Hybrid CORDIC
3. ROMless

20180303

Implementation of Hybrid CORDIC

CORDIC PROC-I : the coarse rotation ROM based

θ_H { ROM look-up operation
addition

CORDIC PROC-II : the fine rotation SHIFT-ADD

θ_L sequence of shift-and-add

no computation of the direction of micro-rotation
the need of a micro-rotation is explicit ($= d_i$)
in the radix-2 representation

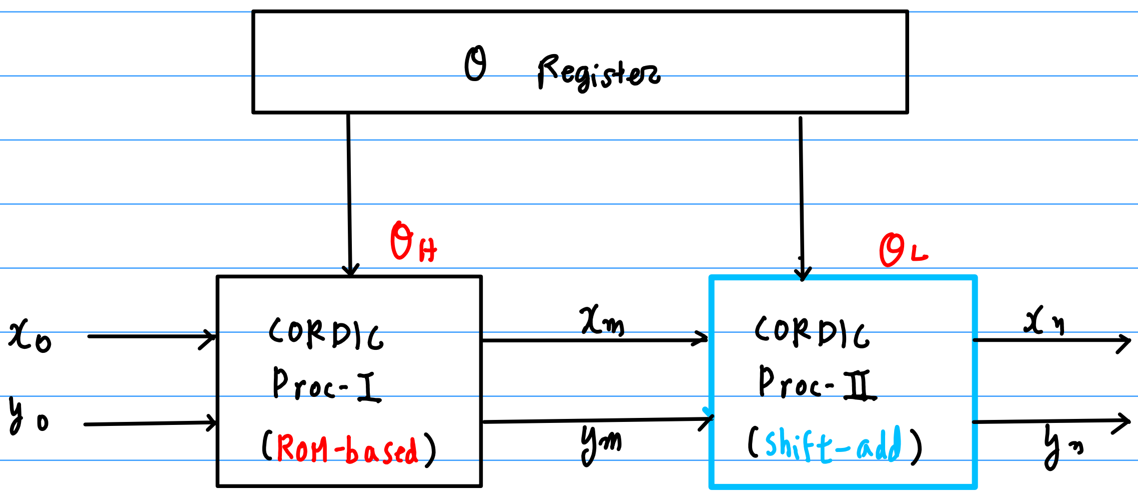
$$\theta_L = \sum_{i=p}^{n-1} d_i \, 2^{-i} \qquad d_i \in \{0, 1\}$$

* In the signed digit notation [Timmermann Low latency]

$$\theta_L = \sum_{i=p}^{n-1} \tilde{b}_i \, 2^{-i} \qquad \tilde{b}_i = \{-1, +1\}$$

$$\theta_L = \sum_{i=p}^{n-1} d_i \, 2^{-i} \qquad d_i \in \{0, 1\} \qquad ① \text{ radix-2 representation}$$

$$= \sum_{i=p}^{n-1} \tilde{b}_i \, 2^{-i} \qquad \tilde{b}_i = \{-1, +1\} \qquad ② \text{ signed digit notation}$$



* the direction is explicit
  → parallel implementation possible

Timmermann, Low Latency time CORDIC algorithm, 1992

* the hybrid decomposition could be used

  ① ROM-based realization of coarse rotation
  → minimize latency

  ② Shift-and-add implementation of fine rotation
  → minimize hardware complexity
  ← no need to find the rotation direction

[23] M. Kuhlmann and K. K. Parhi, "P-CORDIC: A precomputation based rotation CORDIC algorithm," *EURASIP J. Appl. Signal Process.*, vol. 2002, no. 9, pp. 936–943, 2002.

very high precision

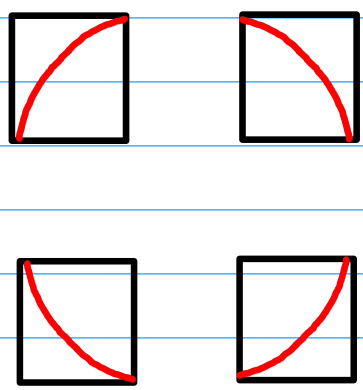
ROM size $n \cdot 2^{n/5}$ bits

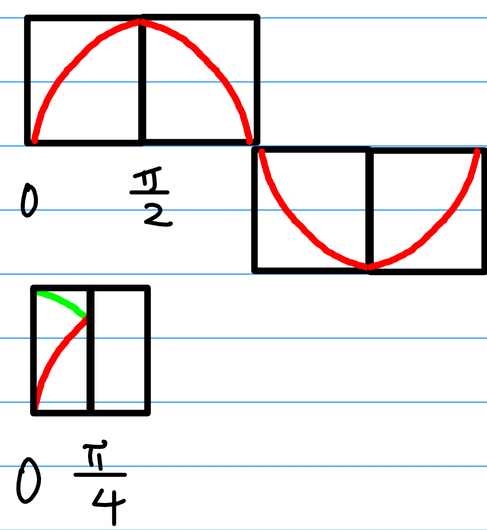
if latency is tolerable
  the conventional CORDIC
    shift-and-add operation

Shift-Add Implementation of <u>Coarse rotation</u>

Using the symmetry properties
of cosine and sine functions
in different quadrants

rotation through arbitrary angle θ
can be mapped from $[0, 2\pi]$
to the first half of the first quadrant $[0, \frac{\pi}{4}]$





$[0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$

Madisetti's approach

assumption arbitrary positive angle

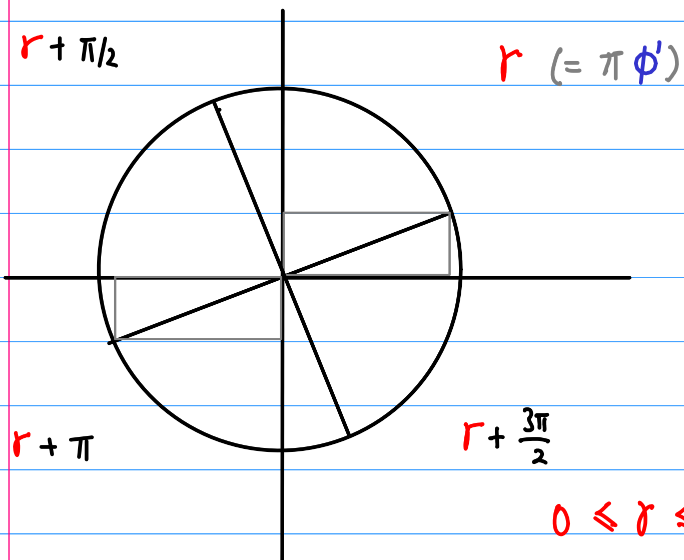
$$\boxed{\theta < 1 \text{ (rad)}}$$

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

$$= \sum_{k=1}^{N} b_k 2^{-k}$$

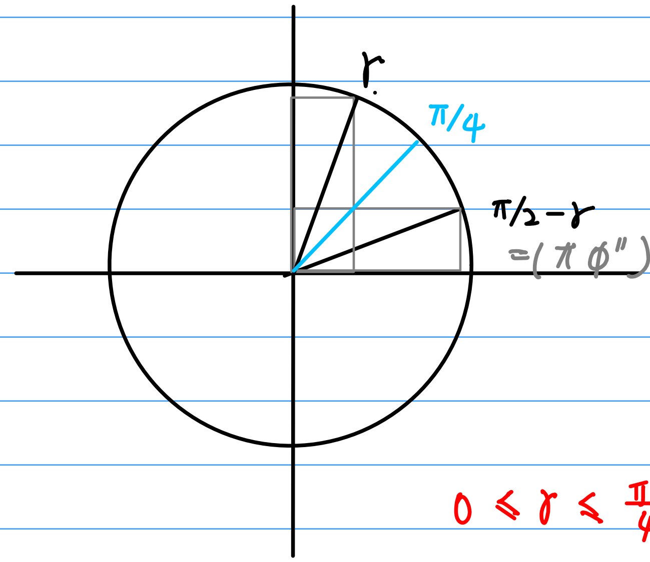
$$= \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Quadrant symmetry



$0 \leqslant \gamma \leqslant \frac{\pi}{2}$

$\pi/4$ mirror



$0 \leqslant \gamma \leqslant \frac{\pi}{4}$

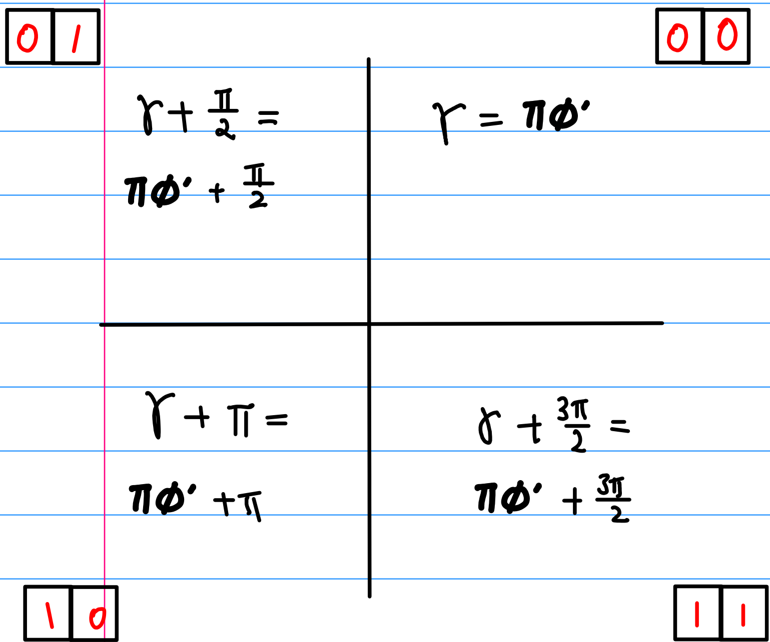
$[-\pi, +\pi] \rightarrow [0, \pi/2]$

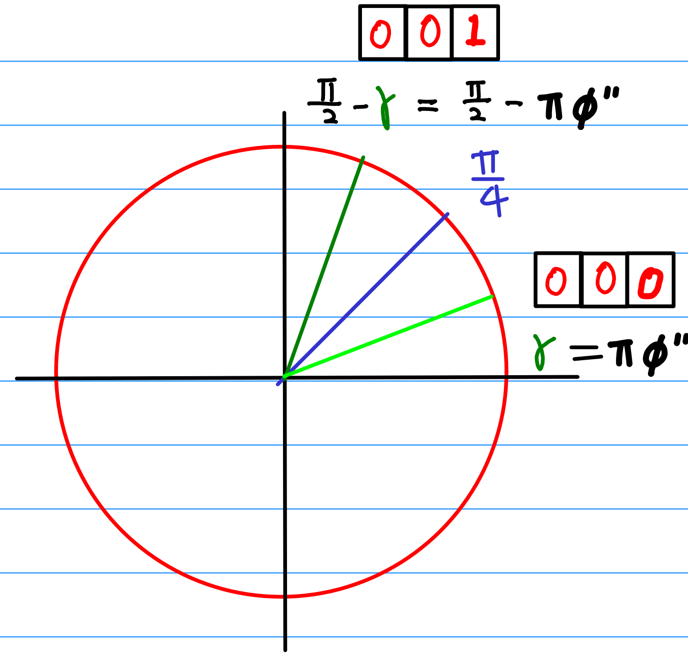
$[0, \pi/2] \rightarrow [0, \pi/4]$

MSB_1	MSB_2
$\boxed{0}$	$\boxed{0}$
0	1
1	0
1	1

MSB_3	
1	$\phi'' = 0.5 - \phi'$
0	$\phi'' = \phi'$

MSB_1	MSB_2	MSB_3

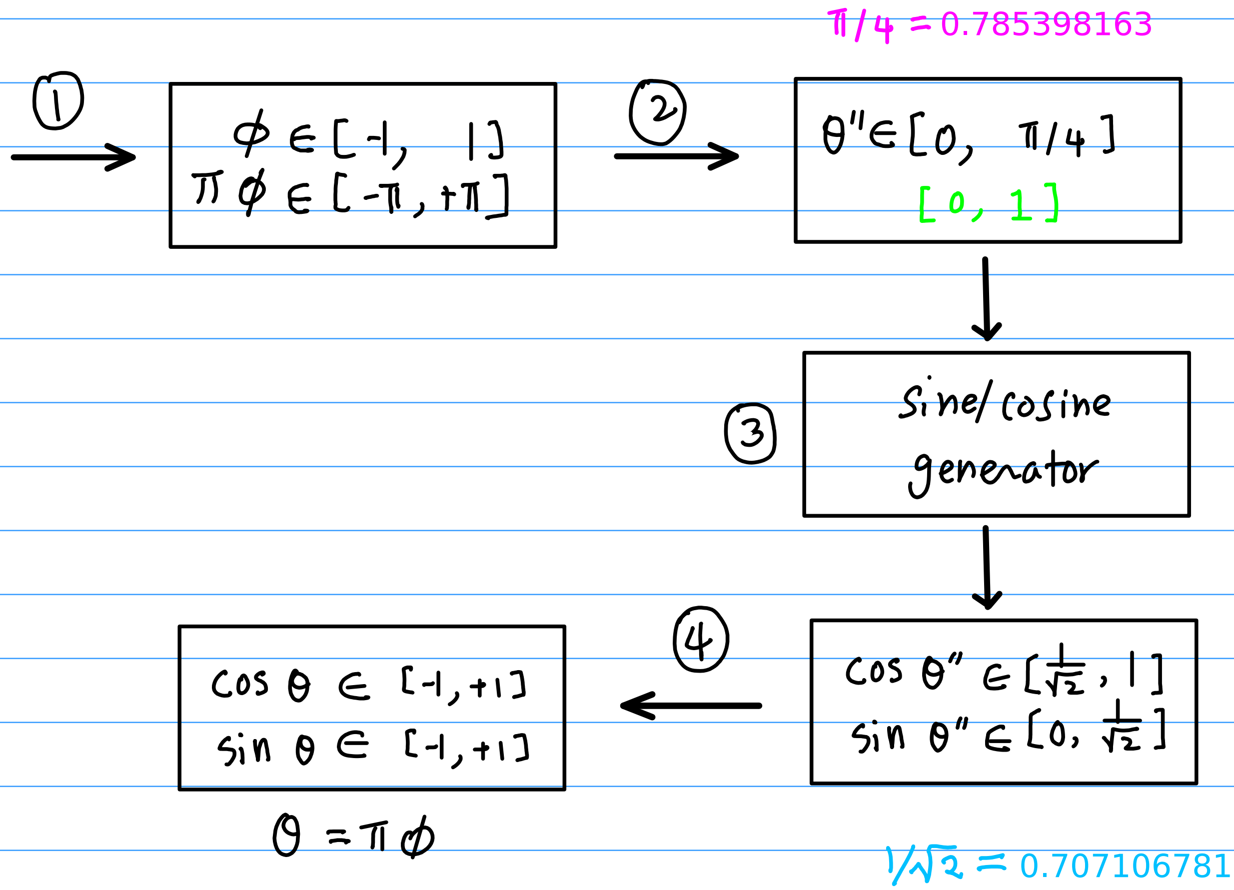




$$\theta = \pi\phi \longrightarrow \theta' = \pi\phi' \longrightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \longrightarrow \theta' = [0, \pi/2] \longrightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

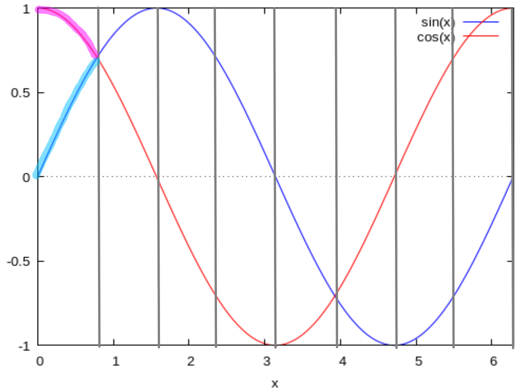


① a phase accumulator	$\phi \in [-1, 1]$
② a radian converter	$\theta'' \in [0, 1]$
③ a sine/cosine generator	$\sin\theta''$, $\cos\theta''$
④ an output stage	$\sin\theta$, $\cos\theta$

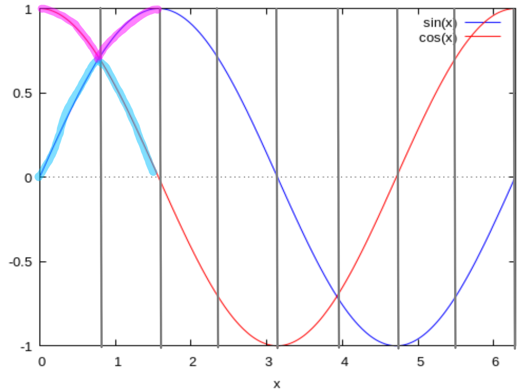
$\theta'' \in [0, \ \pi/4]$ $\quad \cos\theta'' \in [\frac{1}{\sqrt{2}}, 1]$

$\sin\theta'' \in [0, \frac{1}{\sqrt{2}}]$

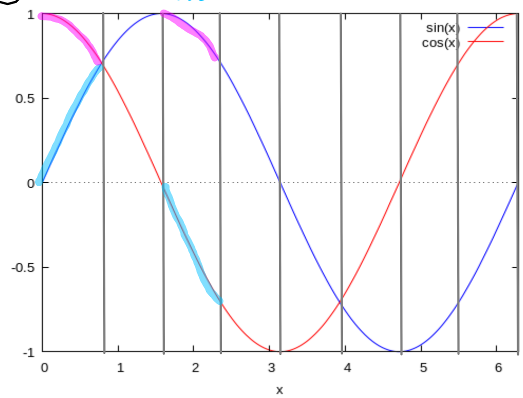
⓪ $\cos\theta$ $\sin\theta$



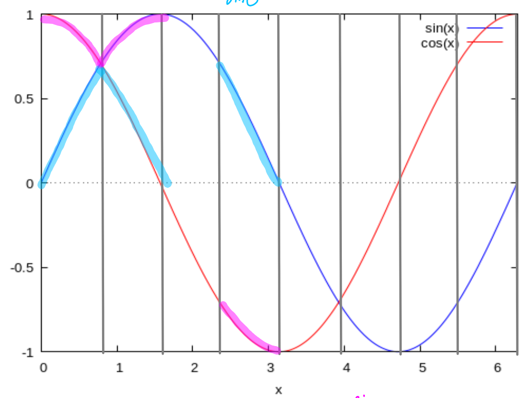
① $\sin\theta$ $\cos\theta$



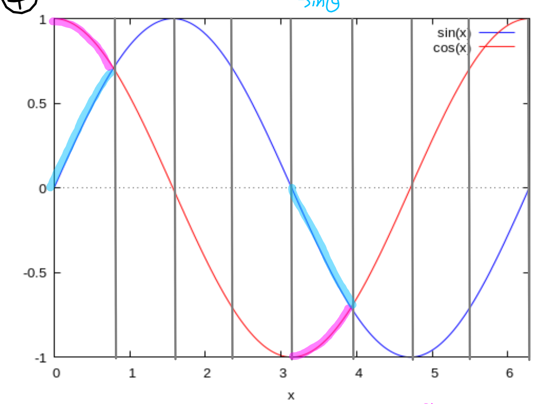
② $-\sin\theta$ $\cos\theta$



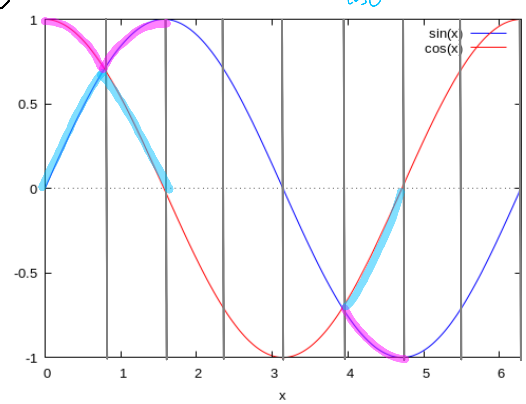
③ $-\cos\theta$ $\sin\theta$



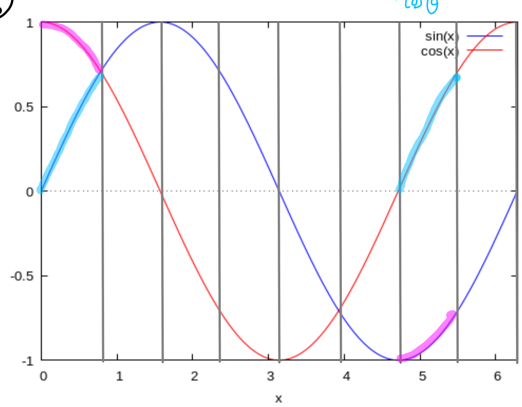
④ $-\cos\theta$ $-\sin\theta$



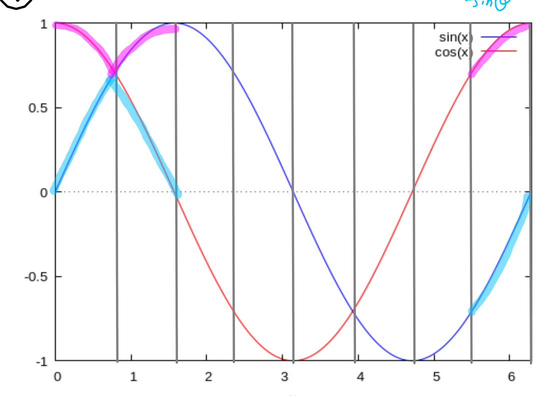
⑤ $-\sin\theta$ $-\cos\theta$



⑥ $\sin\theta$ $-\cos\theta$



⑦ $\cos\theta$ $-\sin\theta$



(+) Subrotation by 2^{-k}

2 equal half rotations by 2^{-k-1}

(+)(+)

(0) Subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

(+)(−) (−)(+)

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

$\theta'' \in [0, \pi/4]$

$[0, 1]$

⋮

fixed

k-th rotation	①	Pos $2^{-(k+1)}$ rotation	Pos $2^{-(k+1)}$ rotation	← $b_k = 1$
	②	Pos $2^{-(k+1)}$ rotation	neg $2^{-(k+1)}$ rotation	← $b_k = 0$

⋮

Combining all the fixed rotations

→ initial fixed rotation

Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0 $\boxed{\sum_{k=2}^{N+1} 2^{-k}}$

a sequence of ⊕/⊖ rotations $\boxed{\sum_{k=2}^{N+1} r_k 2^{-k}}$

$b_k = 1$ $\quad + 2^{-(k+1)}$ pos rotation $\quad r_{k+1} = +1$

$b_k = 0$ $\quad - 2^{-(k+1)}$ neg rotation $\quad r_{k+1} = -1$

$$r_k = (2 b_{k-1} - 1)$$

$2 \cdot 1 - 1 = +1$ $\qquad b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $\qquad b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

Simply replacing $b_k = 0$ with $r_{k+1} = -1$

This recoding maintains

. a constant scaling factor K

$$\boxed{\theta'' \in [0, \pi/4] \quad [0, 1]}$$

initial fixed rotation

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

binary digit representation

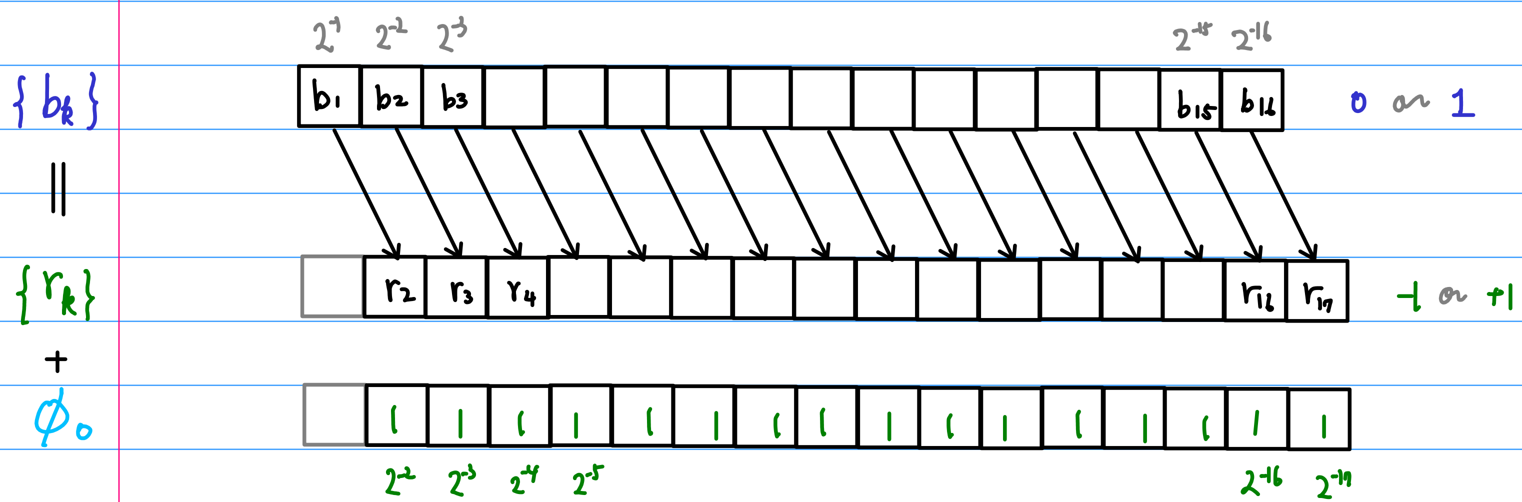
$b_k \in \{0, 1\}$

Signed digit recoding

$r_k \in \{-1, +1\}$

$$\boxed{b_k = (r_{k+1} + 1)/2}$$

$$\boxed{r_k = (2b_{k-1} - 1)}$$

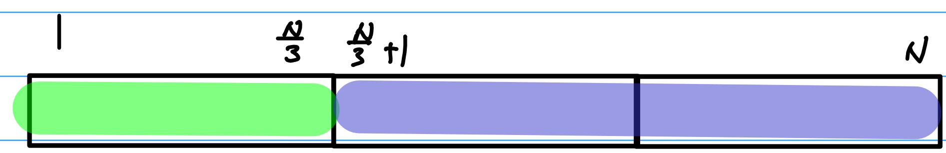


$$\theta = \sum_{k=1}^{N} b_k \, 2^{-k} \qquad \{b_k\} \text{ N-bit Binary } \{0, \; 1\}$$

$$\theta = \phi_0 + \sum_{k=2}^{N+1} r_k \, 2^{-k} \qquad \{r_k\} \text{ N-bit SD } \{-1, \; +1\}$$

$$\theta = \theta_M + \theta_L$$

- Coarse subangle $\quad \theta_M = \sum_{k=1}^{\frac{N}{3}} b_k \, 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k \, 2^{-k}$
- fine subangle $\quad \theta_L = \sum_{k=\frac{N}{3}+1}^{N} b_k \, 2^{-k} = \sum_{k=\frac{N}{3}+1}^{N+1} r_k \, 2^{-k}$



θ_M coarse subangle $\qquad \theta_L$ fine subangle

$$\begin{cases} b_k & k = 1, \cdots, \frac{N}{3} \\ r_k & k = 2, \cdots, \frac{N}{3} \end{cases}$$

Subrotation angle θ_k

① $\boxed{\theta_k = \tan^{-1} 2^{-k}}$ traditional CORDIC

$$\boxed{\tan\theta_k} = \tan\left(\tan^{-1} 2^{-k}\right) = \boxed{2^{-k}}$$

$$\begin{bmatrix} 1 & -\sigma_k \tan(\tan^{-1} 2^{-k}) \\ \sigma_k \tan(\tan^{-1} 2^{-k}) & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -\sigma_k 2^{-k} \\ \sigma_k 2^{-k} & 1 \end{bmatrix}$$

☆ ② $\boxed{\theta_k = 2^{-k}}$ possible because $\theta'' < 1$

$$\boxed{\tan\theta_k} = \boxed{\tan 2^{-k}}$$

$$\begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix} \Rightarrow \begin{cases} \begin{bmatrix} 1 & -\sigma_k \tan(2^{-k}) \\ \sigma_k \tan(2^{-k}) & 1 \end{bmatrix} \quad \theta_M & k \leq \frac{N}{3} \\ \begin{bmatrix} 1 & -\sigma_k 2^{-k} \\ \sigma_k 2^{-k} & 1 \end{bmatrix} \quad \theta_L & k \geq \frac{N}{3}+1 \end{cases}$$

$\tan(2^{-k}) \cong 2^{-k}$ $\quad k \geq k_0$

$\tan(2^{-k})$

> the $\tan\theta_k$ multipliers used in the first few subrotation stages cannot be implemented as simple shift-and-add operations

$$K \begin{bmatrix} 1 & -\tan(r_2 \theta_2) \\ \tan(r_2 \theta_2) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\tan(r_3 \theta_3) \\ \tan(r_3 \theta_3) & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & -\tan(r_N \theta_N) \\ \tan(r_N \theta_N) & 1 \end{bmatrix}$$

$$K = \cos(\theta_2)\cos(\theta_3) \cdots \cos(\theta_N) \qquad \theta_k = 2^{-k}$$

$$\boxed{\begin{aligned} \begin{bmatrix} X_{k+1} \\ Y_{k+1} \end{bmatrix} &= \begin{bmatrix} 1 & -\tan(r_k \theta_k) \\ \tan(r_k \theta_k) & 1 \end{bmatrix} \begin{bmatrix} X_k \\ Y_k \end{bmatrix} \\ &= \begin{bmatrix} X_k - \tan(r_k \theta_k) Y_k \\ Y_k + \tan(r_k \theta_k) X_k \end{bmatrix} \end{aligned}}$$

Sub rotation

$$\boxed{\begin{aligned} X_{k+1} &= X_k - \tan(r_k \theta_k) Y_k \\ Y_{k+1} &= Y_k + \tan(r_k \theta_k) X_k \end{aligned}}$$

$$\tan \theta_k = \tan 2^{-k} = \tan(r_k \theta_k)$$

traditional CORDIC

$$\theta = \sum_{k=2}^{N+1} r_{1k}\,\theta_{1k} = \sum_{k=2}^{N+1} r_{1k}\left(\tan^{-1} 2^{-k}\right) \qquad \theta_{1k} = \left(\tan^{-1} 2^{-k}\right)$$

$$\theta = \sum_{k=2}^{N+1} r_{2k}\,\theta_{2k} = \sum_{k=2}^{N+1} r_{2k}\, 2^{-k} \qquad \theta_{2k} = 2^{-k}$$

$$\theta \qquad \theta_M + \theta_L$$

$$\theta = \sum_{k=2}^{N+1} r_{1k}\,\theta_{1k} = \sum_{k=2}^{\frac{N}{3}} r_{1k}\left(\tan^{-1} 2^{-k}\right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_{1k}\left(\tan^{-1} 2^{-k}\right)$$

$$\theta = \sum_{k=2}^{N+1} r_{2k}\,\theta_{2k} = \sum_{k=2}^{\frac{N}{3}} r_{2k}\, 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_{2k}\, 2^{-k}$$

the same θ_M the same θ_L

different coefficients

Chen, w/o ROM Lookup

$$\begin{cases} X_M = X_0 - Y_0 \tan(\sigma_M \theta_M) \\ Y_M = Y_0 + X_0 \tan(\sigma_M \theta_M) \end{cases}$$

coarse rotation

$$\begin{cases} X = X_M - Y_M \tan(\sigma_L \theta_L) \\ Y = Y_M + X_M \tan(\sigma_L \theta_L) \end{cases}$$

fine rotation

- Shift-and-add
- radix-2 number
- $\theta_L \approx \tan \theta_L$

ROM Lookup Table

To reduce the LUT ROM size

decompose the coarse subangle θ_M

coarse $\quad$ fine

$$X_0 \xrightarrow{\theta_M} X_M \xrightarrow{\theta_L} X$$

$$Y_0 \xrightarrow{\theta_M} Y_M \xrightarrow{\theta_L} Y$$

$$\theta = \theta_M + \theta_L$$

$$= \boxed{\sum_{k=1}^{\frac{N}{3}} b_k \, 2^{-k}} + \boxed{\sum_{k=\frac{N}{3}+1}^{N} b_k \, 2^{-k}}$$

Signed Digit

$r_k \in \{-1, +1\}$

$$\begin{cases} X_M = X_0 - Y_0 \tan(\sigma_M \theta_M) \\ Y_M = Y_0 + X_0 \tan(\sigma_M \theta_M) \end{cases}$$

coarse rotation

$$\begin{cases} X = X_M - Y_M \tan(\sigma_L \theta_L) \\ Y = Y_M + X_M \tan(\sigma_L \theta_L) \end{cases}$$

fine resolution

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_M & \longrightarrow & X \\ Y_0 & \longrightarrow & Y_M & \longrightarrow & Y \end{array}$$

coarse rotation — fine resolution

Modified Coarse - Fine Rotation Method

To reduce the LUT ROM size

decompose the coarse subangle θ_M

$$\theta = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

$$= \theta_M + \theta_L$$

$$\theta_M = \boxed{\sum_{k=1}^{\frac{N}{3}} b_k 2^{-k}} = \boxed{\sum_{k=2}^{\frac{N}{3}} r_k . 2^{-k}}$$

* target

$$\theta_M = \theta_H + \theta_L$$

$\theta_{Mk} = 2^{-k}$
recoding

$\theta_{Hk} = \tan^{-1} 2^{-k}$
traditional
complex rotation

$\theta_{Lk} = \theta_{Mk} - \theta_{Hk}$
$= 2^{-k} - \tan^{-1} 2^{-k}$

$$\boxed{\theta_{Lk} = \theta_{Mk} - \theta_{Hk}}$$

Modified Coarse - Fine Rotation Method

* target

$$\theta_M = \theta_H + \theta_L$$

$\theta_{Mk} = 2^{-k}$
recoding

$\theta_{Hk} = \tan^{-1} 2^{-k}$
traditional
complex rotation

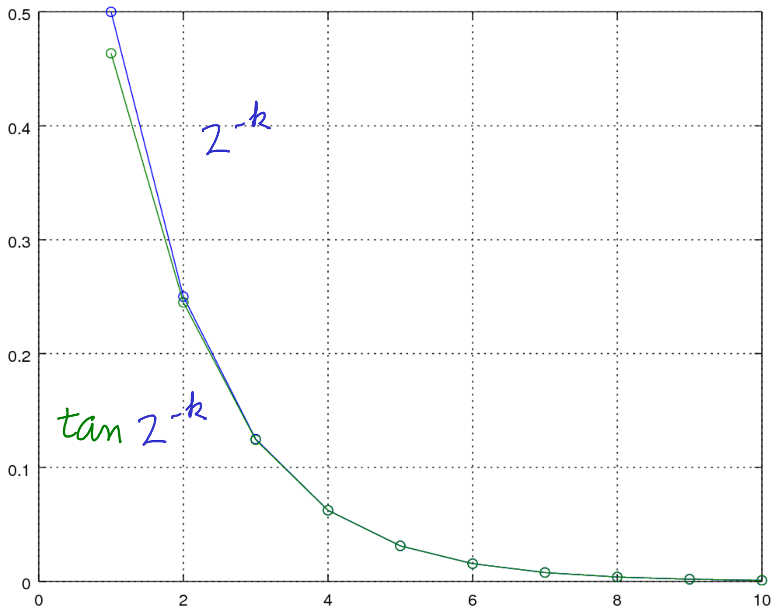
$$\theta_{Lk} = \theta_{Mk} - \theta_{Hk} = 2^{-k} - \tan^{-1} 2^{-k}$$

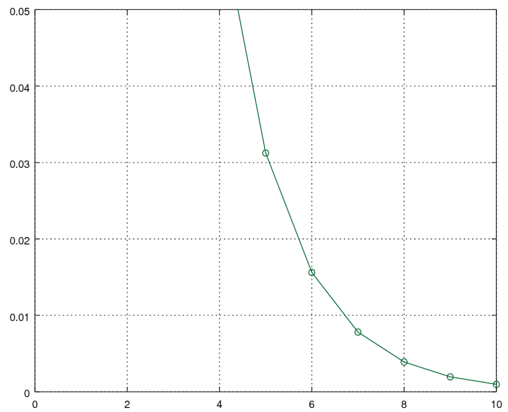
$$\boxed{\theta_{Lk} = \theta_{Mk} - \theta_{Hk}}$$

$$\boxed{\theta_M = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Mk}} = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk}$$

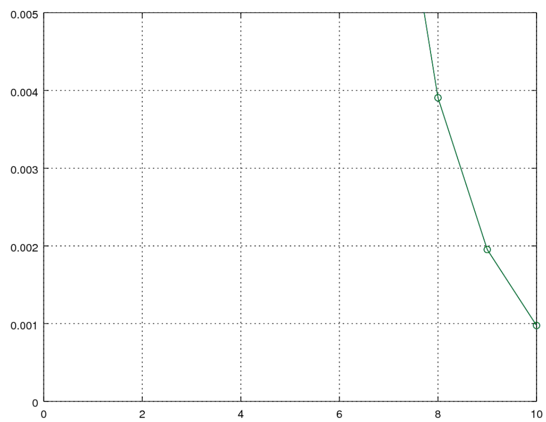
$$= \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k \tan^{-1} 2^{-k} + \sum_{k=2}^{\frac{N}{3}} r_k \left(2^{-k} - \tan^{-1} 2^{-k}\right)$$

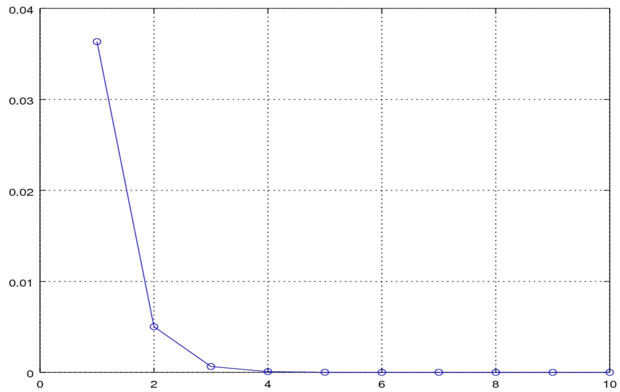
sum(y1-y2)
ans = 0.042112

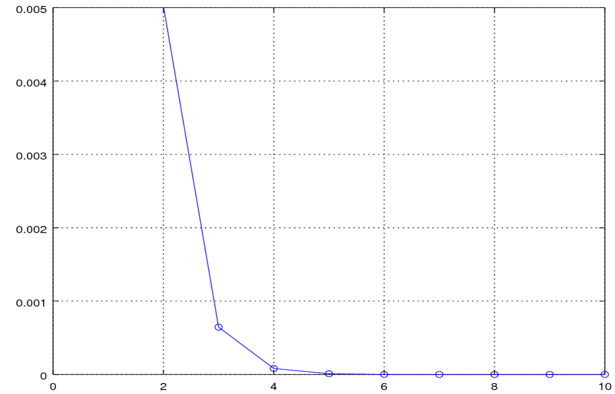


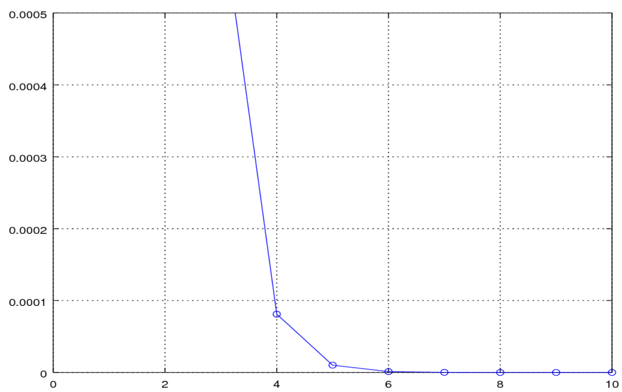


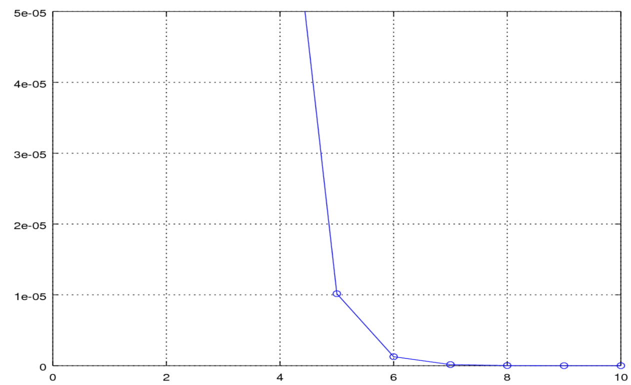

```
41 k= 1:10
42 y1 = 2.^(-k);
43 y1
44 y2 = atan(y1)
45 y3 = y2 - y1
46 k
47 plot(k, y1)
48 plot(k, y3)
49 y3 = y1 - y2
50 plot(k, y3)
51 plot(k, y3, 'o-')
52 grid on
53 axis([0, 10, 0, 0.005])
54 axis([0, 10, 0, 0.0005])
55 axis([0, 10, 0, 0.00005])
56 axis([0, 10, 0, 0.000005])
57 axis([0, 10, 0, 0.0000005])
58 axis([0, 10, 0, 0.00000005])
```

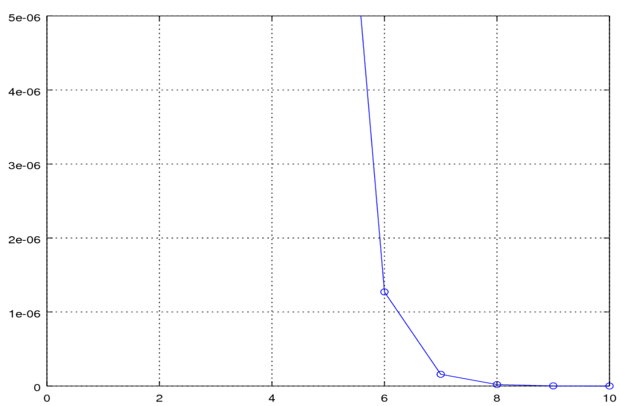



0.04
0.03
0.02
0.01
0
0 2 4 6 8 10


0.005
0.004
0.003
0.002
0.001
0
0 2 4 6 8 10


0.0005
0.0004
0.0003
0.0002
0.0001
0
0 2 4 6 8 10


5e-05
4e-05
3e-05
2e-05
1e-05
0
0 2 4 6 8 10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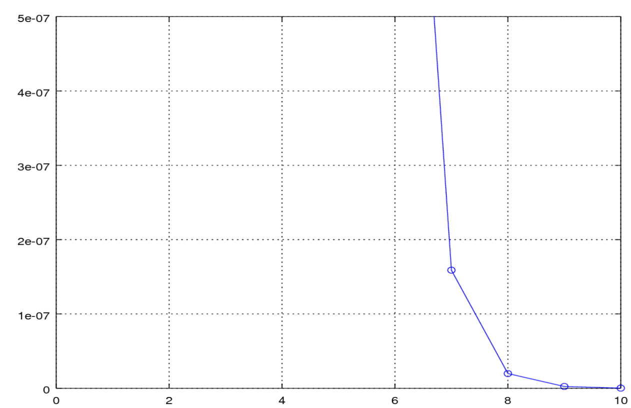
4e-06
3e-06
2e-06
1e-06
0
0 2 4 6 8 10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4e-07
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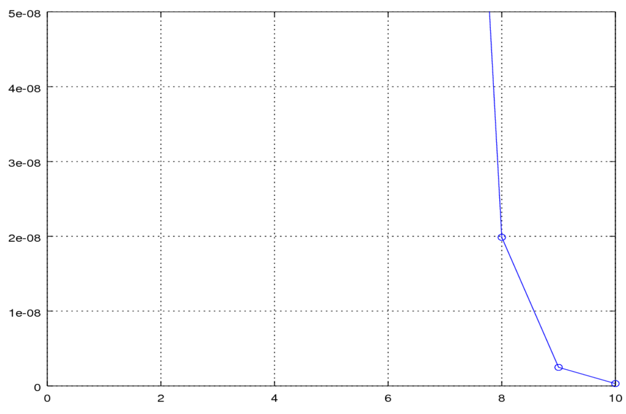
2e-07
1e-07
0
0 2 4 6 8 10


5e-08
4e-08
3e-08
2e-08
1e-08
0
0 2 4 6 8 10

$$\theta = \theta_M + \theta_L$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Mk} + \theta_L$$

$$= \left[\sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk} \right] + \theta_L$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \left[\sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk} + \theta_L \right]$$

$$\theta_{\Sigma L} = \left[\sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk} + \theta_L \right]$$

$$\theta_{Lk} = 2^{-k} - \tan^{-1} 2^{-k}$$

$$\theta_{Hk} = \tan^{-1} 2^{-k}$$

$$\theta = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \theta_{\Sigma L}$$

$$\theta_M = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Mk} = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk}$$

$$\boxed{\theta_{Lk} = \theta_{Mk} - \theta_{Hk}} = 2^{-k} - \tan^{-1} 2^{-k}$$

$\theta_{Mk} = 2^{-k}$ recoding

$\theta_{Hk} = \tan^{-1} 2^{-k}$ traditional CORDIC rotation

$$\theta = \sum_{k=2}^{\frac{N}{3}} r_k \theta_{Hk} + \theta_{\Sigma L} \qquad \theta_{Hk} = \tan^{-1} 2^{-k}$$

$$\theta_{\Sigma L} = \left[\sum_{k=2}^{\frac{N}{3}} r_k \theta_{Lk} + \theta_L \right] \qquad \theta_{Lk} = 2^{-k} - \tan^{-1} 2^{-k}$$

θ_{Hk} inherently a power of 2

$\theta_{\Sigma L}$ ① small enough

② with a carry in the $\frac{N}{3}$-th stage

① Small enough

Shift-and-add

② a carry in the $\frac{N}{3}$-th stage

ⓐ $\theta_{H\frac{N}{3}}$ should be rotated again to realize the carry

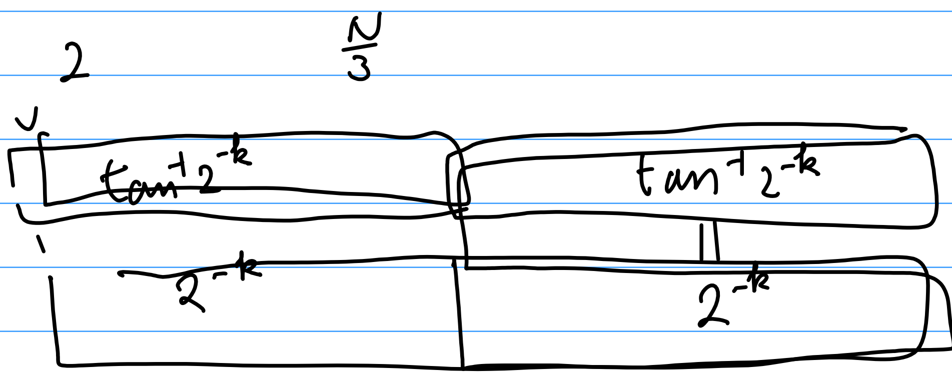
ⓑ Shift-and-add

Can be realized by a sequence of shift-and-add in the radix-2

$\begin{cases} \text{ROM LUT (X)} \\ \text{real multiplication (X)} \end{cases}$

$$\theta = \sum_{k=2}^{\frac{N}{3}} r_k \, \theta_{Hk} + \theta_{\Sigma L} \qquad \theta_{Hk} = \tan^{-1} 2^{-k}$$

$$\theta_{\Sigma L} = \left[\sum_{k=2}^{\frac{N}{3}} r_k \, \theta_{Lk} + \theta_L \right] \qquad \theta_{Lk} = 2^{-k} - \tan^{-1} 2^{-k}$$



$$\theta = \sum_{k=2}^{N+1} r_{1k}\,\theta_{1k} = \sum_{k=2}^{\frac{N}{3}} r_{1k}\left(\tan^{-1} 2^{-k}\right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_{1k}\left(\tan^{-1} 2^{-k}\right)$$

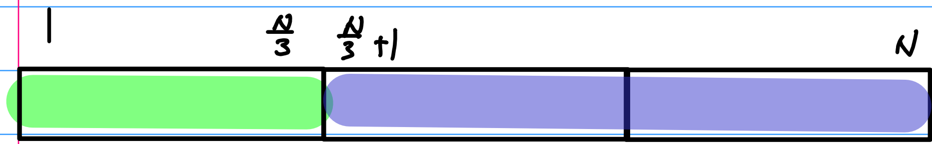
$$\theta = \sum_{k=2}^{N+1} r_{2k}\,\theta_{2k} = \sum_{k=2}^{\frac{N}{3}} r_{2k}\, 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_{2k}\, 2^{-k}$$

$$\sum_{k=2}^{\frac{N}{3}} r_{2k}\left(2^{-k} - \tan^{-1} 2^{-k}\right)$$

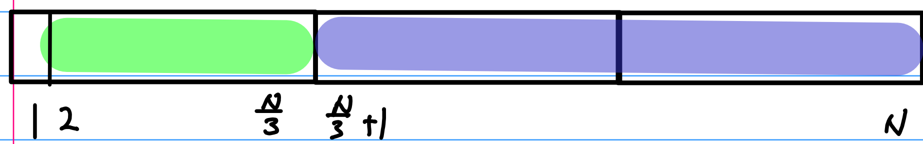
$$+ \sum_{k=2}^{\frac{N}{3}} r_{2k}\left(\tan^{-1} 2^{-k}\right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_{2k}\left(\tan^{-1} 2^{-k}\right)$$

$$\theta = \sum_{k=2}^{N+1} r_{2k}\,\theta_{2k} = \sum_{k=2}^{\frac{N}{3}} r_{2k}\, 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_{2k}\, 2^{-k}$$

coarse subangle θ_M fine subangle θ_L



$$\begin{cases} b_k & k = 1, \dots, \frac{N}{3} \\ r_k & k = 2, \dots, \frac{N}{3} \end{cases}$$



$$\theta_M = \sum_{k=1}^{\frac{N}{3}} b_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k}$$

$$\boxed{2^{-k}} = \boxed{\tan^{-1}(2^{-k})} + \boxed{\left[2^{-k} - \tan^{-1}(2^{-k})\right]}$$

$$\sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k \tan^{-1}(2^{-k}) + \sum_{k=2}^{\frac{N}{3}} r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$$

$$\sum_{k=2}^{\frac{N}{3}} \theta_{Mk} = \sum_{k=2}^{\frac{N}{3}} \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} \theta_{Lk}$$

$$\theta_{Lk} = \theta_{Mk} - \theta_{Hk}$$

$\theta_{Mk} = r_k 2^{-k}$ $\theta_{Hk} = r_k \tan^{-1}(2^{-k})$ $\theta_{Lk} = r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$

error correction term

$$\theta = \overset{\text{coarse}}{\theta_M} + \overset{\text{fine}}{\theta_L}$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk} + \theta_{Lk} \right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk} \right) + \boxed{\sum_{k=2}^{\frac{N}{3}} \left(r_k \left[2^{-k} - \tan^{-1}(2^{-k}) \right] \right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k}) \right) + \boxed{\sum_{k=2}^{N+1} \left(r_k 2^{-k} \right) - \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k}) \right)}$$

$$\theta = \theta_M + \theta_L$$

$$= \boxed{\sum_{k=2}^{\frac{N}{3}} r_k \, 2^{-k}} + \boxed{\sum_{k=\frac{N}{3}+1}^{N+1} r_k \, 2^{-k}}$$

Signed Digit

$r_k \in \{-1, +1\}$

$$\theta_M = \sum_{k=2}^{\frac{N}{3}} \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} \theta_{Lk}$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k \tan^{-1}(2^{-k}) + \sum_{k=2}^{\frac{N}{3}} r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$$

θ_M $\quad \{ 2^{-k} \}$

θ_{Hk} $\quad \{ \tan^{-1}(2^{-k}) \}$

θ_{Lk} $\quad$ error terms

$\theta_{\Sigma L}$

$$= \boxed{\sum_{k=2}^{\frac{N}{3}} \left(r_k \left[2^{-k} - \tan^{-1}(2^{-k}) \right] \right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}}$$

$$= \boxed{\sum_{k=2}^{N+1} \left(r_k 2^{-k} \right) - \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k}) \right)}$$

· $\theta_{\Sigma L}$ small enough
a carry in the $\frac{N}{3}$-th stage

small enough → realized by a sequence of shift-and-add op's

Otherwise $\theta_{H\frac{N}{3}}$ should be rotated again to realize the carry

the remain of $\theta_{\Sigma L}$ can be realized by a sequence of shift-and-add operations

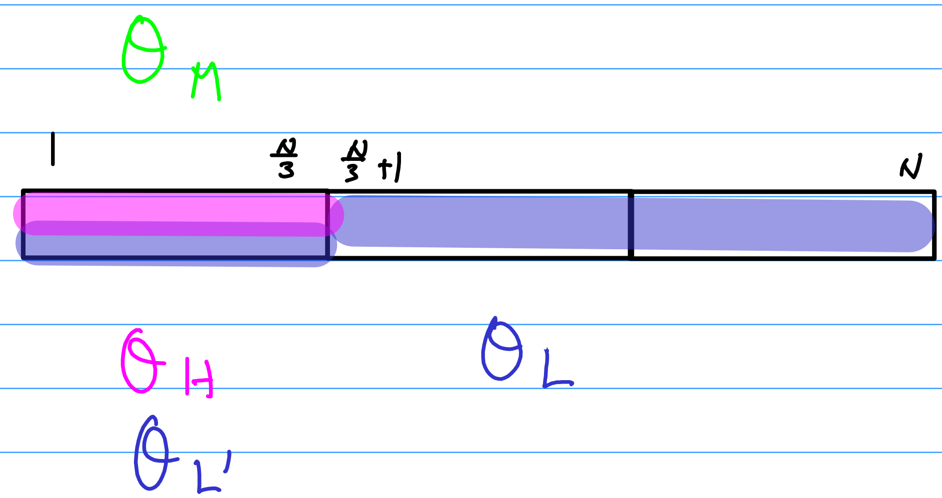
$\phi_0 +$

$$\theta_M = \sum_{k=1}^{\frac{N}{3}} b_k 2^{-k} \longrightarrow \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} \theta_{Mk}$$

$$\theta_L = \sum_{k=\frac{N}{3}+1}^{N} b_k 2^{-k} \longrightarrow \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k} = \sum_{k=\frac{N}{3}+1}^{N+1} \theta_{Lk}$$

$$\theta_M = \sum_{k=2}^{\frac{N}{3}} \theta_{Mk} = \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk} + \theta_{Lk}\right)$$

$$\theta_L = \sum_{k=\frac{N}{3}+1}^{N+1} \theta_{Lk}$$



Chen's coarse-fine approach.

$$\theta = \underset{\text{coarse}}{\theta_M} + \underset{\text{fine}}{\theta_L}$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

50 year's CORDIC

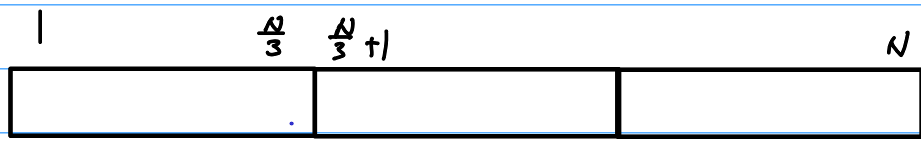
Swartzlander's Hybrid CORDIC approach

$\theta = \theta_H + \theta_L$ coarse & fine subangles

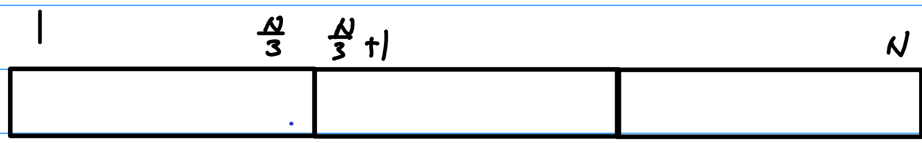
$$\theta_H = \sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i} \qquad \sigma_i \in \{1, -1\}$$

$$\theta_L = \sum_{i=p}^{n-1} d_i \, 2^{-i} \qquad d_i \in \{0, 1\}$$

$\sigma_k \left(\tan^{-1} 2^{-k}\right)$ $\qquad$ $\sigma_k = \{-1, +1\}$



$b_k \, 2^{-k}$ $\qquad$ $b_k \in \{0, 1\}$



ϕ_0, $\; r_k \, 2^{-k}$ $\qquad$ $r_k \in \{-1, +1\}$

to reduce the ROM size

ϕ_0 : initial angle

$$\theta = \theta_M + \theta_L$$

$$\theta_M = \sum_{k=2}^{N/3} \theta_{Mk} = \sum_{k=2}^{N/3} (\theta_{Hk} + \theta_{Lk})$$

$$\theta_{Mk} = 2^{-k} \qquad \tan \theta_{Hk} = 2^{-k}$$

$$\theta_{Lk} = \theta_{Mk} - \theta_{Hk} = \theta_{Mk} - \tan^{-1} 2^{-k}$$

Architecture

$$\begin{cases} X_M = X_0 - \boxed{Y_0 \cdot \tan(\sigma_n \theta_M)} \\ Y_M = Y_0 + \boxed{X_0 \cdot \tan(\sigma_n \theta_M)} \end{cases}$$

coarse rotation

multiplier → bottleneck in computation

a positive angle θ (< 1 rad)

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k the bits corresponding to
the $(N+1)$-bit fractional binary number
Sign + N-bit fraction

$b_k \in \{0, 1\}$ $\quad\quad$ $\theta_k = 2^{-k}$
positive $\quad\quad$ $\theta_0 = 0$

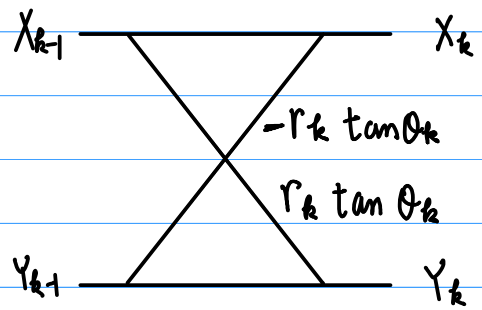
$b_k \in \{0, 1\} \Rightarrow r_k \in \{-1, +1\}$
recoded

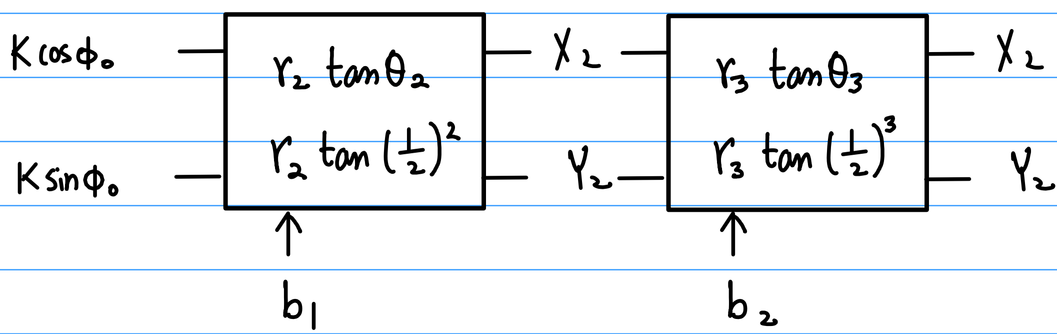
$$X_{k+1} = X_k - (r_k \tan \theta_k) Y_k$$
$$Y_{k+1} = Y_k + (r_k \tan \theta_k) X_k$$

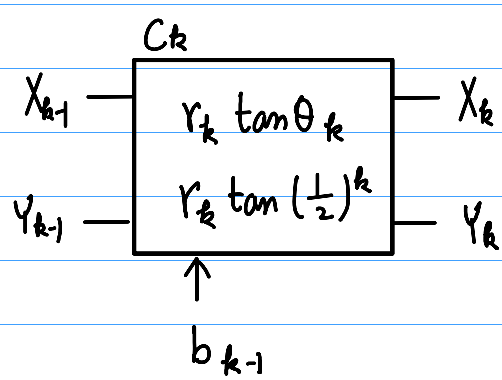
$$X_k = X_{k-1} - (r_{k-1} \tan \theta_{k-1} Y_{k-1}$$
$$Y_k = Y_{k-1} + (r_{k-1} \tan \theta_{k-1} X_{k-1}$$

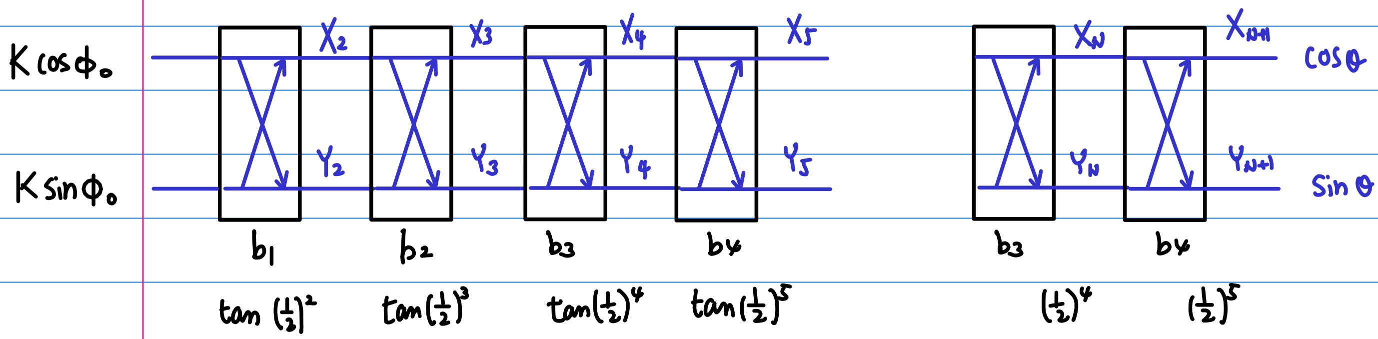
$$X_{k+1} = X_k - (r_k \tan \theta_k) Y_k$$
$$Y_{k+1} = Y_k + (r_k \tan \theta_k) X_k$$

$$X_k = X_{k-1} - (r_k \tan \theta_k) Y_{k-1}$$
$$Y_k = Y_{k-1} + (r_k \tan \theta_k) X_{k-1}$$









$$\theta = \sum_{k=1}^{N} b_k 2^{-k} \quad = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

binary digit representation

$b_k \in \{0, 1\}$

Signed digit recoding

$r_k \in \{-1, +1\}$

$$\boxed{b_k = (r_k + 1)/2}$$

$$\boxed{r_k = (2b_{k-1} - 1)}$$

the coarse-fine partition
could be applied for reducing
the number of micro-rotations
necessary for fine rotations

To implement the coarse rotation
through **shift-add** operation

the coarse sub angle θ_M
in terms of elementary rotations of the form $\tan^{-1} 2^{-i}$

$$\theta_M = \sum_{i=1}^{p-1} d_i \, 2^{-i} \qquad = \sum_{i=1}^{p-1} \left(\sigma_i \tan^{-1}(2^{-i}) - \theta_{L_i} \right)$$

$$\sum_{i=1}^{p-1} \boxed{d_i \, 2^{-i}} \qquad \qquad \sum_{i=1}^{p-1} \boxed{\sigma_i \tan^{-1}(2^{-i})}$$

shift-add   coarse rotation

θ_{L_i} : correction term

Chen, w/o ROM Lookup

$$\theta_M = \sum_{k=1}^{N/3} b_k 2^k$$

$$\theta_L = \sum_{i=N/3+1}^{N} b_k 2^k$$

$$\theta = \theta_H + \theta_L$$

$$= \theta_M + \tilde{\theta}_L$$

$$= \sum_{i=1}^{p-1} \sigma_i \tan^{-1}(2^{-i}) + \sum_{i=p}^{n-1} d_i 2^{-i}$$

$$= \sum_{i=1}^{p-1} d_i 2^{-i} + \boxed{\sum_{i=1}^{p-1} \theta_{L_i} + \sum_{i=p}^{n-1} d_i 2^{-i}}$$

$$\boxed{\sum_{i=1}^{p-1} \theta_{L_i} + \theta_L}$$

$$\boxed{\tilde{\theta}_L}$$

$$\theta = \boxed{\theta_H} + \boxed{\theta_L} \quad \text{(higher \& lower parts)}$$

$$= \boxed{\sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i}} + \boxed{\sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$\sigma_i \in \{1, -1\} \qquad d_i \in \{0, 1\}$$

$$\boxed{\theta_M} + \boxed{\tilde{\theta}_L}$$

$$= \boxed{\sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i} - \sum_{i=1}^{p-1} \theta_{Li}} + \boxed{\sum_{i=1}^{p-1} \theta_{Li} + \sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$= \boxed{\sum_{i=1}^{p-1} d_i \, 2^{-i}} + \boxed{\sum_{i=1}^{p-1} \theta_{Li} + \sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$\tilde{\theta}_L = \theta_L + \sum_{i=2}^{N/3} \theta_{L_i}$$

$$\boxed{(-\theta_0) + \sum_{j=1}^{m-1} \theta_j 2^{j-1}} + \boxed{\sum_{j=m}^{N} \theta_j 2^{-j}}$$

$$\theta_M \approx \sum_{i=2}^{n/3} d_i 2^{-i} = \sum_{i=2}^{N/3} \left(\sigma_i \tan^{-1}(2^{-i}) + \theta_{L_i} \right)$$