Hybrid CORDIC
1. Overview

20171007

radix -2 decomposition

any rotation angle θ

~ linear combination of angles

⑥ the radix-2 based set

$\{2^{-i}\}, \quad i = 1, 2, \cdots, n-1$

$\sum_{i=0}^{n-1} b_i 2^{-i}, \quad b_i \in \{0, 1\}$

↳ determines whether a micro-rotation is to be performed or not.

- not widely used ($\Leftarrow$ no hardware benefit)

⑥ the elementary angle set (EAS)

$\{\tan^{-1} 2^{-i}\}, \quad i = 1, 2, \cdots, n-1$

- can be used in implementing shift-and-add operations

$\{2^{-i} : i \in \{1, 2, \cdots, n-1\}\}$

$\{\tan^{-1}(2^{-i}) : i \in \{1, 2, \cdots, n-1\}\}$

Coarse-Fine Decomposition

for sufficiently large j

$\alpha_j = \tan^{-1}(2^{-j}) \approx 2^{-j}$ when $\boxed{j > \lceil \frac{n}{3} \rceil - 1}$

the elementary angle set

$\{ \tan^{-1}(2^{-j}) \}$ for most significant part
$\{ (2^{-j}) \}$ for less significant part

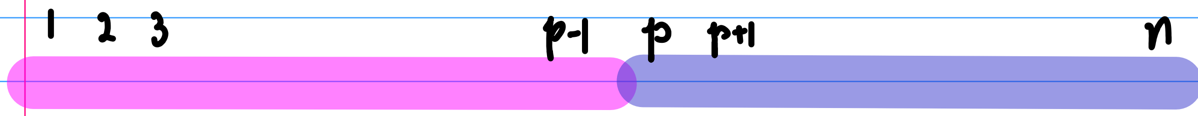
radix set $S = S_1 \cup S_2$

$S_1 = \{ \tan^{-1}(2^{-i}) \; ; \; i \in [1, 2, \cdots, p-1] \}$

$S_2 = \{ (2^{-i}) \; : \; i \in [p, p+1, \cdots, n-1] \}$

the rotation angle is partitioned

$$\theta = \theta_H + \theta_L$$



$S_1 = \{ \tan^{-1}(2^{-i}) \}$ $\qquad$ $S_2 = \{ 2^{-i} \}$

$p > \lceil \frac{n}{3} \rceil - 1$

$$\theta = \boxed{\begin{array}{c} \theta_H \\ \Cap \\ S_1 \\ \sigma_i \in \{1, -1\} \end{array}} + \boxed{\begin{array}{c} \theta_L \\ \Cap \\ S_2 \\ d_i \in \{\overset{*}{0}, 1\} \end{array}}$$

$$\theta_H = \sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i} \qquad \sigma_i \in \{1, -1\}$$

$$\theta_L = \sum_{i=p}^{n-1} d_i \, 2^{-i} \qquad d_i \in \{\overset{*}{0}, 1\}$$

rotation angle decomposition

$$\theta = \theta_H + \theta_L \qquad \text{coarse \& fine subangles}$$

$$\theta_H = \sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i} \qquad \sigma_i \in \{1, -1\}$$

$$\theta_L = \sum_{i=p}^{n-1} d_i \, 2^{-i} \qquad d_i \in \{0, 1\}$$

Wang & Swartzlander, Hybrid CORDIC algorithms
IEEE Computers, 1997

ATR (Arc Tangent Radix)

$$\{\alpha_0, \alpha_1, \cdots, \alpha_{N-1}\} = \{\tan^{-1} 2^0, \tan^{-1} 2^{-1}, \cdots, \tan^{-1} 2^{-N+1}\}$$

CORDIC convergence theorem

$$\alpha_i - \sum_{j=i+1}^{N-1} \alpha_j < \alpha_{N-1}$$

The Hybrid Radix sets

Mixed-Hybrid Circular ATR

$$\{\underbrace{\tan^{-1} 2^0, \tan^{-1} 2^{-1}, \cdots, \tan^{-1} 2^{-n+1}}_{\text{most significant part}}, \underbrace{2^{-n}, 2^{-n+1}, \cdots, 2^{-N+1}}_{\text{least significant}}\}$$

most significant part — n-bits

least significant — N - n bits

$$\theta_H = \sum_{i=0}^{n-1} \theta_i 2^{-i} \qquad \theta_L = \sum_{i=n}^{N-1} \theta_i 2^{-i}$$

Partitioned-Hybrid Circular ATR

$$\{\underbrace{\tan^{-1} 2^{-n+1}}_{\text{most significant part}}, \underbrace{2^{-n}, 2^{-n+1}, \cdots, 2^{-N+1}}_{\text{least significant}}\}$$

Wang & Swartzlander, Hybrid CORDIC algorithms
IEEE Computers, 1997

Partitioned - Hybrid Circular ATR

most significant part

$\tan^{-1} 2^{-n+1} \cdot \sigma_{n-1}$

How many $\tan^{-1} 2^{-n+1}$

compressed into only one radix

to formally describe the single iteration θ_H

in a single step

lookup table

σ_{n-1} the rotation direction

to represent any possible angle θ_H

by compressing the circular ATR iterations

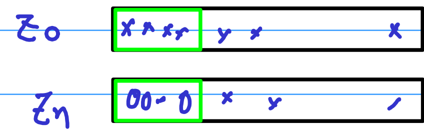
from the initial angle

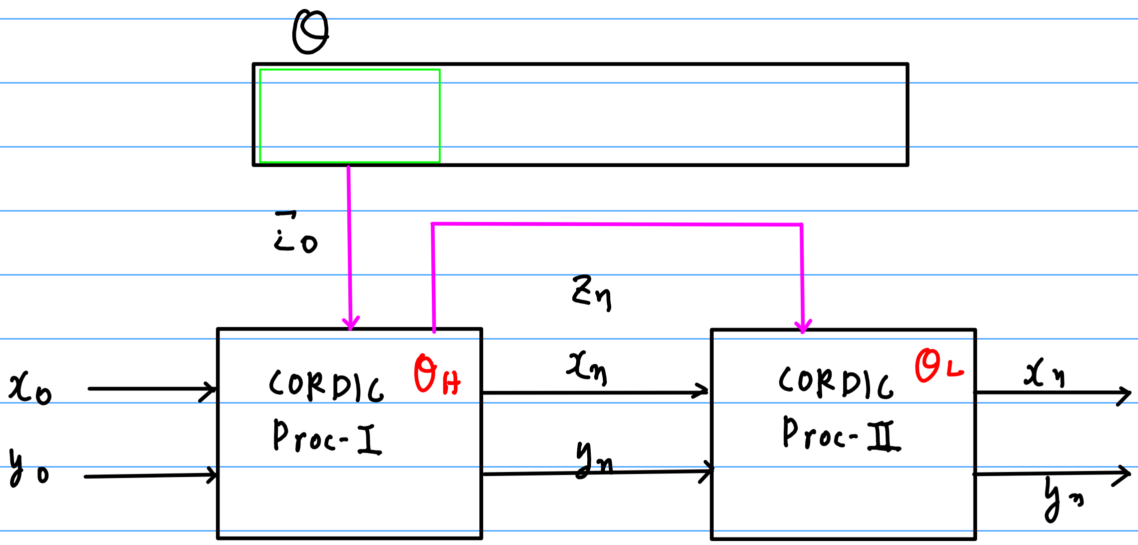
to the (n-1) th angle

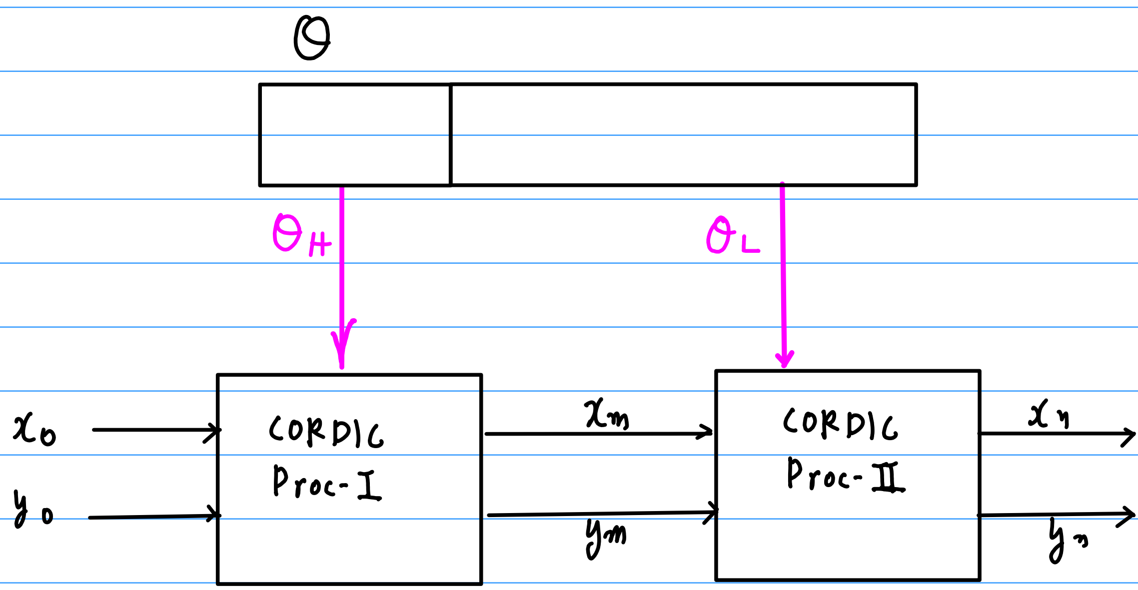
$$\boxed{\sigma_{n-1} = \frac{\theta_H}{\tan^{-1} 2^{-n+1}} = \frac{\sum_{i=0}^{n-1} \theta_i 2^{-i}}{\tan^{-1} 2^{-n+1}}}$$

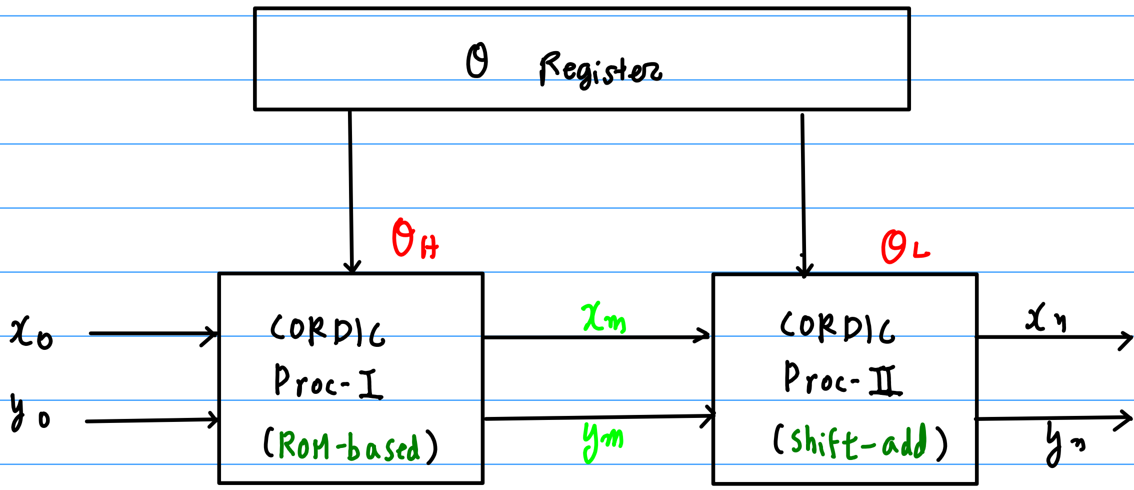
Wang & Swartzlander, Hybrid CORDIC algorithms
IEEE Computers, 1997

$$\theta_H = \sum_{i=0}^{n-1} \theta_i 2^{-i}$$









two cascade stages

coarse rotation

$$\begin{bmatrix} x_m \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & -\tan(\theta_H) \\ \tan(\theta_H) & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Fine rotation

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & -\tan(\theta_L) \\ \tan(\theta_L) & 1 \end{bmatrix} \begin{bmatrix} x_m \\ y_m \end{bmatrix}$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} \leftarrow \begin{bmatrix} x_m \\ y_m \end{bmatrix} \leftarrow \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

Implementation of Hybrid CORDIC

CORDIC PROC-I : the coarse rotation ROM based

θ_H { ROM look-up operation
addition

CORDIC PROC-II : the fine rotation SHIFT-ADD

θ_L sequence of shift-and-add
no computation of the direction of micro-rotation
the need of a micro-rotation is explicit ($= d_i$)
in the radix-2 representation

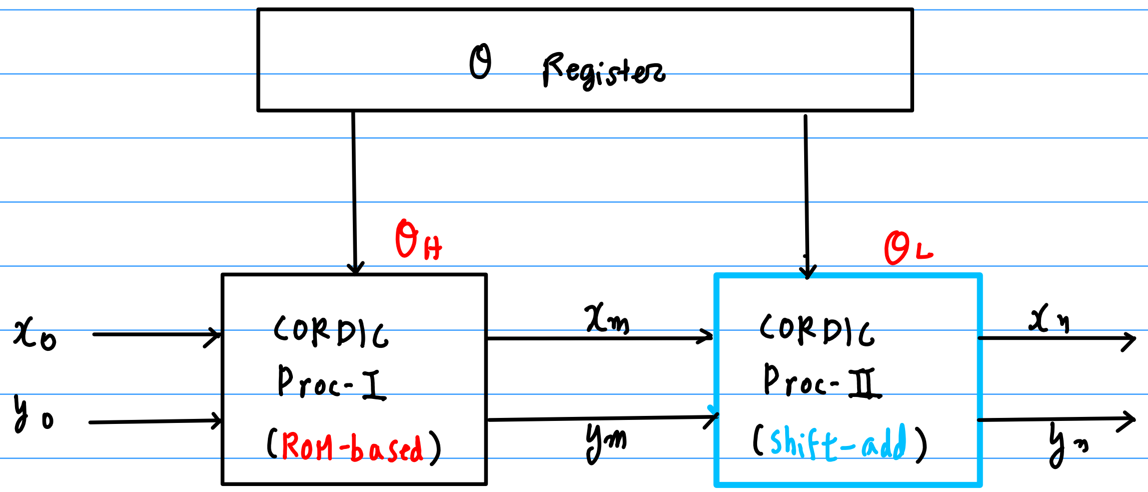
$$\theta_L = \sum_{i=p}^{n-1} d_i 2^{-i} \qquad d_i \in \{0, 1\}$$

* In the signed digit notation [Timmermann Low' latency]

$$\theta_L = \sum_{i=p}^{n-1} \tilde{b}_i 2^{-i} \qquad \tilde{b}_i = \{-1, +1\}$$

$$\theta_L = \sum_{i=p}^{n-1} d_i\, 2^{-i} \qquad d_i \in \{0, 1\} \qquad \text{① radix-2 representation}$$

$$= \sum_{i=p}^{n-1} \tilde{b}_i\, 2^{-i} \qquad \tilde{b}_i = \{-1, +1\} \qquad \text{② signed digit notation}$$



* the direction is explicit
  → parallel implementation possible

Timmermann, Low Latency time CORDIC algorithm, 1992

* the hybrid decomposition could be used

  ① ROM-based realization of coarse rotation
     → minimize latency

  ② Shift-and-add implementation of fine rotation
     → minimize hardware complexity
     ← no need to find the rotation direction

[23] M. Kuhlmann and K. K. Parhi, "P-CORDIC: A precomputation based rotation CORDIC algorithm," *EURASIP J. Appl. Signal Process.*, vol. 2002, no. 9, pp. 936–943, 2002.

very high precision

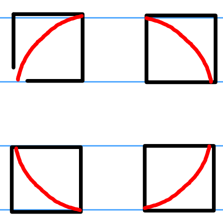
ROM size $n \cdot 2^{n/5}$ bits

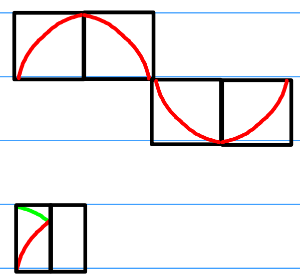
if latency is tolerable
   the conventional CORDIC
         shift-and-add operation

Shift-Add Implementation of Coarse rotation

Using the symmetry properties
of cosine and sine functions
in different quadrants

rotation through arbitrary angle θ
can be mapped from $[0, 2\pi]$
to the first half of the first quadrant $[0, \frac{\pi}{4}]$





$[0, 2\pi] \rightarrow [0, \frac{\pi}{4}]$

Madisetti's approach

assumption arbitrary positive angle

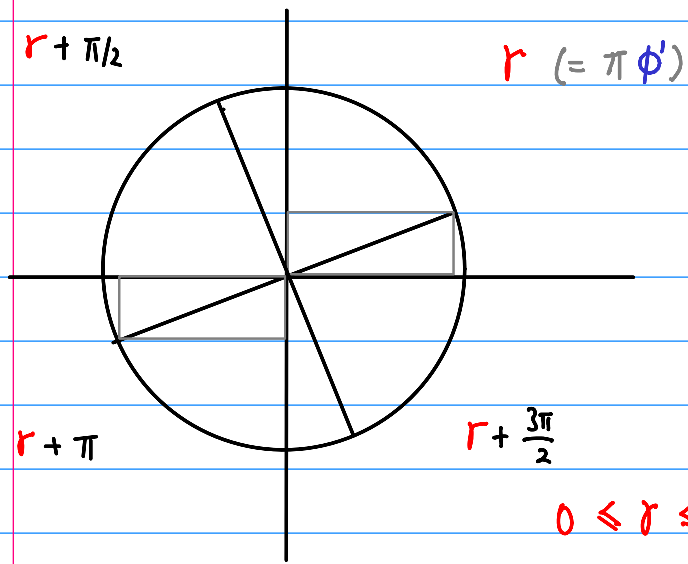
$$\boxed{\theta < 1 \text{ rad}}$$

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

$$= \sum_{k=1}^{N} b_k 2^{-k}$$

$$= \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

Quadrant symmetry

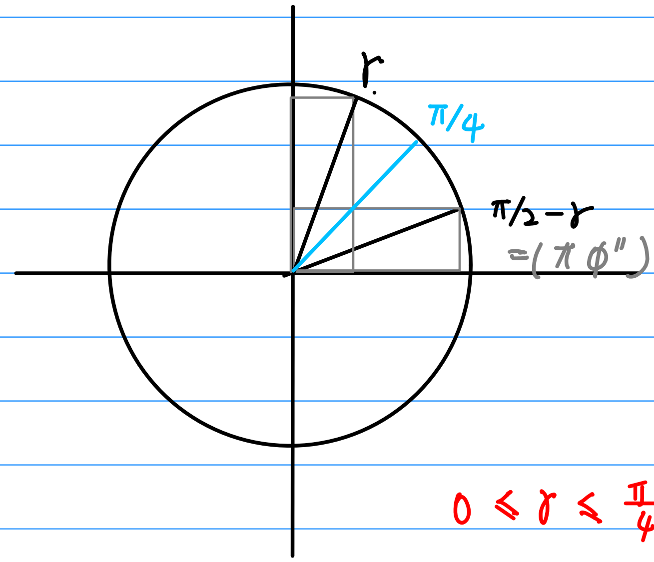


$0 \leqslant r \leqslant \frac{\pi}{2}$

$[-\pi, +\pi] \rightarrow [0, \pi/2]$

MSB_1	MSB_2	MSB_3
0	0	
0	1	
1	0	
1	1	

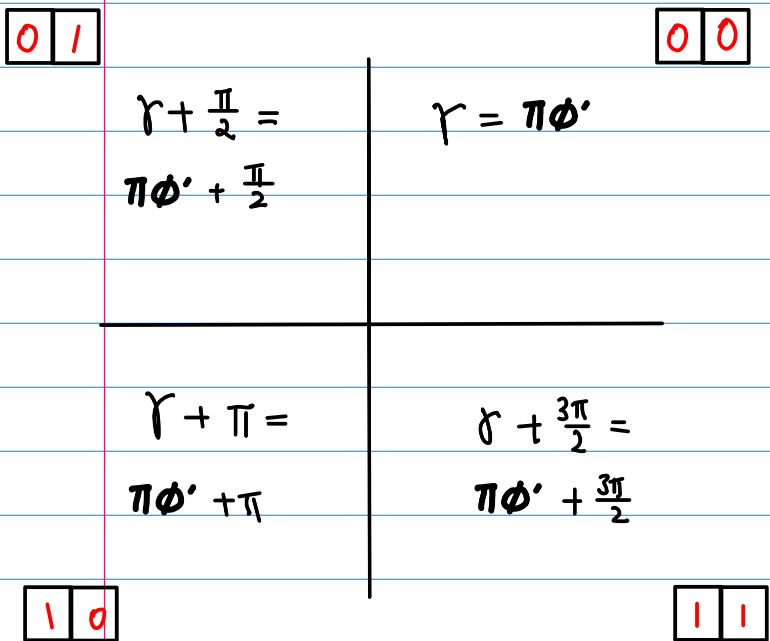
π/4 mirror

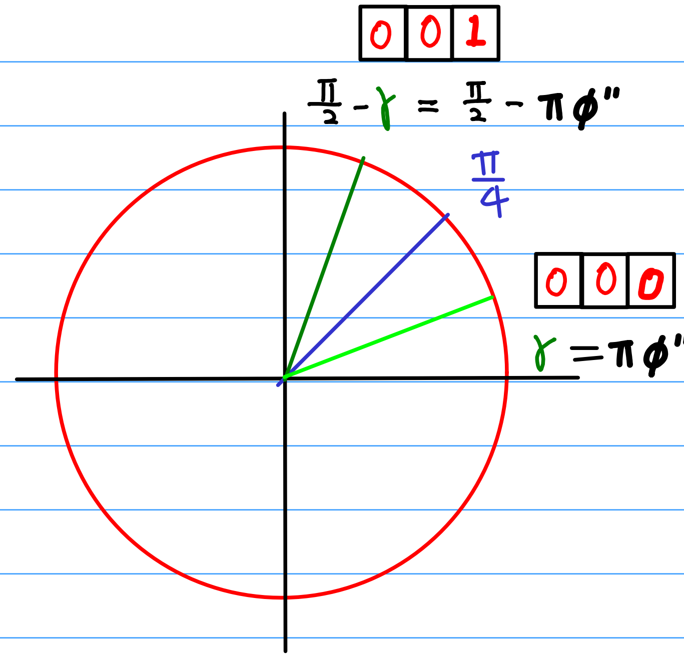


$0 \leqslant r \leqslant \frac{\pi}{4}$

$[0, \pi/2] \rightarrow [0, \pi/4]$

MSB_3	
1	$\phi'' = 0.5 - \phi'$
0	$\phi'' = \phi'$





$$\theta = \pi\phi \longrightarrow \theta' = \pi\phi' \longrightarrow \theta'' = \pi\phi''$$

$$\theta \in [-\pi, +\pi] \longrightarrow \theta' = [0, \pi/2] \longrightarrow \theta'' = [0, \pi/4]$$

$$\phi \in [-1, +1] \longrightarrow \phi' = [0, 0.5] \longrightarrow \phi'' = [0, 0.25]$$

argument: signed normalized by π angle $[-1, 1]$

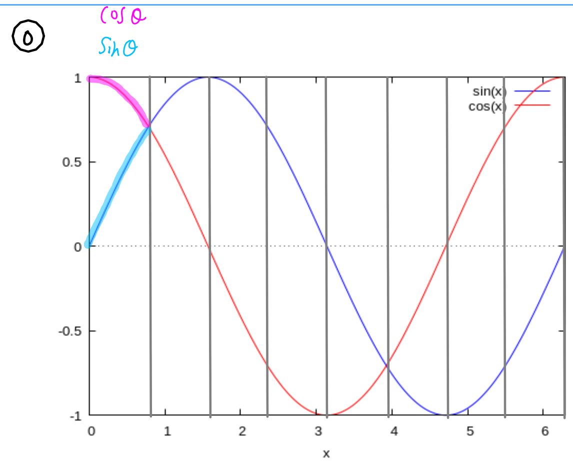
binary representation of a radian angle required

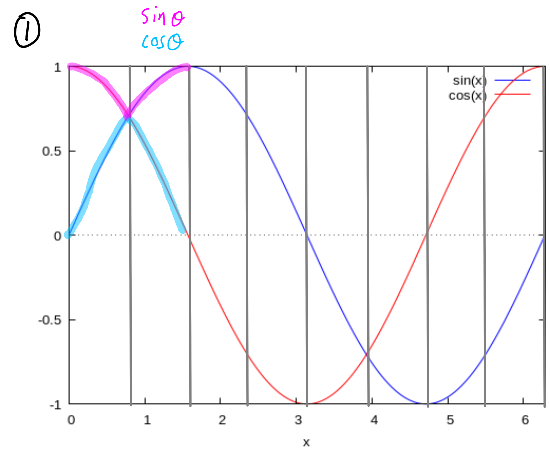
$$[-1, 1] \xrightarrow[\phi]{} [0, \pi/4] \xrightarrow[\theta]{} \text{Sine/cosine generator}$$

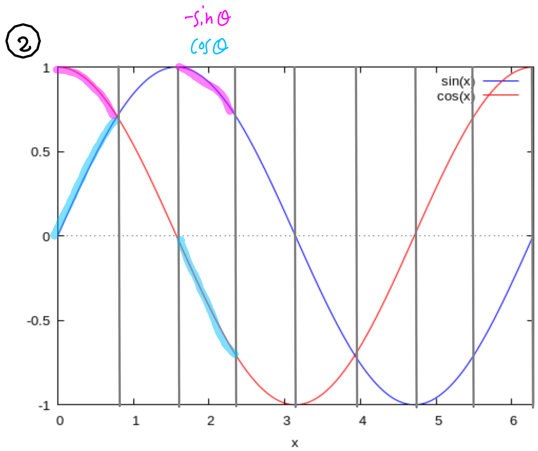
$$\pi\phi \longleftarrow$$

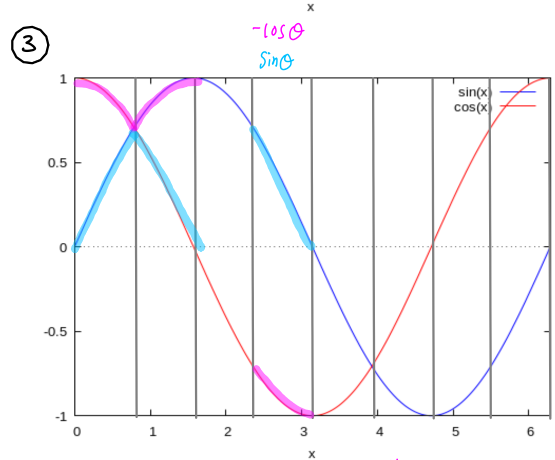
1. a phase accumulator ϕ $[-1, 1]$
2. a radian converter $\phi \rightarrow \theta$
3. a sine/cosine generator $\sin\theta$, $\cos\theta$
4. an output stage $\sin\theta$, $\cos\theta$

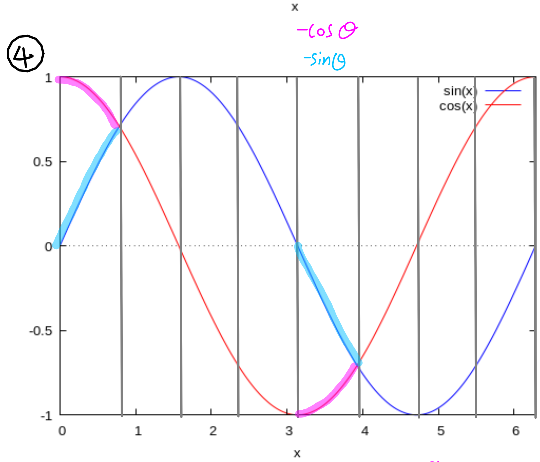
$$\sin\theta \downarrow \sin\pi\phi \qquad \cos\theta \downarrow \cos\pi\phi$$

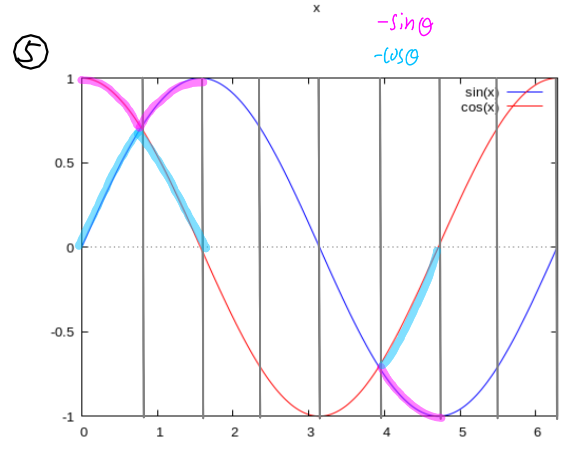

0
cos θ
sin θ
sin(x)
cos(x)
x

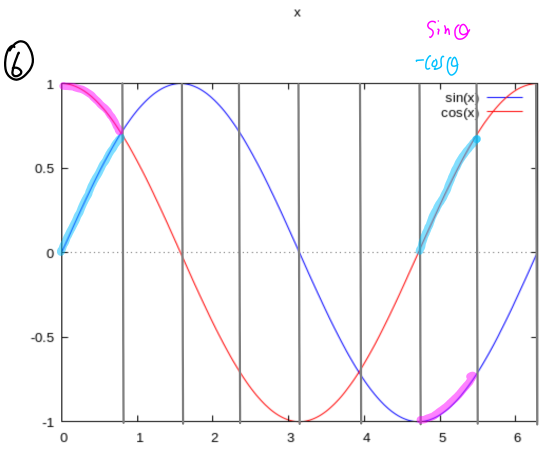

1
sin θ
cos θ
sin(x)
cos(x)
x

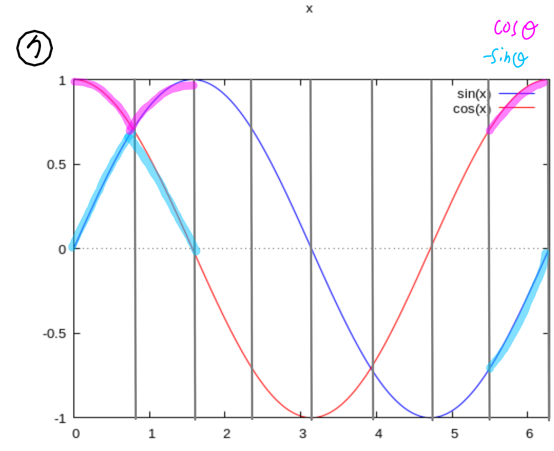

2
-sin θ
cos θ
sin(x)
cos(x)
x


3
-cos θ
sin θ
sin(x)
cos(x)
x


4
-cos θ
-sin θ
sin(x)
cos(x)
x


5
-sin θ
-cos θ
sin(x)
cos(x)
x


6
sin θ
-cos θ
sin(x)
cos(x)
x


7
cos θ
-sin θ
sin(x)
cos(x)
x

+ subrotation by 2^{-k}

2 equal half rotations by 2^{-k-1}

+ +

0 subrotation

2 equal opposite half rotations by $\pm 2^{-k-1}$

+ − − +

Binary Representation

$b_k = 1$: rotation by 2^{-k}

$b_k = 0$: zero rotation

⋮

fixed

k-th rotation	pos 2^{-k-1} rotation	pos 2^{-k-1} rotation	← $b_k = 1$
	pos 2^{-k-1} rotation	neg 2^{-k-1} rotation	← $b_k = 0$

⋮

Combining all the fixed rotations

→ initial fixed rotation

	$k=1$	$k=2$	$k=3.$		$k=N$
	b_1 2^{-1}	b_2 2^{-2}	b_3 2^{-3}		b_N 2^{-N}
fixed pos $\Rightarrow$	$+2^{-2}$	$+2^{-3}$	$+2^{-4}$		$+2^{-N-1}$
$(b_i=1)$ pos $\Rightarrow$	$(b_1=1)$ $+2^{-2}$	$(b_2=1)$ $+2^{-3}$	$(b_3=1)$ $+2^{-4}$		$(b_N=1)$ $+2^{-N-1}$
$(b_i=0)$ neg $\Rightarrow$	$(b_1=0)$ -2^{-2}	$(b_2=0)$ -2^{-3}	$(b_3=0)$ -2^{-4}		$(b_N=0)$ -2^{-N-1}

initial fixed rotation

$$\phi_0 = \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{N+1}} = \sum_{k=2}^{N+1} \frac{1}{2^k}$$

$$= \frac{\frac{1}{2^2}\left(1 - \frac{1}{2^N}\right)}{\left(1 - \frac{1}{2}\right)} = \frac{1}{2}\left(1 - \frac{1}{2^N}\right) = \frac{1}{2} - \frac{1}{2^{N+1}}$$

Signed Digit Recoding

the rotation after recoding

— a fixed initial rotation ϕ_0 $\boxed{\sum_{k=2}^{N+1} 2^{-k}}$

a sequence of ⊕/⊖ rotations $\boxed{\sum_{k=2}^{N+1} r_k 2^{-k}}$

$b_k = 1$ $+2^{-k-1}$ pos rotation $r_k = +1$

$b_k = 0$ -2^{-k-1} neg rotation $r_k = -1$

$r_k = (2b_{k-1} - 1)$

$2 \cdot 1 - 1 = +1$ $b_{k-1} = 1 \longrightarrow r_k = +1$

$2 \cdot 0 - 1 = -1$ $b_{k-1} = 0 \longrightarrow r_k = -1$

The recoding need not be explicitly performed

Simply replacing $b_k = 0$ with -1

This recoding maintains

a constant scaling factor K

$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

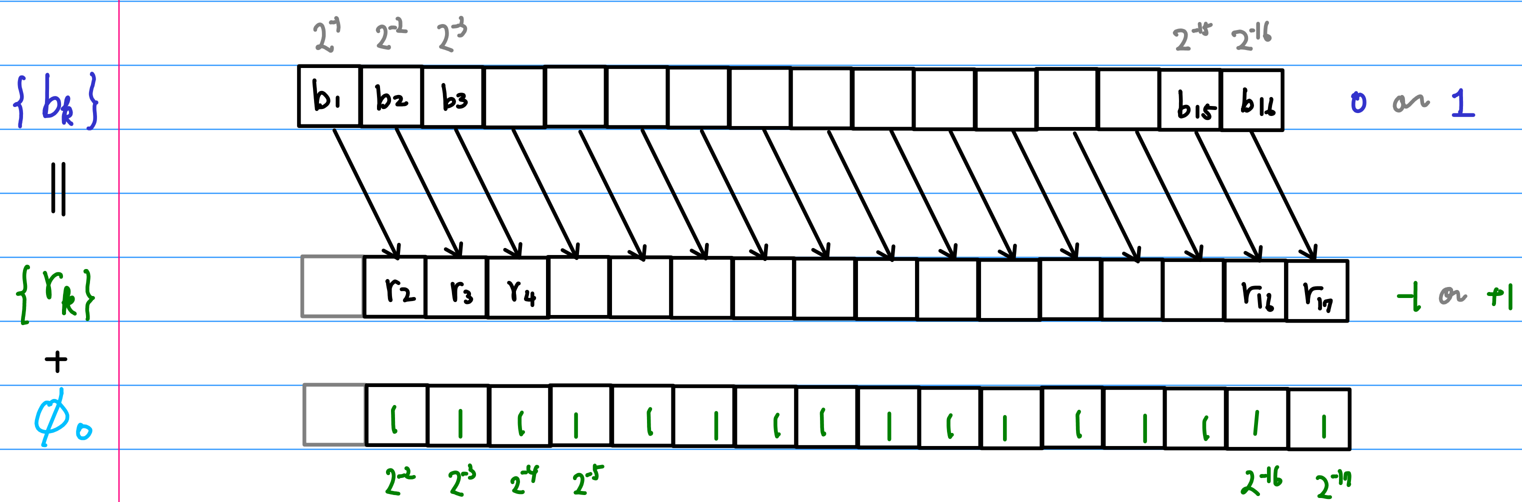
binary digit representation

$b_k \in \{0, 1\}$

Signed digit recoding

$r_k \in \{-1, +1\}$

$$\boxed{b_k = (r_k + 1)/2} \qquad \boxed{r_k = (2 b_{k-1} - 1)}$$

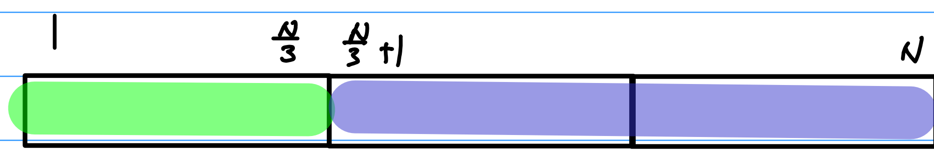


$$\theta = \sum_{k=1}^{N} b_k \, 2^{-k} \qquad \{b_k\} \text{ N-bit Binary } \qquad \{0, \; 1\}$$

$$\theta = \phi_0 + \sum_{k=2}^{N+1} r_k \, 2^{-k} \qquad \{r_k\} \text{ N-bit SD } \qquad \{-1, \; +1\}$$

$$\theta = \theta_M + \theta_L$$

- Coarse subangle $\quad \theta_M = \sum_{k=1}^{\frac{N}{3}} b_k \, 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k \, 2^{-k}$
- fine subangle $\quad \theta_L = \sum_{k=\frac{N}{3}+1}^{N} b_k \, 2^{-k} = \sum_{k=\frac{N}{3}+1}^{N+1} r_k \, 2^{-k}$



θ_M coarse subangle $\qquad \theta_L$ fine subangle

$$\begin{cases} b_k & k = 1, \cdots, \frac{N}{3} \\ r_k & k = 2, \cdots, \frac{N}{3} \end{cases}$$

Chen, w/o ROM Lookup

$$\begin{cases} X_M = X_0 - Y_0 \tan(\sigma_M \theta_M) \\ Y_M = Y_0 + X_0 \tan(\sigma_M \theta_M) \end{cases}$$

coarse rotation

$$\begin{cases} X = X_M - Y_M \tan(\sigma_L \theta_L) \\ Y = Y_M + X_M \tan(\sigma_L \theta_L) \end{cases}$$

fine rotation

- Shift-and-add
- radix-2 number
- $\theta_L \approx \tan \theta_L$

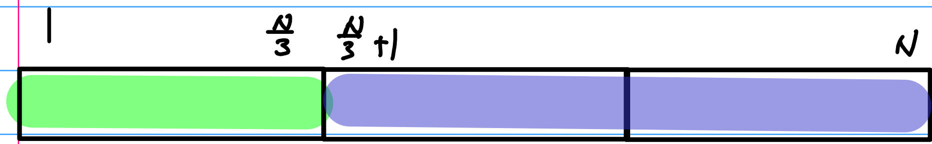
ROM Lookup Table

To reduce the LUT ROM size

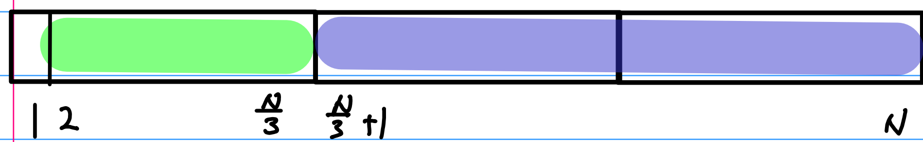
decompose the coarse subangle θ_M

$$\boxed{\begin{array}{ccccc} & \text{coarse} & & \text{fine} & \\ & \theta_M & & \theta_L & \\ X_0 & \longrightarrow & X_M & \longrightarrow & X \\ Y_0 & \longrightarrow & Y_M & \longrightarrow & Y \end{array}}$$

coarse subangle θ_M fine subangle θ_L



$$\begin{cases} b_k & k = 1, \dots, \frac{N}{3} \\ r_k & k = 2, \dots, \frac{N}{3} \end{cases}$$



$$\theta_M = \sum_{k=1}^{\frac{N}{3}} b_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k}$$

$$\boxed{2^{-k}} = \boxed{\tan^{-1}(2^{-k})} + \boxed{\left[2^{-k} - \tan^{-1}(2^{-k})\right]}$$

$$\sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} r_k \tan^{-1}(2^{-k}) + \sum_{k=2}^{\frac{N}{3}} r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$$

$$\sum_{k=2}^{\frac{N}{3}} \theta_{Mk} = \sum_{k=2}^{\frac{N}{3}} \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} \underbrace{\theta_{Lk}}_{= \theta_{Mk} - \theta_{Hk}}$$

$\theta_{Mk} = r_k 2^{-k}$ $\theta_{Hk} = r_k \tan^{-1}(2^{-k})$ $\theta_{Lk} = r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$

error correction term

coarse fine

$$\theta = \theta_M + \theta_L$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk} + \theta_{Lk}\right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk}\right) + \boxed{\sum_{k=2}^{\frac{N}{3}} \left(r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]\right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}}$$

$$= \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k})\right) + \boxed{\sum_{k=2}^{N+1} \left(r_k 2^{-k}\right) - \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k})\right)}$$

$$\theta = \theta_M + \theta_L$$

$$= \boxed{\sum_{k=2}^{\frac{N}{3}} r_k 2^{-k}} + \boxed{\sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}}$$

Signed Digit

$r_k \in \{-1, +1\}$

$$\theta_M = \sum_{k=2}^{\frac{N}{3}} \theta_{Hk} + \sum_{k=2}^{\frac{N}{3}} \theta_{Lk}$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k \tan^{-1}(2^{-k}) + \sum_{k=2}^{\frac{N}{3}} r_k \left[2^{-k} - \tan^{-1}(2^{-k})\right]$$

θ_M $\{ 2^{-k} \}$

θ_{Hk} $\{ \tan^{-1}(2^{-k}) \}$

θ_{Lk} error terms

$\theta_{\Sigma L}$

$$= \boxed{\sum_{k=2}^{\frac{N}{3}} \left(r_k \left[2^{-k} - \tan^{-1}(2^{-k}) \right] \right) + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}}$$

$$= \boxed{\sum_{k=2}^{N+1} \left(r_k 2^{-k} \right) - \sum_{k=2}^{\frac{N}{3}} \left(r_k \tan^{-1}(2^{-k}) \right)}$$

$\cdot\, \theta_{\Sigma L}$ small enough

a carry in the $\frac{N}{3}$-th stage

small enough → realized by a sequence of shift-and-add op's

Otherwise $\theta_{H\frac{N}{3}}$ should be rotated again to realize the carry

the remain of $\theta_{\Sigma L}$ can be realized by a sequence of shift-and-add operations

Chen, w/o ROM Lookup

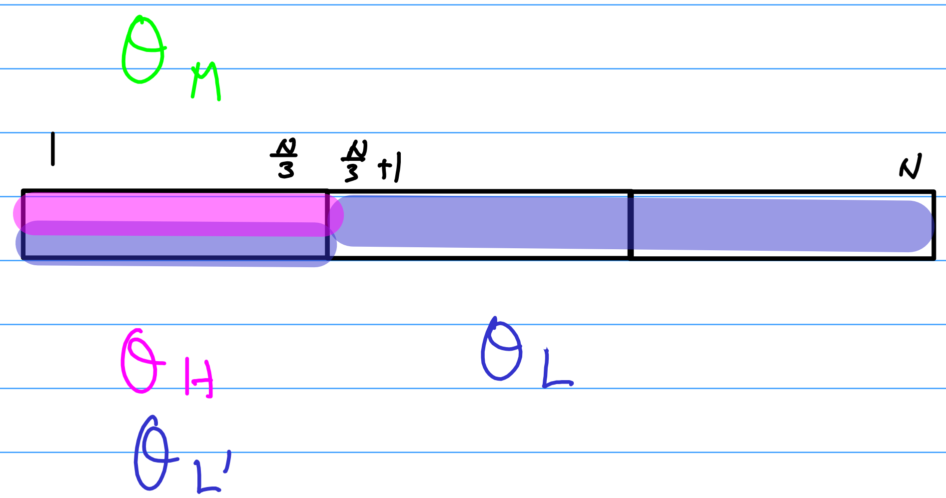
$\phi_0 +$

$$\theta_M = \sum_{k=1}^{\frac{N}{3}} b_k 2^{-k} \longrightarrow \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} = \sum_{k=2}^{\frac{N}{3}} \theta_{Mk}$$

$$\theta_L = \sum_{k=\frac{N}{3}+1}^{N} b_k 2^{-k} \longrightarrow \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k} = \sum_{k=\frac{N}{3}+1}^{N+1} \theta_{Lk}$$

$$\theta_M = \sum_{k=2}^{\frac{N}{3}} \theta_{Mk} = \sum_{k=2}^{\frac{N}{3}} \left(\theta_{Hk} + \theta_{Lk}\right)$$

$$\theta_L = \sum_{k=\frac{N}{3}+1}^{N+1} \theta_{Lk}$$



Chen's coarse-fine approach.

$$\theta = \underbrace{\theta_M}_{\text{coarse}} + \underbrace{\theta_L}_{\text{fine}}$$

$$= \sum_{k=2}^{\frac{N}{3}} r_k 2^{-k} + \sum_{k=\frac{N}{3}+1}^{N+1} r_k 2^{-k}$$

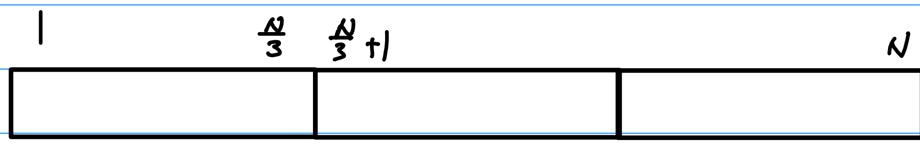
50 year's CORDIC

Swartzlander's Hybrid CORDIC approach

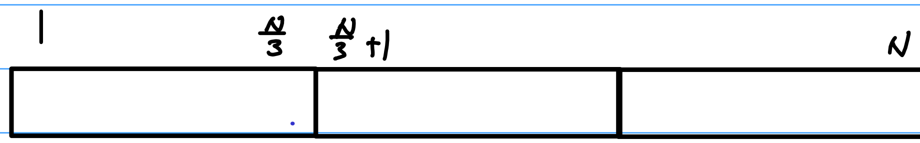
$\theta = \theta_H + \theta_L$ coarse & fine subangles

$\theta_H = \sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i}$ $\sigma_i \in \{1, -1\}$

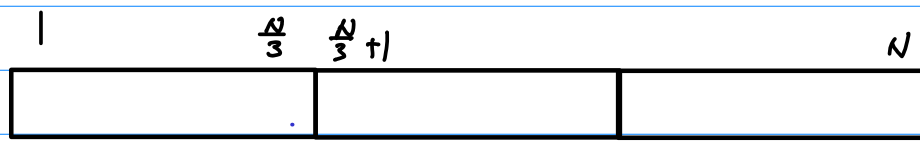
$\theta_L = \sum_{i=p}^{n-1} d_i 2^{-i}$ $d_i \in \{0, 1\}$



$\sigma_k \left(\tan^{-1} 2^{-k} \right)$ $\qquad$ $\sigma_k = \{-1, +1\}$



$b_k 2^{-k}$ $\qquad$ $b_k \in \{0, 1\}$



ϕ_0, $r_k 2^{-k}$ $\qquad$ $r_k \in \{-1, +1\}$

Chen, w/o ROM Lookup

to reduce the ROM size

ϕ_0 : initial angle

$$\theta = \theta_M + \theta_L$$

$$\theta_M = \sum_{k=2}^{N/3} \theta_{Mk} = \sum_{k=2}^{N/3} (\theta_{Hk} + \theta_{Lk})$$

$\parallel$ $\qquad\qquad\qquad$ $\Downarrow$

2^{-k} $\qquad\qquad\qquad$ $\tan C_{\cdot k} = 2^{-k}$

$$\theta_{Lk} = \theta_{Mk} - \theta_{Hk} = \theta_{Mk} - \tan^{-1} 2^{-k}$$

Architecture

$$\begin{cases} X_M = X_0 - \boxed{Y_0 \cdot \tan(\sigma_n \theta_M)} \\ Y_M = Y_0 + \boxed{X_0 \cdot \tan(\sigma_n \theta_M)} \end{cases}$$

coarse rotation

multiplier → bottleneck in computation

a positive angle θ (<1 rad)

$$\theta = \sum_{k=1}^{N} b_k \theta_k$$

b_k the bits corresponding to the $(N+1)$-bit fractional binary number

Sign + N-bit fraction

$b_k \in \{0, 1\}$ $\qquad$ $\theta_k = 2^{-k}$

positive $\qquad$ $\theta_0 = 0$

$b_k \in \{0, 1\} \Rightarrow r_k \in \{-1, +1\}$

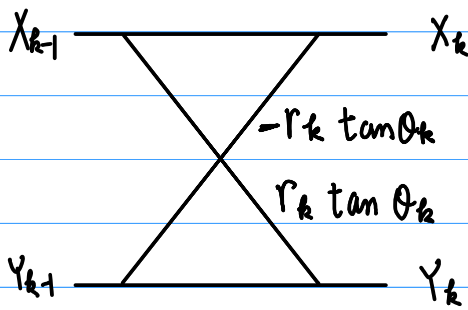
recoded

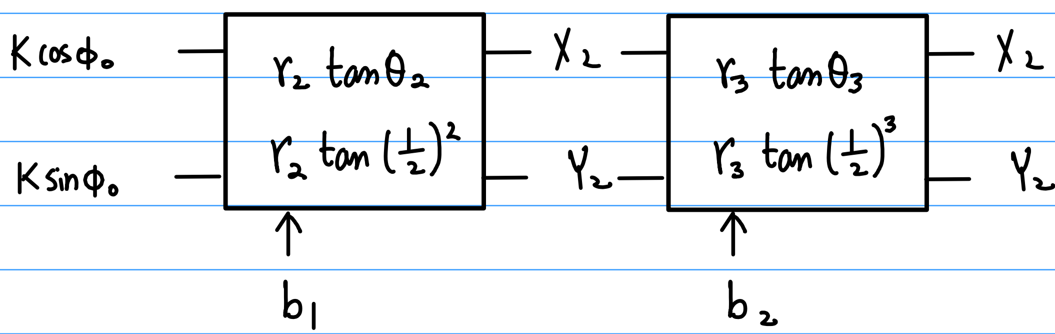
$$X_{k+1} = X_k - (r_k \tan\theta_k)\, Y_k$$
$$Y_{k+1} = Y_k + (r_k \tan\theta_k)\, X_k$$

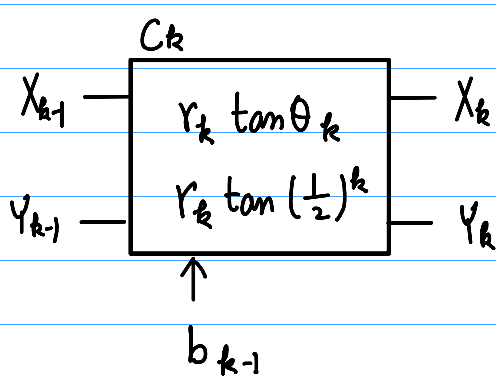
$$X_k = X_{k-1} - (r_{k-1} \tan\theta_{k-1}\, Y_{k-1}$$
$$Y_k = Y_{k-1} + (r_{k-1} \tan\theta_{k-1}\, X_{k-1}$$

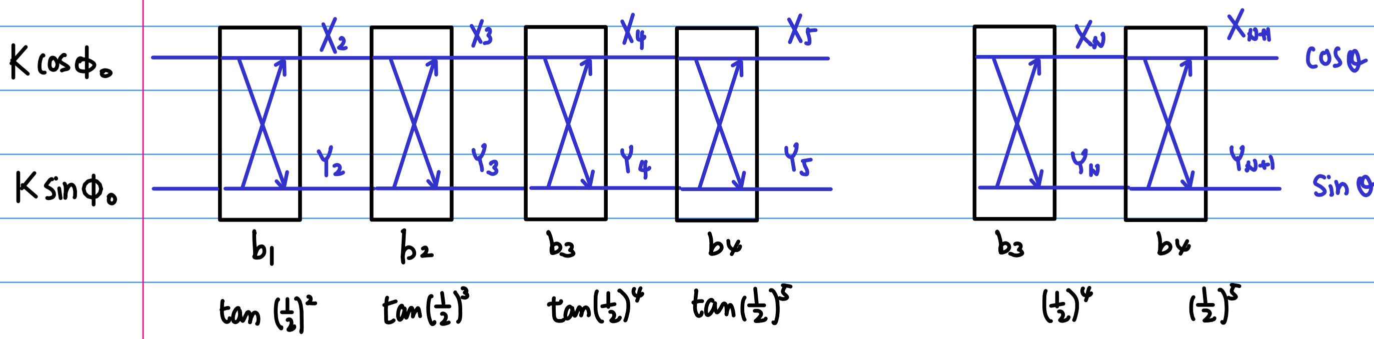
$$X_{k+1} = X_k - (r_k \tan\theta_k)\, Y_k$$
$$Y_{k+1} = Y_k + (r_k \tan\theta_k)\, X_k$$

$$X_k = X_{k-1} - (r_k \tan\theta_k)\, Y_{k-1}$$
$$Y_k = Y_{k-1} + (r_k \tan\theta_k)\, X_{k-1}$$









$$\theta = \sum_{k=1}^{N} b_k 2^{-k} = \phi_0 + \sum_{k=2}^{N+1} r_k 2^{-k}$$

binary digit representation

$b_k \in \{0, 1\}$

Signed digit recoding

$r_k \in \{-1, +1\}$

$$\boxed{b_k = (r_k + 1)/2}$$

$$\boxed{r_k = (2b_{k-1} - 1)}$$

the coarse-fine partition
could be applied for reducing
the number of micro-rotations
necessary for fine rotations

To implement the coarse rotation
through **shift-add** operation

the coarse sub angle θ_M
in terms of elementary rotations of the form $\tan^{-1} 2^{-i}$

$$\theta_M = \sum_{i=1}^{p-1} d_i \, 2^{-i} \qquad = \sum_{i=1}^{p-1} \left(\sigma_i \tan^{-1}(2^{-i}) - \theta_{L_i} \right)$$

$$\sum_{i=1}^{p-1} \boxed{d_i \, 2^{-i}} \qquad \qquad \sum_{i=1}^{p-1} \boxed{\sigma_i \tan^{-1}(2^{-i})}$$

shift-add coarse rotation

θ_{L_i} : correction term

Chen, w/o ROM Lookup

$$\theta_M = \sum_{k=1}^{N/3} b_k 2^k$$

$$\theta_L = \sum_{i=N/3+1}^{N} b_k 2^k$$

$$\theta = \theta_H + \theta_L$$
$$= \theta_M + \tilde{\theta}_L$$
$$= \sum_{i=1}^{p-1} \sigma_i \tan^{-1}(2^{-i}) + \sum_{i=p}^{n-1} d_i 2^{-i}$$
$$= \sum_{i=1}^{p-1} d_i 2^{-i} + \boxed{\sum_{i=1}^{p-1} \theta_{L_i} + \sum_{i=p}^{n-1} d_i 2^{-i}}$$
$$\boxed{\sum_{i=1}^{p-1} \theta_{L_i} + \theta_L}$$
$$\boxed{\tilde{\theta}_L}$$

$$\theta = \boxed{\theta_H} + \boxed{\theta_L} \quad \text{(higher \& lower parts)}$$

$$= \boxed{\sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i}} + \boxed{\sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$\sigma_i \in \{1, -1\} \qquad d_i \in \{0, 1\}$$

$$\boxed{\theta_M} + \boxed{\tilde{\theta}_L}$$

$$= \boxed{\sum_{i=1}^{p-1} \sigma_i \tan^{-1} 2^{-i} - \sum_{i=1}^{p-1} \theta_{Li}} + \boxed{\sum_{i=1}^{p-1} \theta_{Li} + \sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$= \boxed{\sum_{i=1}^{p-1} d_i \, 2^{-i}} + \boxed{\sum_{i=1}^{p-1} \theta_{Li} + \sum_{i=p}^{n-1} d_i \, 2^{-i}}$$

$$\tilde{\theta}_L = \theta_L + \sum_{i=2}^{N/3} \theta_{L_i} \qquad \boxed{(-\theta_0) + \sum_{j=1}^{m-1} \theta_j 2^{j-1}} + \boxed{\sum_{j=m}^{N} \theta_j 2^{-j}}$$

$$\theta_M \simeq \sum_{i=2}^{n/3} d_i \, 2^{-i} = \sum_{i=2}^{N/3} \left(\sigma_i \tan^{-1}(2^{-i}) + \theta_{L_i} \right)$$

[24] D. Fu and A. N. Willson, Jr., “A high-speed processor for digital sine/cosine generation and angle rotation,” in *Conf. Rec. 32nd Asilomar Conf. on Signals, Syst. & Computers*, Nov. 1998, vol. 1, pp. 177–181.

[25]

[25] C.-Y. Chen and W.-C. Liu, “Architecture for CORDIC algorithm realization without ROM lookup tables,” in *Proc. 2003 Int. Symp. on Circuits Syst., ISCAS’03*, May 2003, vol. 4, pp. 544–547.

both coarse and fine rotations
can be implemented by a sequence of
shift-add operations
without ROM look-up or
real multiplication

PROC-I : the conventional CORDIC
the first 1/3 iteration

{ the residual angle
the intermediate rotated vector

reduced latency implementation
sine-cosine generation

[24] D. Fu and A. N. Willson, Jr., “A high-speed processor for digital sine/cosine generation and angle rotation,” in *Conf. Rec. 32nd Asilomar Conf. on Signals, Syst. & Computers*, Nov. 1998, vol. 1, pp. 177–181.

[25] C.-Y. Chen and W.-C. Liu, “Architecture for CORDIC algorithm realization without ROM lookup tables,” in *Proc. 2003 Int. Symp. on Circuits Syst., ISCAS’03*, May 2003, vol. 4, pp. 544–547.

[26] D. Fu and A. N. Willson, Jr., “A two-stage angle-rotation architecture and its error analysis for efficient digital mixer implementation,” *IEEE Trans.Circuits Syst. I: Reg. Papers*, vol. 53, no. 3, pp. 604–614, Mar. 2006.

[27] S. Ravichandran and V. Asari, “Implementation of unidirectional CORDIC algorithm using precomputed rotation bits,” in *45th Midwest Symp. on Circuits Syst., 2002. MWSCAS 2002*, Aug. 2002, vol. 3, pp. 453–456.

[28] C.-Y. Chen and C.-Y. Lin, “High-resolution architecture for CORDIC algorithm realization,” in *Proc. Int. Conf. on Commun., Circuits Syst., ICCCS’06*, June 2006, vol. 1, pp. 579–582.

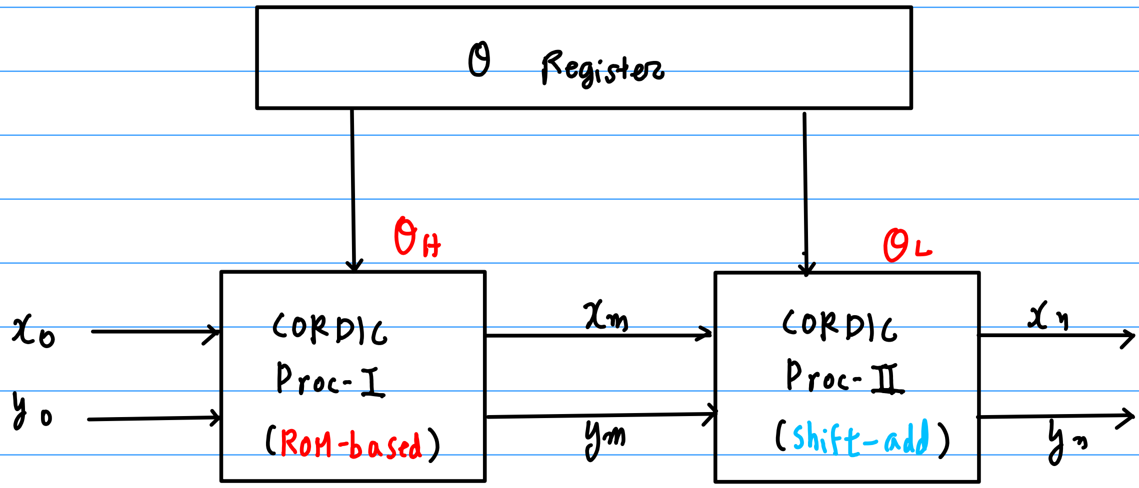
High speed, High precision [24], [26]

[29] D. D. Hwang, D. Fu, and A. N. Willson, Jr., “A 400-MHz processor for the conversion of rectangular to polar coordinates in 0.25-/mu-m CMOS,” *IEEE J. Solid-State Circuits*, vol. 38, no. 10, pp. 1771–1775, Oct. 2003.

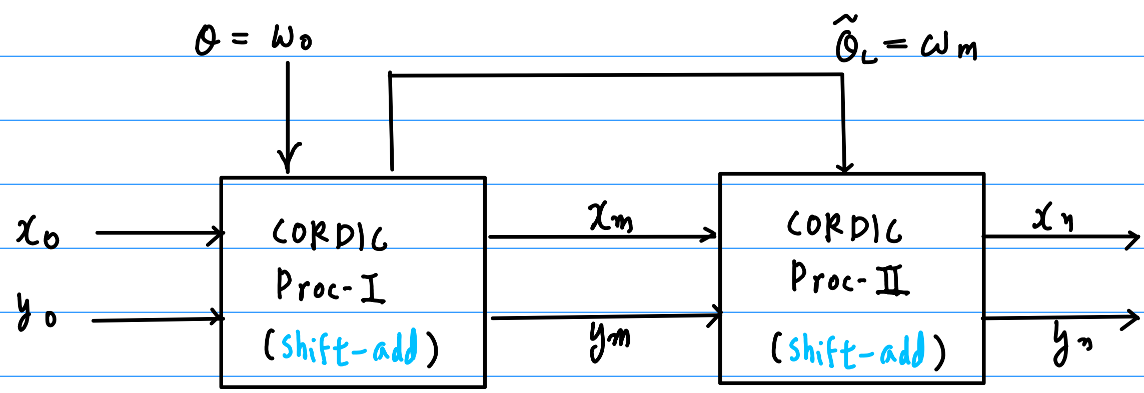
[30] S.-W. Lee, K.-S. Kwon, and I.-C. Park, “Pipelined cartesian-to-polar coordinate conversion based on SRT division,” *IEEE Trans. Circuits Syst. II: Express Briefs*, vol. 54, no. 8, pp. 680–684, Aug. 2007.

ordinates

[10] Hybrid CORDIC, Wang, Swartzlander



[24] Fu & Willson Sine / Cosine Generation



$$\theta_H = \sum_{i=1}^{p-1} d_i 2^{-i} = \sum_{i=1}^{p-1} \left(\sigma_i \tan^{-1} 2^{-i} - \underline{\theta_{L_i}} \right)$$

↳ correction term

$$\theta = \sum_{i=1}^{p-1} d_i 2^{-i} + \tilde{\theta}_L$$

$$\tilde{\theta}_L = \theta_L + \sum_{i=1}^{p-1} \theta_{L_i}$$

Wilson ROM based Sine/Cosine Generation

[24] Fu & Willson Sine / Cosine Generation

ROM-based

For high resolution, ROM size grows exponentially

quater-wave symmetry

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\phi \ [0, 2\pi] \longrightarrow [0, \frac{\pi}{4}]$$

conditionally interchanging inputs X_0 & Y_0
Conditionally interchanging and negating outputs X & Y

$$X = X_0 \cos\phi - Y_0 \sin\phi$$
$$Y = Y_0 \cos\phi + X_0 \sin\phi$$

Madisetti VLSI arch

Microrotation Angle Recording

$i = 1, \cdots, m-1 \Rightarrow \tan^{-1}(2^{-i}) \neq 2^{-i}$

decompose each positional binary weight

$\boxed{2^{-i}}$, $i = 1, 2, \cdots, m-1$ into

$$\begin{cases} \text{the combination of significant } \boxed{\tan^{-1}(2^{-j})} \text{ terms} \\ \text{plus error terms } \boxed{e_i} \end{cases}$$

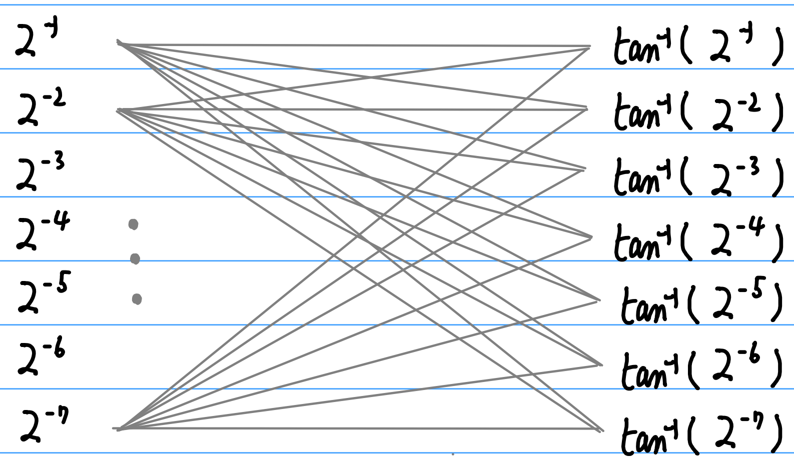
$$\begin{cases} \text{collecting all the other} \\ \text{insignificant values of} \\ \tan^{-1}(2^{-j}),\ j > m \end{cases}$$

$N = 24, \Rightarrow m = \lceil (N - \log_2 3)/3 \rceil = 8$

$m = 8$

micro rotation angle

2^{-i} as a linear combination of $\tan^{-1}(2^{-j})$



$2^{-1} = \tan^{-1}(2^{-1}) + \tan^{-1}(2^{-5}) + \tan^{-1}(2^{-8}) + e_1$

$2^{-2} = \tan^{-1}(2^{-2}) + \tan^{-1}(2^{-8}) + e_2$

$2^{-3} = \tan^{-1}(2^{-3}) + e_3$

$2^{-4} = \tan^{-1}(2^{-4}) + e_4$

$2^{-5} = \tan^{-1}(2^{-5}) + e_5$

$2^{-6} = \tan^{-1}(2^{-6}) + e_6$

$2^{-7} = \tan^{-1}(2^{-7}) + e_7$

$$\theta_H = (-\theta_0) + 2^{-1} + \sum_{k=2}^{m} r_k 2^{-k} - 2^{-m}$$

$$= (-\theta_0) + 2^{-1} + \sum_{k=2}^{8} r_k 2^{-k} - 2^{-8}$$

$$= (1-2\theta_0)2^{-1} + \sum_{k=2}^{8} r_k 2^{-k} - 2^{-8}$$

$$r_k = (2\theta_k - 1) \qquad r_k \in \{-1, +1\}$$

the first 8 rotation directions are computed concurrently

$$\sigma_1 = (1-2\theta_0)$$

$$\sigma_k = r_k = (2\theta_{k-1} - 1), \quad k = 2, \cdots, 8$$

$$\sigma_1 = (2\theta_0 - 1)\cdot(-1)$$

$$\sigma_2 = (2\theta_1 - 1)$$

$$\sigma_3 = (2\theta_2 - 1)$$

$$\sigma_4 = (2\theta_3 - 1)$$

$$\sigma_5 = (2\theta_4 - 1)$$

$$\sigma_6 = (2\theta_5 - 1)$$

$$\sigma_7 = (2\theta_6 - 1)$$

$$\sigma_8 = (2\theta_7 - 1)$$

$$\theta_{17} = (-\theta_0) + 2^{-1} + \sum_{k=2}^{m} r_k 2^{-k} - 2^{-m}$$

$$= (1-2\theta_0)2^{-1} + \sum_{k=2}^{m} r_k 2^{-k} - 2^{-m}$$

$$= (1-2\theta_0)2^{-1} + \sum_{k=2}^{8} r_k 2^{-k} - 2^{-8}$$

$$r_k = (2\theta_{k-1} - 1) \in \{-1, +1\}$$

the first 8 rotation directions

$$\sigma_1 = (1-2\theta_0)$$
$$\sigma_2 = (2\theta_1 - 1) = r_2$$
$$\sigma_3 = (2\theta_2 - 1) = r_3$$
$$\sigma_4 = (2\theta_3 - 1) = r_4$$
$$\sigma_5 = (2\theta_4 - 1) = r_5$$
$$\sigma_6 = (2\theta_5 - 1) = r_6$$
$$\sigma_7 = (2\theta_6 - 1) = r_7$$
$$\sigma_8 = (2\theta_7 - 1) = r_8$$

all the signed error terms $\sigma_i e_i$ $i = 1, \cdots, 7$
and the last term -2^{-8} are added to θ_L

the corrected lower part $\hat{\theta}_L$ — 2's complement form

$$\hat{\theta}_L = \theta_L + \sum_{i=1}^{7} \sigma_i e_i - 2^{-8}$$

$$= (-\hat{\theta}_7)\, 2^{-7} + \sum_{k=8}^{24} \hat{\theta}_k 2^{-k}$$

$\hat{\theta}_k \in \{0, 1\}$

$k = 8, \cdots, 24$

$$2^{-1} = \tan^{-1}(2^{-1}) + \tan^{-1}(2^{-5}) + \tan^{-1}(2^{-8}) \quad + e_1 \times \sigma_1$$
$$2^{-2} = \tan^{-1}(2^{-2}) + \tan^{-1}(2^{-8}) \quad + e_2 \times \sigma_2$$
$$2^{-3} = \tan^{-1}(2^{-3}) \quad + e_3 \times \sigma_3$$
$$2^{-4} = \tan^{-1}(2^{-4}) \quad + e_4 \times \sigma_4$$
$$2^{-5} = \tan^{-1}(2^{-5}) \quad + e_5 \times \sigma_5$$
$$2^{-6} = \tan^{-1}(2^{-6}) \quad + e_6 \times \sigma_6$$
$$2^{-7} = \tan^{-1}(2^{-7}) \quad + e_7 \times \sigma_7$$

$$\theta_H = (1-2\theta_0)2^{-1} + \sum_{k=2}^{8} r_k 2^{-k} - 2^{-8} \quad + \quad \rightarrow \theta_L$$

Corrected lower part angle

$$\hat{\theta}_L = \theta_L + \sum_{i=1}^{7} e_i \cdot \sigma_i - 2^{-8}$$

all the signed error terms $\sigma_i e_i$ $i = 1, \cdots, 7$
and the last term -2^{-8}
are added to θ_L
generating the corrected lower part $\hat{\theta}_L$

$$\hat{\theta}_L = \theta_L + \sum_{i=1}^{7} e_i \cdot \sigma_i - 2^{-8}$$

$$= (1-\hat{\theta}_7) 2^{-7} + \sum_{k=8}^{24} \hat{\theta}_k 2^{-k}, \qquad \hat{\theta}_k \in \{0, 1\}$$

$$k = 8, \cdots, 24$$

$$|\hat{\theta}_L| \leq 2^{-7}$$

$$\tan^{-1} 2^{-i} = 2^{-i} \qquad i \geq 8$$

$$\hat{\theta}_L = \theta_L + \sum_{i=1}^{7} e_i \cdot \sigma_i - 2^{-8}$$

$$= (1-\hat{\theta}_7) 2^{-7} + \sum_{k=8}^{24} \hat{\theta}_k 2^{-k}, \qquad \hat{\theta}_k \in \{0, 1\}$$

$$k = 8, \cdots, 24$$

$$\hat{\theta}_L = (-\hat{\theta}_7) 2^{-7} + \sum_{k=8}^{24} \hat{\theta}_k 2^{-k}$$

$$= (-\hat{\theta}_7) 2^{-7} + \sum_{k=9}^{25} (2\theta_{k-1} - 1) 2^{-k} + 2^{-8} - 2^{-25}$$

$$= (1 - 2\hat{\theta}_7) 2^{-8} + \sum_{k=9}^{25} \hat{r}_k 2^{-k} - 2^{-25}$$

$$\hat{r}_k = (2\hat{\theta}_k - 1) \in \{1, -1\}$$

$$\hat{\sigma}_8 = (1 - 2\hat{\theta}_7)$$

$$\hat{\sigma}_k = \hat{r}_k = (2\hat{\theta}_k - 1), \qquad k = 9, \cdots, 25$$

for the last $2N/3$ iterations

CSA Carry Save Adder

4:2 compressor