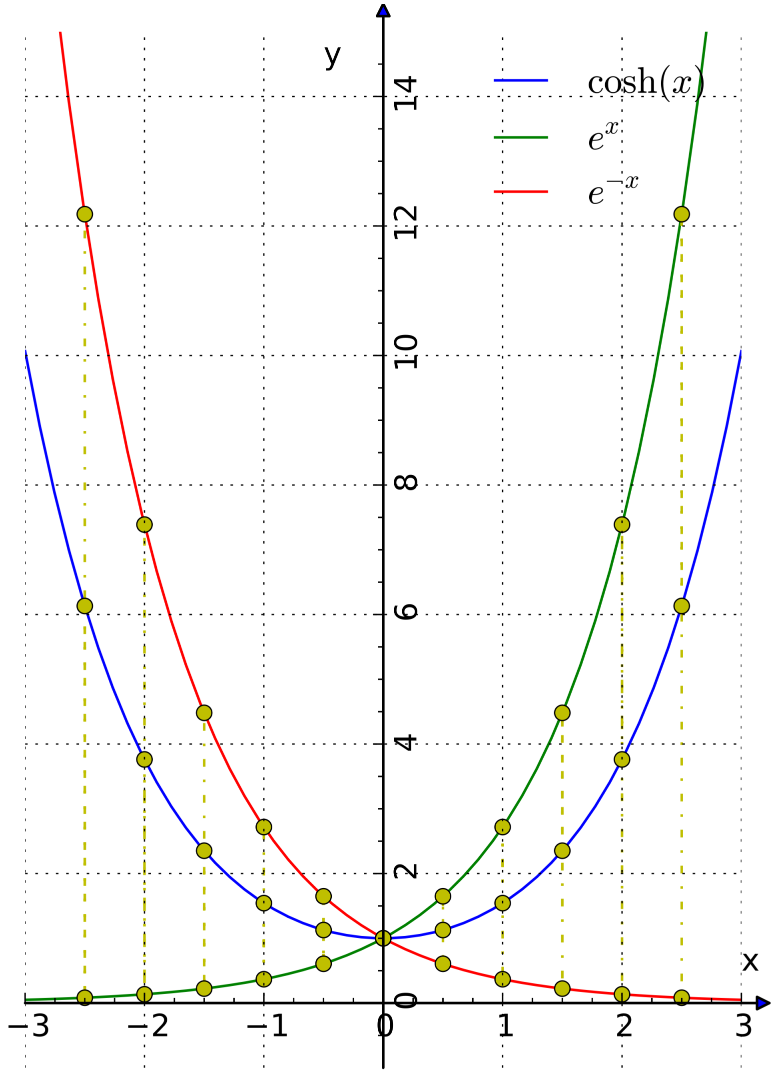


$$\cos^2\alpha + \sin^2\alpha = 1$$

$$\cosh^2\alpha - \sinh^2\alpha = 1$$

http://en.wikipedia.org/

cosh(x)

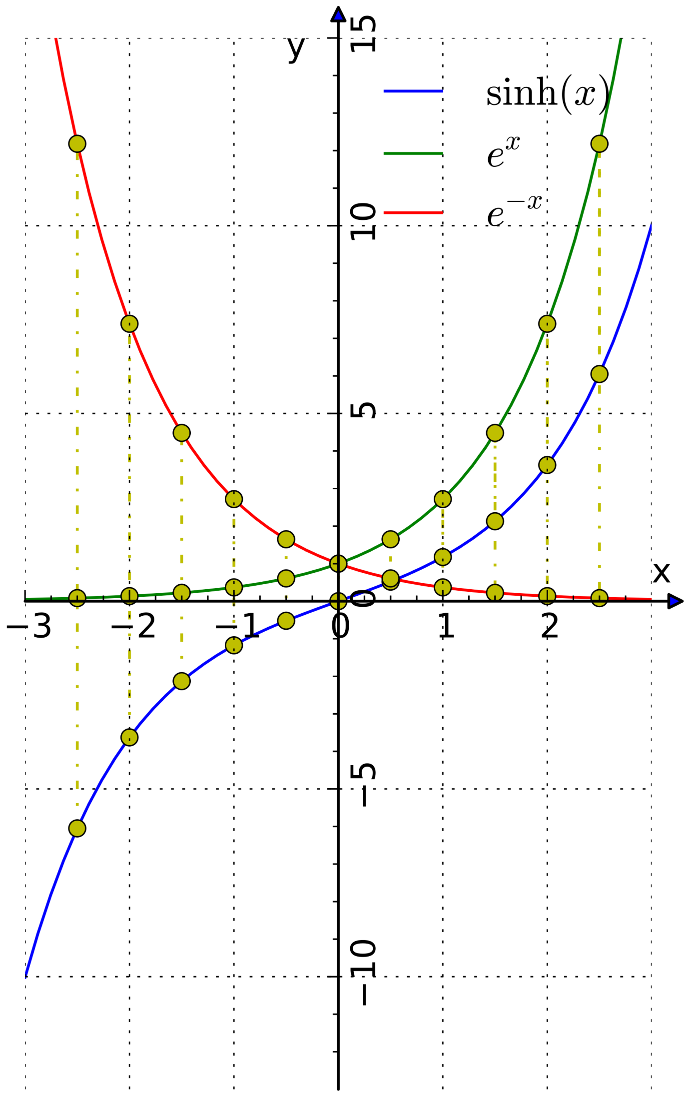


$$\cosh\alpha = \frac{1}{2}\left(e^{+\alpha} + e^{-\alpha}\right)$$

$$\cos\theta = \frac{1}{2}\left(e^{+j\theta} + e^{-j\theta}\right)$$

http://en.wikipedia.org/

sinh(x)

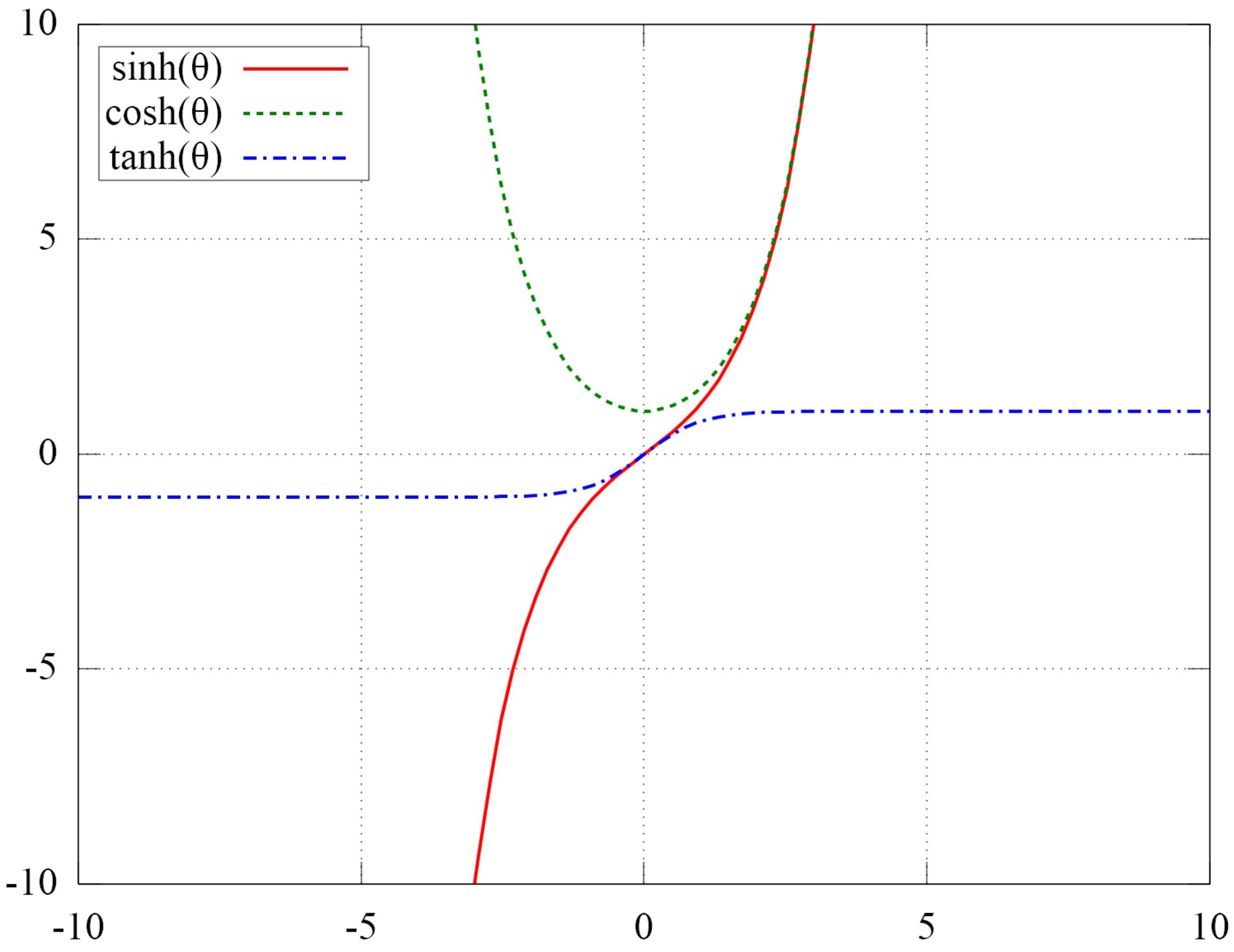


$$\sinh\alpha = \frac{1}{2}\left(e^{+\alpha} - e^{-\alpha}\right)$$

$$\sin\theta = \frac{1}{2j}\left(e^{+j\theta} - e^{-j\theta}\right)$$

http://en.wikipedia.org/

Definitions of Hyperbolic Functions



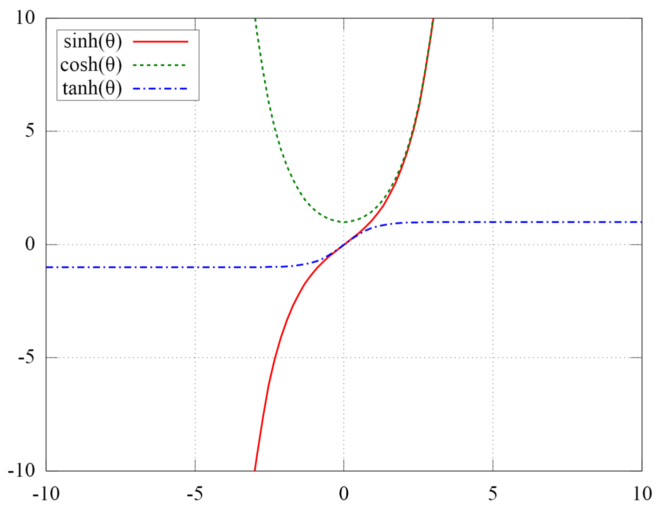
$$\cosh\alpha = \frac{1}{2}(e^{\alpha} + e^{-\alpha})$$

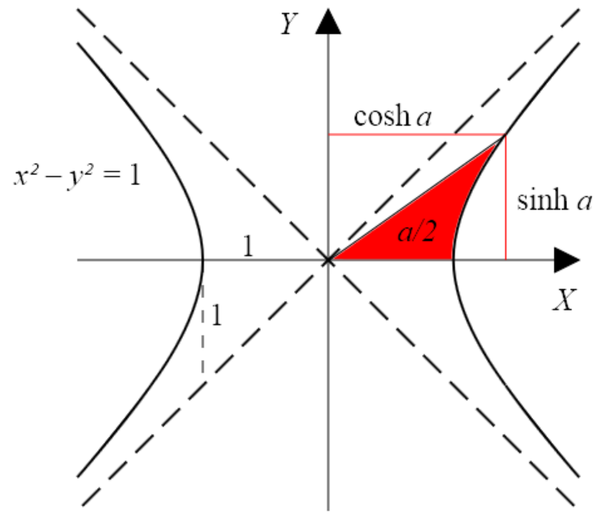
$$\sinh\alpha = \frac{1}{2}(e^{\alpha} - e^{-\alpha})$$

$$\tanh\alpha = \frac{(e^{\alpha} - e^{-\alpha})}{(e^{\alpha} + e^{-\alpha})}$$

http://en.wikipedia.org/

Definitions of Hyperbolic Functions





$$\cosh\alpha = \frac{1}{2}(e^{\alpha} + e^{-\alpha})$$

$$\sinh\alpha = \frac{1}{2}(e^{\alpha} - e^{-\alpha})$$

$$\tanh\alpha = \frac{(e^{\alpha} - e^{-\alpha})}{(e^{\alpha} + e^{-\alpha})}$$

$$x^2 - y^2 = 1$$

$$\cosh^2\alpha - \sinh^2\alpha = 1$$

$$\frac{1}{4}(e^{\alpha} + e^{-\alpha})^2 - \frac{1}{4}(e^{\alpha} - e^{-\alpha})^2 = 1$$

$(\cosh\alpha, \sinh\alpha)$

$\downarrow$ x $\quad$ $\downarrow$ y

http://en.wikipedia.org/

Hyperbolic vs. Trigonometric Functions

Trigonometric Function		*Hyperbolic Function*
$i\,x$		x
$e^{+ix} = \cos x + i \sin x$	⟺	$e^{+x} = \cosh x + \sinh x$
$e^{-ix} = \cos x - i \sin x$	⟺	$e^{-x} = \cosh x - \sinh x$
$\cos x = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right)$	⟺	$\cosh x = \frac{1}{2}\left(e^{+x} + e^{-x}\right)$
$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right)$	⟺	$\sinh x = \frac{1}{2}\left(e^{+x} - e^{-x}\right)$
$\tan x = \frac{1}{i}\frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)}$	⟺	$\tanh x = \frac{\left(e^{+x} - e^{-x}\right)}{\left(e^{+x} + e^{-x}\right)}$

Trigonometric functions with imaginary arguments

$i\,x \rightarrow x$

$$\cos x = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right) \quad \Leftrightarrow \quad \mathbf{cosh}\, x = \frac{1}{2}\left(e^{+x} + e^{-x}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right) \quad \Leftrightarrow \quad \mathbf{sinh}\, x = \frac{1}{2}\left(e^{+x} - e^{-x}\right)$$

$$\tan x = \frac{1}{i}\frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)} \quad \Leftrightarrow \quad \mathbf{tanh}\, x = \frac{\left(e^{+x} - e^{-x}\right)}{\left(e^{+x} + e^{-x}\right)}$$

$$\cos i\,x = \mathbf{cosh}\, x$$

$$\sin i\,x = i\,\mathbf{sinh}\, x$$

$$\tan i\,x = i\,\mathbf{tanh}\, x$$

$$\cos i\,x = \frac{1}{2}\left(e^{-x} + e^{+x}\right) \qquad \mathbf{cosh}\, x = \frac{1}{2}\left(e^{+x} + e^{-x}\right)$$

$$\sin i\,x = \frac{1}{2i}\left(e^{-x} - e^{+x}\right) \qquad \mathbf{sinh}\, x = \frac{1}{2}\left(e^{+x} - e^{-x}\right)$$

$$\tan i\,x = \frac{1}{i}\frac{\left(e^{-x} - e^{+x}\right)}{\left(e^{-x} + e^{+x}\right)} \qquad \mathbf{tanh}\, x = \frac{\left(e^{+x} - e^{-x}\right)}{\left(e^{+x} + e^{-x}\right)}$$

Hyperbolic functions with imaginary arguments

$$\cos x = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right)$$
$$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right)$$
$$\tan x = \frac{1}{i}\frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)}$$

$$\cosh x = \frac{1}{2}\left(e^{+x} + e^{-x}\right)$$
$$\sinh x = \frac{1}{2}\left(e^{+x} - e^{-x}\right)$$
$$\tanh x = \frac{\left(e^{+x} - e^{-x}\right)}{\left(e^{+x} + e^{-x}\right)}$$

$$x \leftarrow ix$$

$$\cos x = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right)$$
$$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right)$$
$$\tan x = \frac{1}{i}\frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)}$$

$$\cosh ix = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right)$$
$$\sinh ix = \frac{1}{2}\left(e^{+ix} - e^{-ix}\right)$$
$$\tanh ix = \frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)}$$

$$\cosh ix = \cos x$$
$$\sinh ix = i\sin x$$
$$\tanh ix = i\tan x$$

With imaginary arguments

$$\cos x = \frac{1}{2}\left(e^{+ix} + e^{-ix}\right) \quad \Leftrightarrow \quad \cosh x = \frac{1}{2}\left(e^{+x} + e^{-x}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{+ix} - e^{-ix}\right) \quad \Leftrightarrow \quad \sinh x = \frac{1}{2}\left(e^{+x} - e^{-x}\right)$$

$$\tan x = \frac{1}{i}\frac{\left(e^{+ix} - e^{-ix}\right)}{\left(e^{+ix} + e^{-ix}\right)} \quad \Leftrightarrow \quad \tanh x = \frac{\left(e^{+x} - e^{-x}\right)}{\left(e^{+x} + e^{-x}\right)}$$

x

$$\cos ix = \cosh x$$

$$\sin ix = i \sinh x$$

$$\tan ix = i \tanh x$$

$$\cosh ix = \cos x$$

$$\sinh ix = i \sin x$$

$$\tanh ix = i \tan x$$

ix

Euler Formula

Euler Formula

$$e^{+ix} = \cos x + i \sin x$$

$$e^{-ix} = \cos x - i \sin x$$

Euler Formula

$$e^{+ix} = \cosh ix + \sinh ix$$

$$e^{-ix} = \cosh ix - \sinh ix$$

$$\cos ix = \cosh x \qquad \cosh ix = \cos x$$

$$\sin ix = i \sinh x \qquad \sinh ix = i \sin x$$

$$\tan ix = i \tanh x \qquad \tanh ix = i \tan x$$

Modulus of sin(z) – (1)

$$\sin(z) = \sin(x+iy)$$
$$= \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$|\sin(z)|^2 = \sin(z)\,\overline{\sin(z)}$$
$$= \frac{1}{2i}\left(e^{+i(x+iy)} - e^{-i(x+iy)}\right)\frac{-1}{2i}\left(e^{-i(x-iy)} - e^{+i(x-iy)}\right)$$
$$= \frac{1}{4}\left(e^{-y+ix} - e^{+y-ix}\right)\left(e^{-y-ix} - e^{+y+ix}\right)$$
$$= \frac{1}{4}\left(e^{-2y} - e^{+2ix} - e^{-2ix} + e^{+2y}\right)$$
$$= \frac{1}{4}\left(e^{+2y} - 2 + e^{-2y} - e^{+2ix} + 2 - e^{-2ix}\right)$$
$$= +\frac{1}{4}\left(e^{+2y} - 2 + e^{-2y}\right) - \frac{1}{4}\left(e^{+2ix} - 2 + e^{-2ix}\right)$$
$$= \left[\frac{1}{2}\left(e^{+y} - e^{-y}\right)\right]^2 + \left[\frac{1}{2i}\left(e^{+ix} - e^{-ix}\right)\right]^2$$
$$= \sin^2(x) + \sinh^2(y)$$

Modulus of sin(z) – (2)

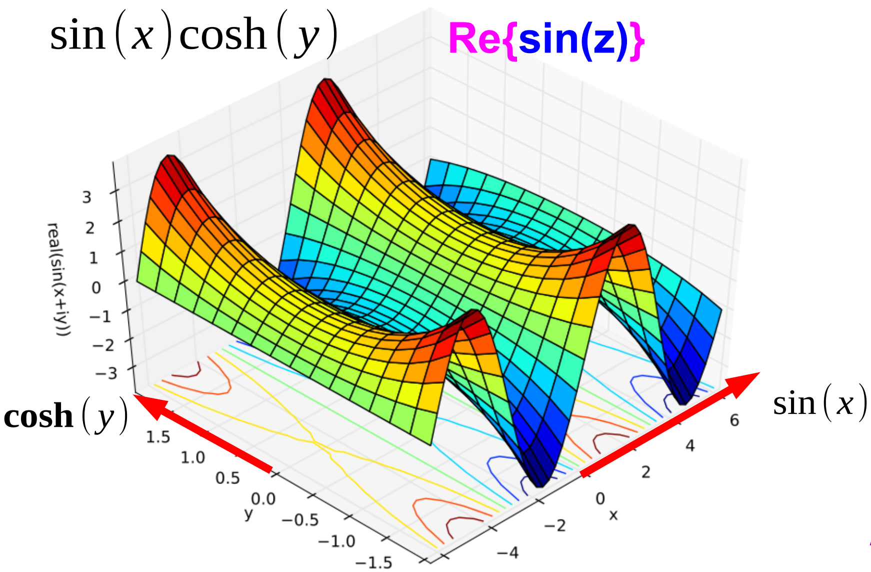
$$\sin(z) = \sin(x + i\,y)$$
$$= \sin(x)\cos(i\,y) + \cos(x)\sin(i\,y)$$
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

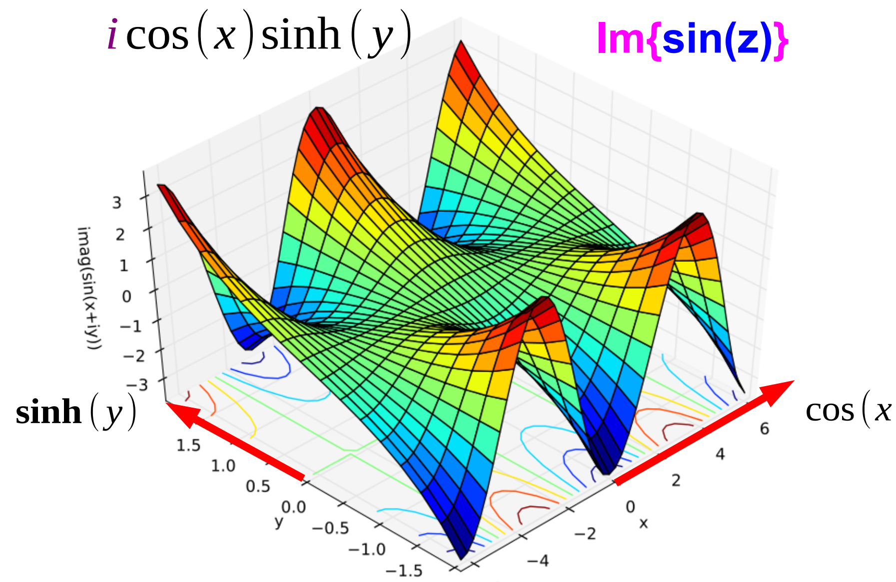
$$|\sin(z)|^2 = |\sin(x)\cosh(y) + i\cos(x)\sinh(y)|^2$$
$$= \sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)$$
$$= \sin^2(x)(1 + \sinh^2(y)) + (1 - \sin^2(x))\sinh^2(y)$$
$$= \sin^2(x) + \sin^2(x)\sinh^2(y) + \sinh^2(y) - \sin^2(x)\sinh^2(y)$$
$$= \sin^2(x) + \sinh^2(y)$$

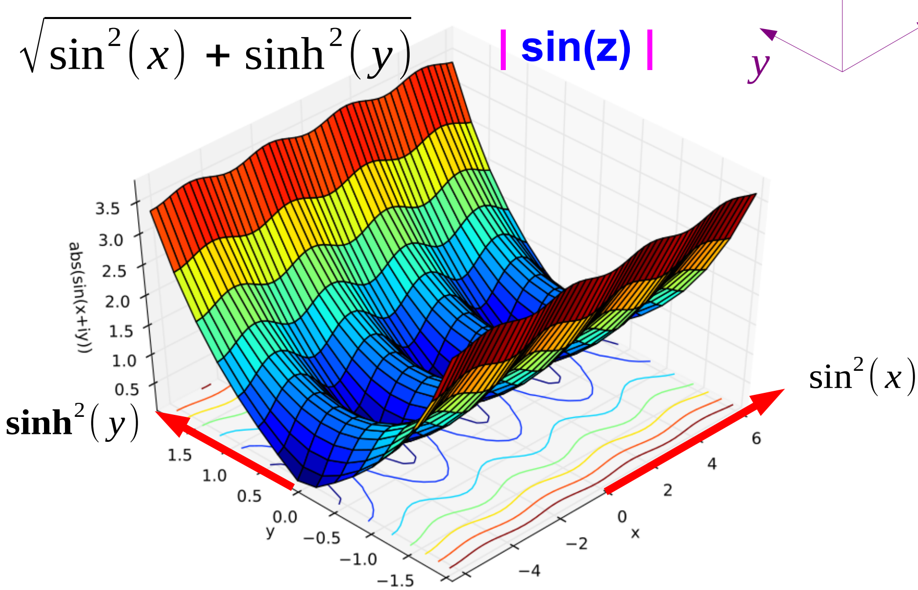
$$\mathbf{cosh}^2\alpha - \mathbf{sinh}^2\alpha = 1$$

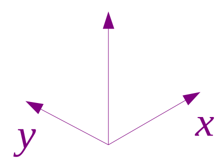
$$\cos^2\alpha + \sin^2\alpha = 1$$

Graphs of sin(z)









$$\sin(z) = \sin(x + i\,y)$$

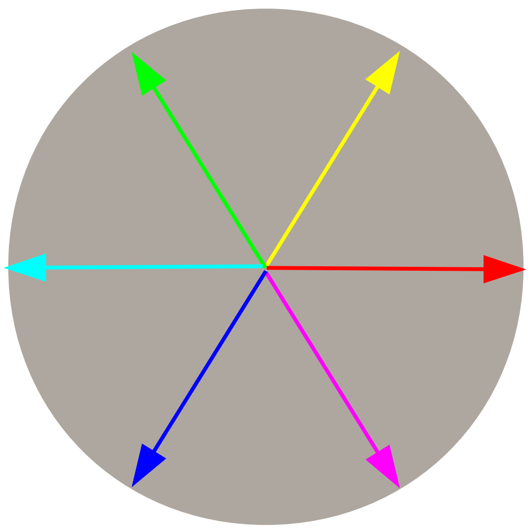
$$= \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$$

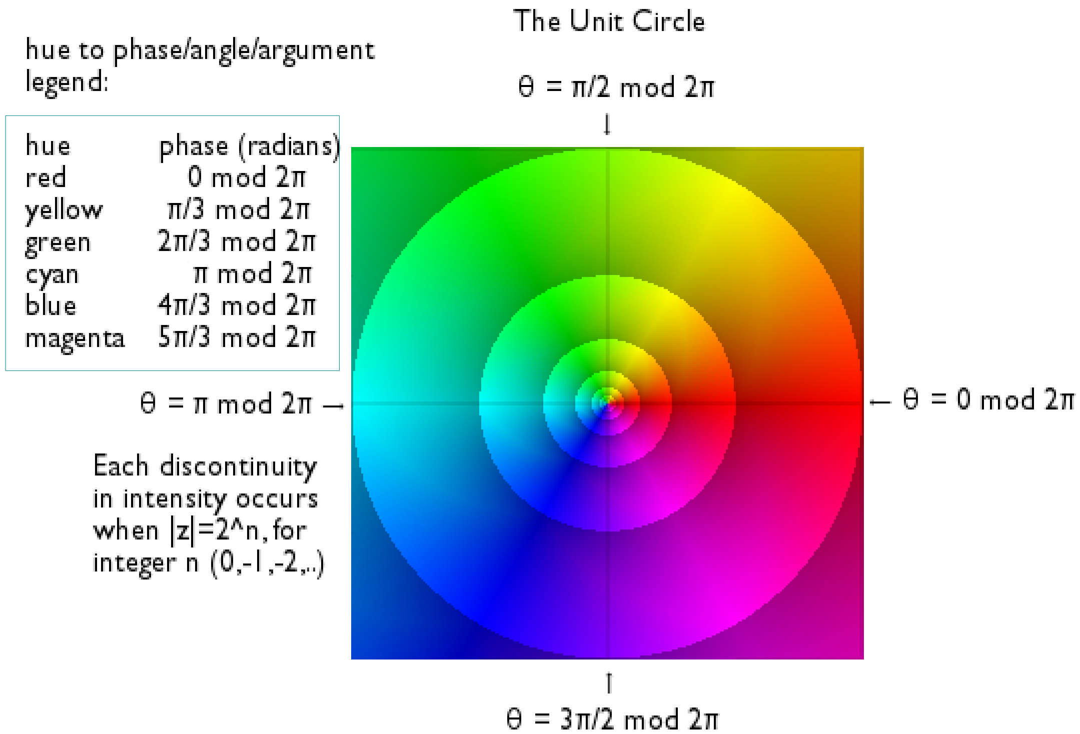
http://en.wikipedia.org/

Domain ↑ z

Coloring ↑ $f(z)$



Argument of f(z)

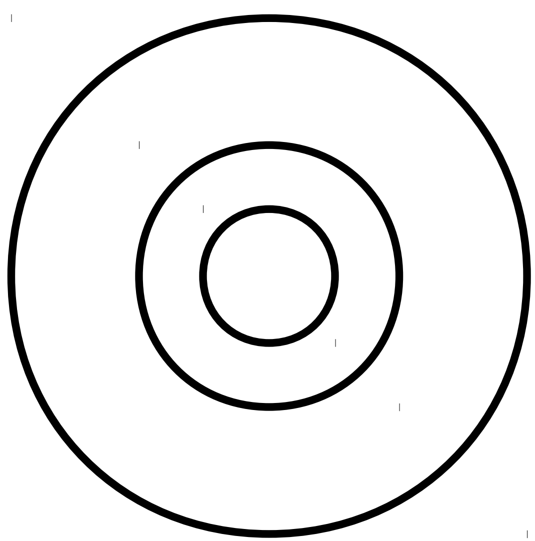


http://en.wikipedia.org/

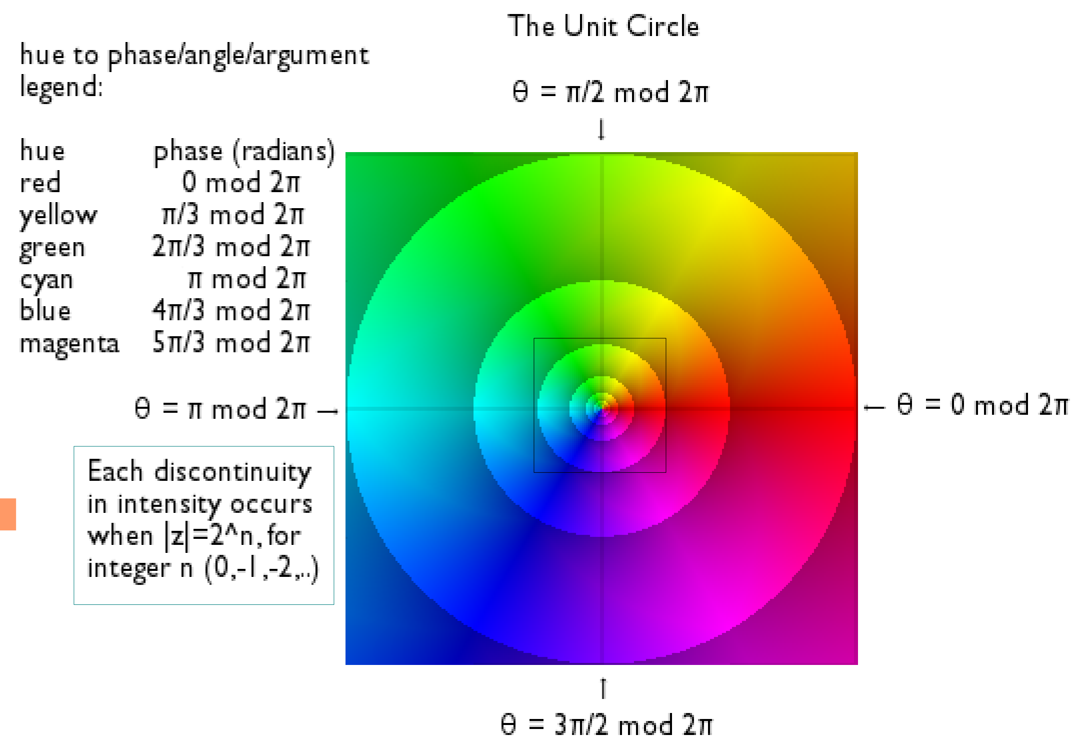
Domain Coloring – Modulus

Domain ← z

Coloring ← $f(z)$



Modulus of f(z)



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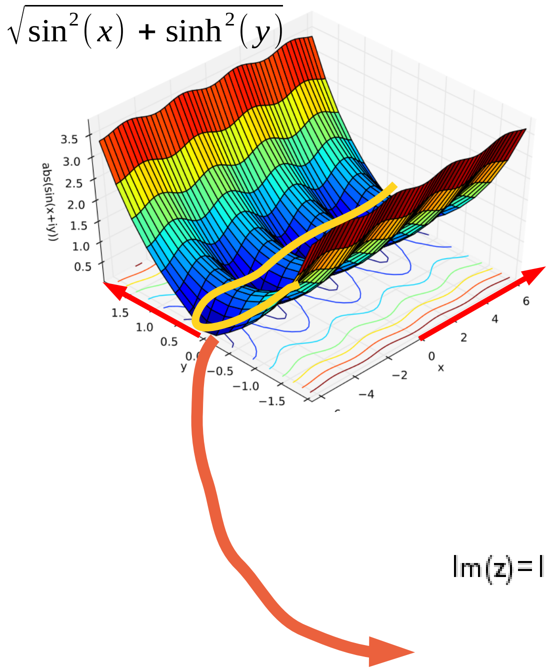
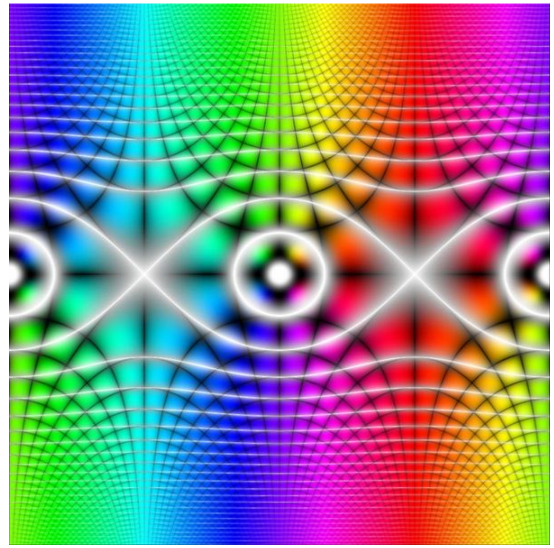
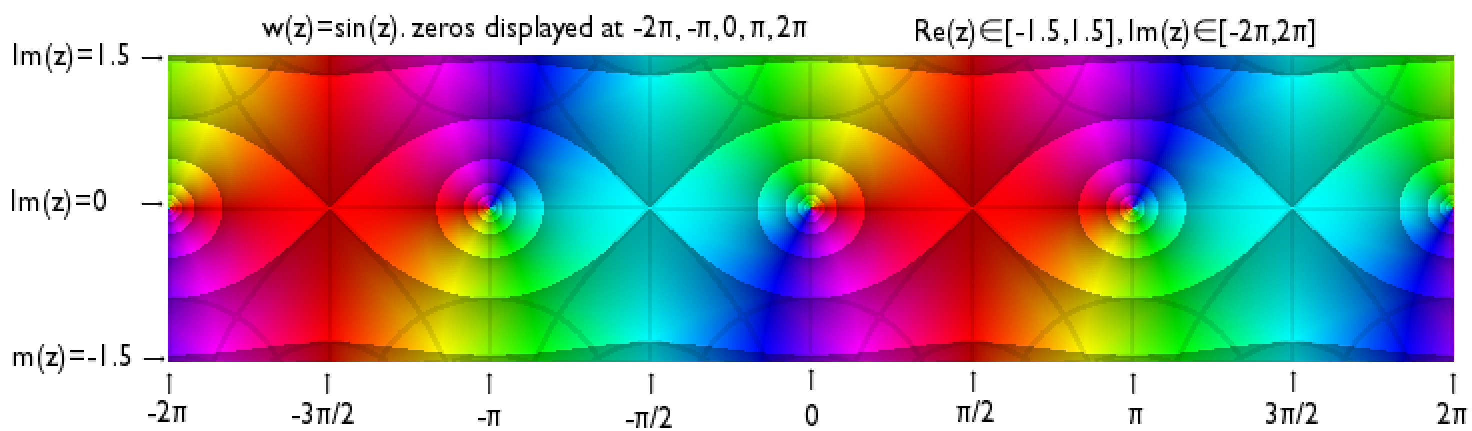
Domain Coloring of sin(z)



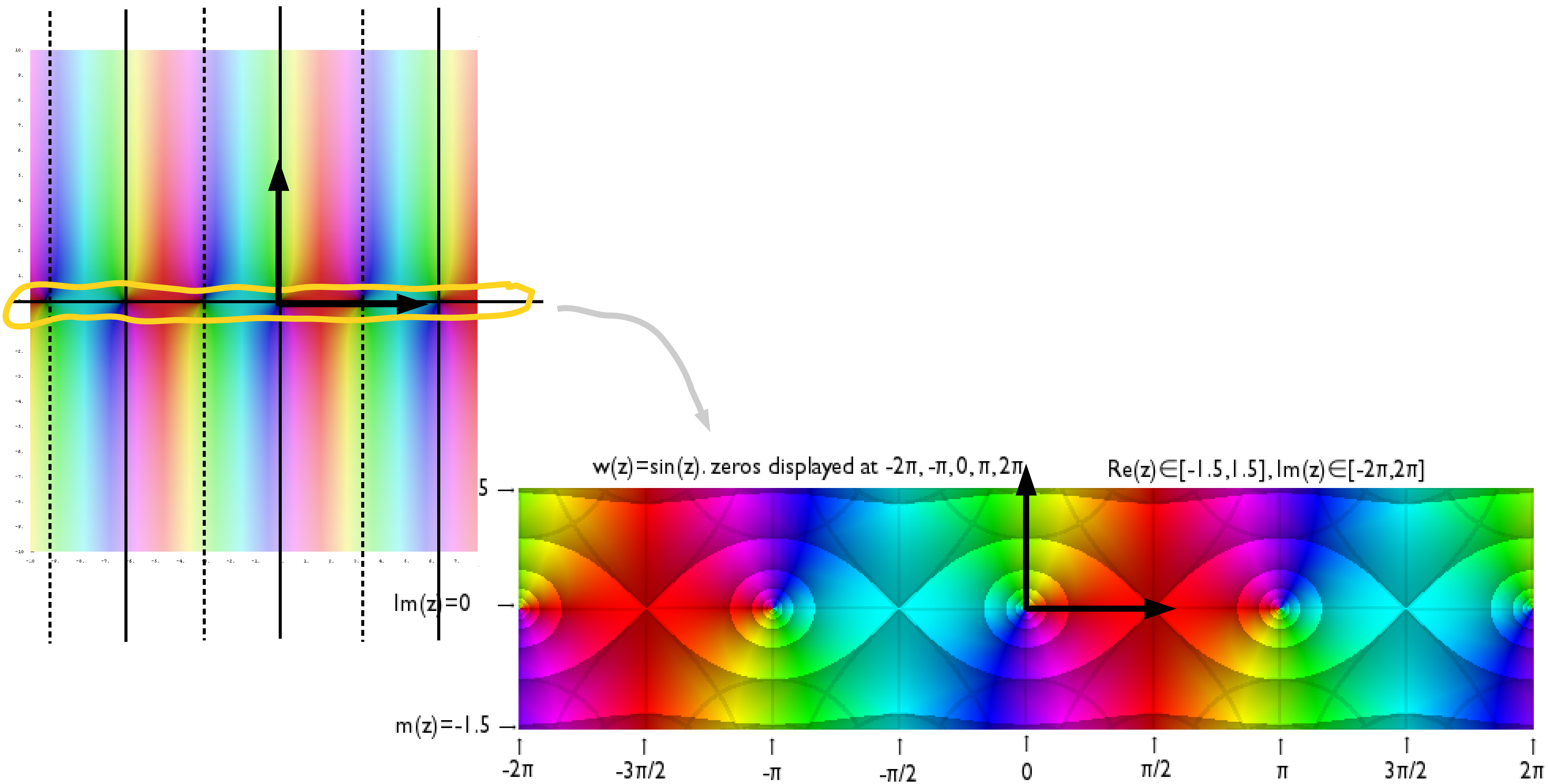


Domain coloring of sin(z) over (-π,π) on x and y axes. Brightness indicates absolute magnitude, saturation represents imaginary and real magnitude.



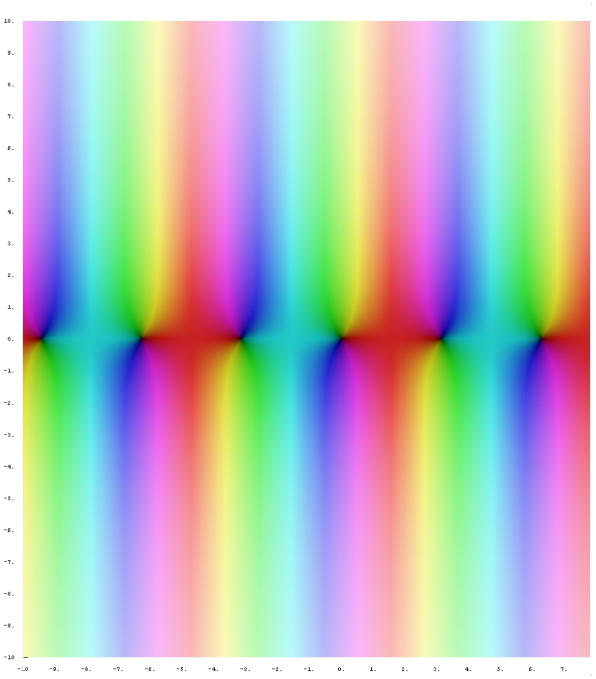
http://en.wikipedia.org/

Another domain coloring of sin(z)

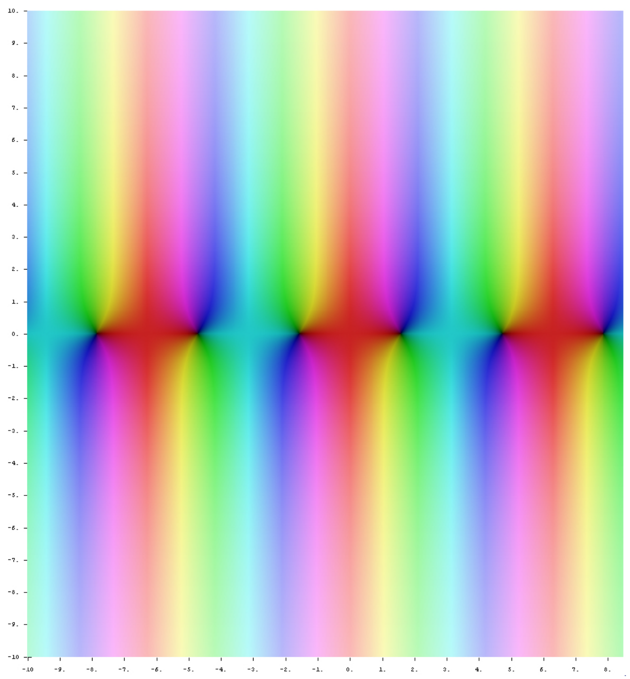


http://en.wikipedia.org/

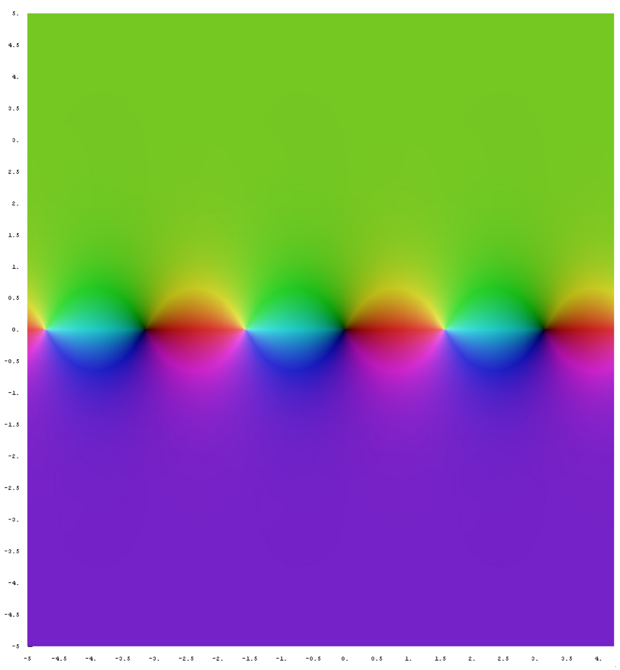
Domain Coloring of sin(x), cos(x), tan(x)



$\sin(z) = \sin(x + i\,y)$



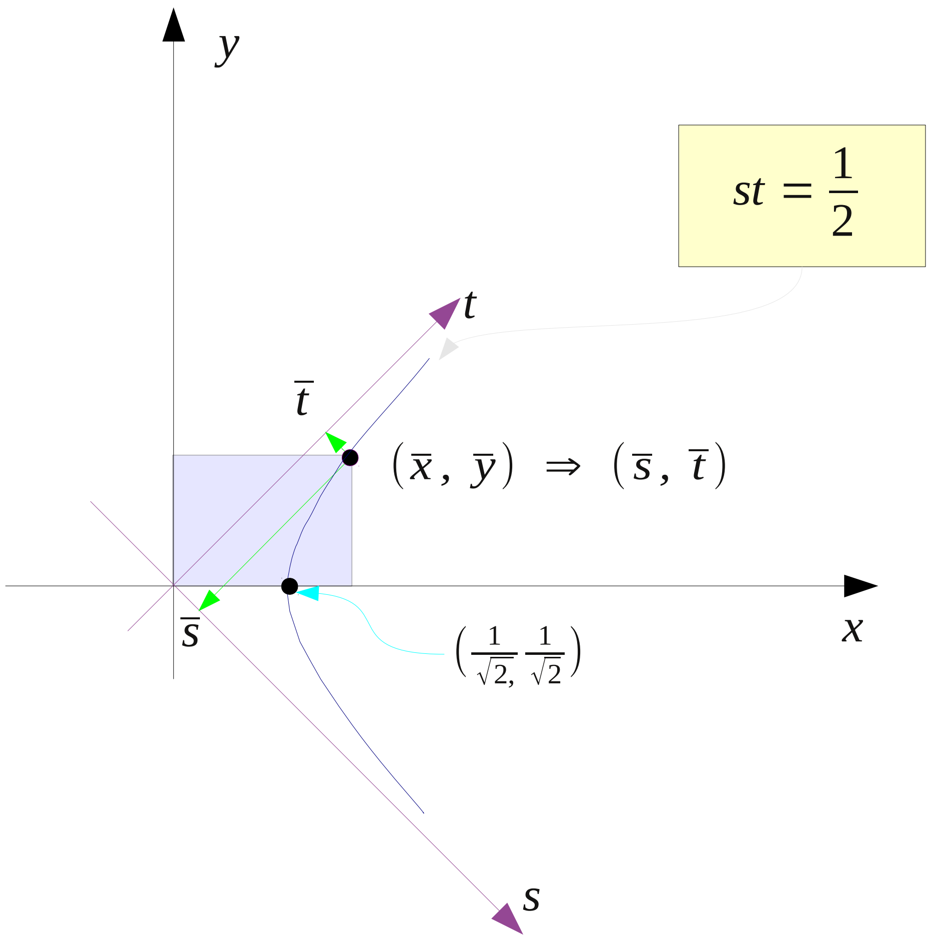
$\cos(z) = \cos(x + i\,y)$



$\tan(z) = \tan(x + i\,y)$

http://en.wikipedia.org/

Coordinates Changes



$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x - y = s\sqrt{2}$$

$$x + y = t\sqrt{2}$$

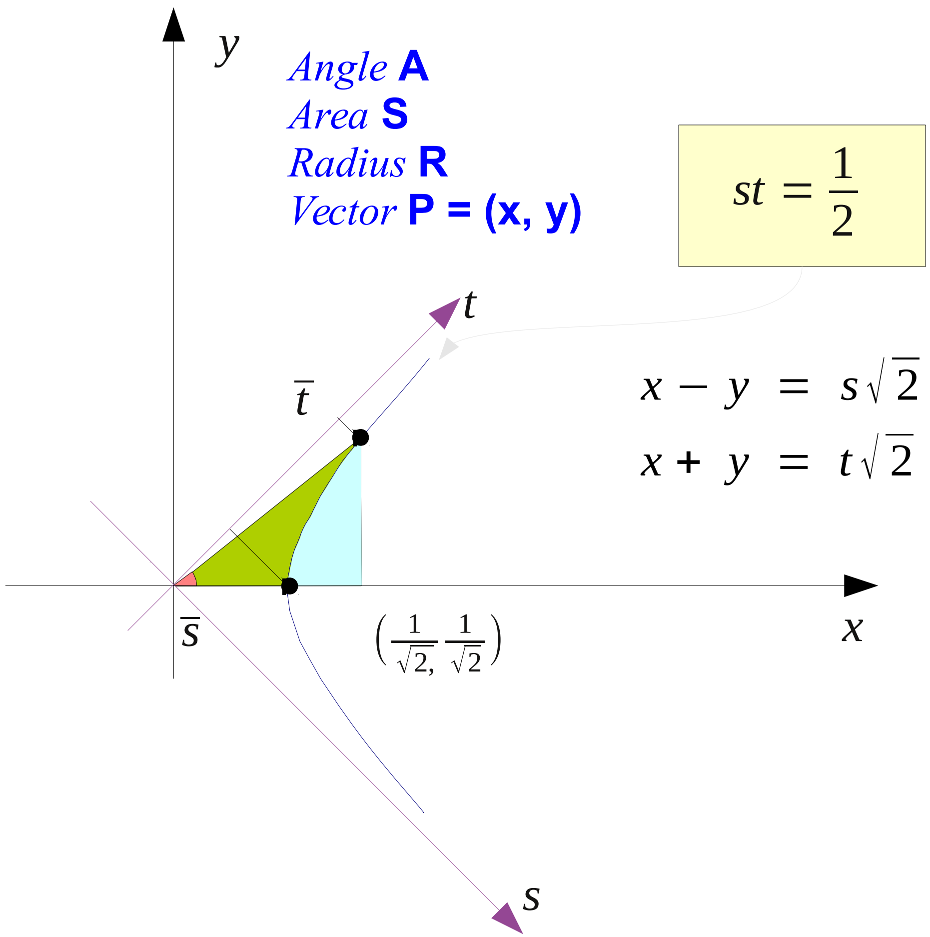
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\left(\frac{-\pi}{4}\right) & -\sin\left(\frac{-\pi}{4}\right) \\ \sin\left(\frac{-\pi}{4}\right) & \cos\left(\frac{-\pi}{4}\right) \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix}$$

$$+s + t = x\sqrt{2}$$

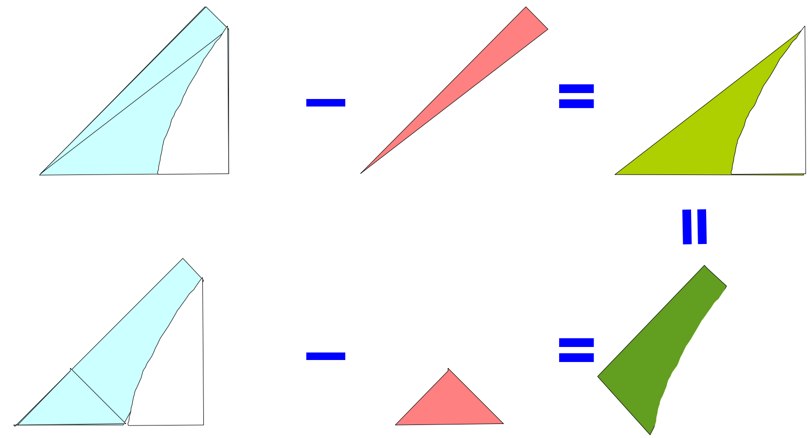
$$-s + t = y\sqrt{2}$$

Area:S Angle:A → 2S = A (1)



$$x - y = s\sqrt{2}$$

$$x + y = t\sqrt{2}$$



$$S = \int_{\frac{1}{\sqrt{2}}}^{t} \frac{1}{2s}\, ds = \int_{\frac{1}{\sqrt{2}}}^{\frac{(x+y)}{\sqrt{2}}} \frac{1}{2s}\, ds$$

$$2S = 2\int_{\frac{1}{\sqrt{2}}}^{\frac{(x+y)}{\sqrt{2}}} \frac{ds}{2s}$$

$$= \ln(x+y) = \ln\left(x \pm \sqrt{x^2 - 1}\right)$$

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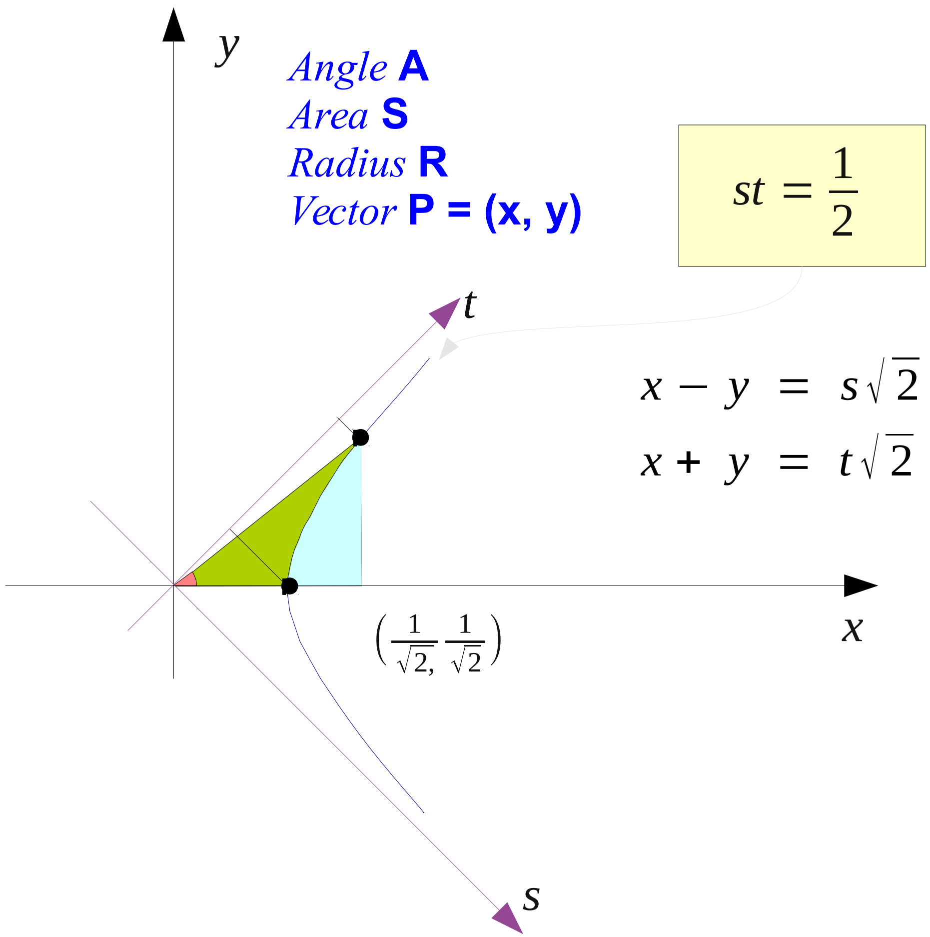
Area:S Angle:A → 2S = A (2)

$$2S = \ln(x + y) = \ln\left(x \pm \sqrt{x^2 - 1}\right)$$



$$st = \frac{1}{2}$$

$$x - y = s\sqrt{2}$$

$$x + y = t\sqrt{2}$$

$$\cosh A = x$$

$$A = \cosh^{-1} x$$

$$\cosh A = \frac{1}{2}(e^A + e^{-A}) = x$$

$$2x = u + \frac{1}{u} \qquad (u = e^A)$$

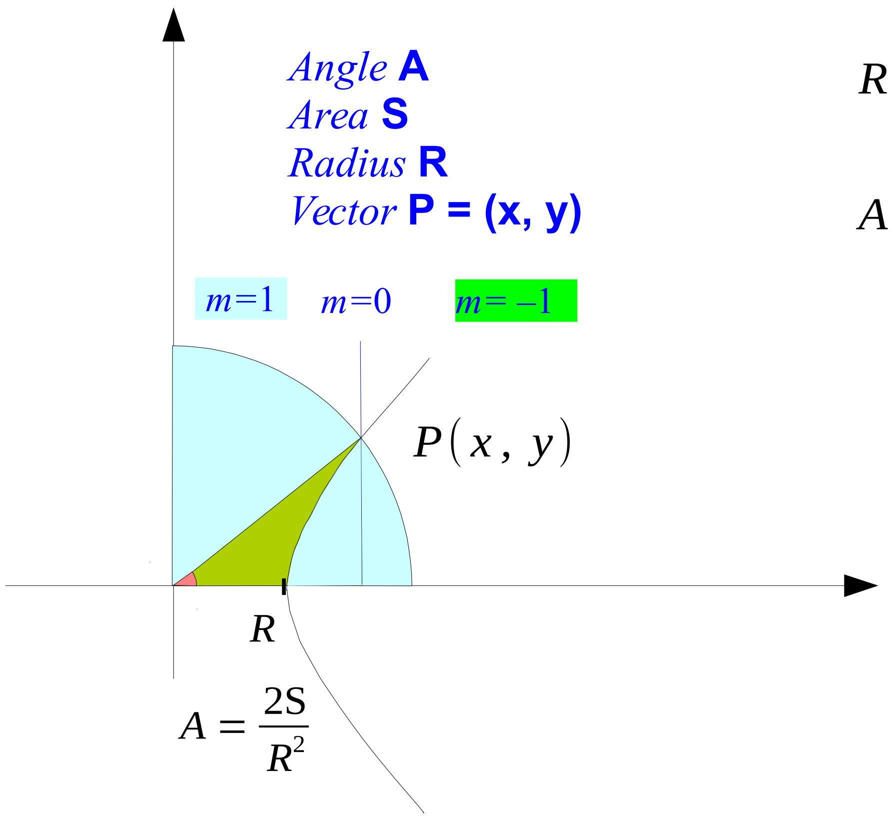
$$u^2 - 2xu + 1 = 0$$

$$u = x \pm \sqrt{x^2 - 1} \Rightarrow e^A$$

$$A = \ln\left(x \pm \sqrt{x^2 - 1}\right)$$

$$\boxed{2S = A} = \ln\left(x \pm \sqrt{x^2 - 1}\right)$$

Area:S Angle:A → 2S = A (3)



$$R = (x^2 + y^2)^{1/2}$$

$$A = \tan^{-1}\left(\frac{y}{x}\right)$$

$$R = (x^2 - y^2)^{1/2}$$

$$A = -\tan^{-1}\left(-\frac{y}{x}\right)$$

References

[1] http://en.wikipedia.org/
[2] J. S. Walther, A Unified Algorithm for Elementary Functions
[3] J. Calvert, http://mysite.du.edu/~jcalvert/math/hyperb.htm