CTFT of Rectangular Pulse Functions (3B)

- CTFT of a Rectangular Pulse
- CTFT of a Shifted Rectangular Pulse
- Spectrum Plots of the CTFT of a Rectangular Pulse
- Spectrum Plots of the CTFT of a Shifted Rectangular Pulse

Young W. Lim
11/9/13

Please send corrections (or suggestions) to youngwlim@hotmail.com.

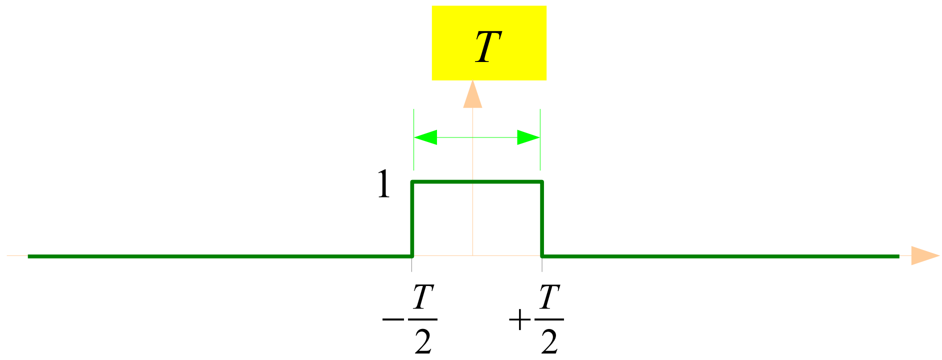
This document was produced by using OpenOffice and Octave.

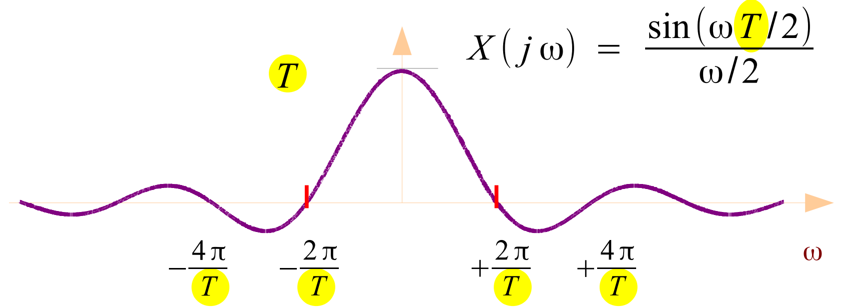
- CTFT of a Rectangular Pulse
- CTFT of a Shifted Rectangular Pulse
- Spectrum Plots of the CTFT of a Rectangular Pulse
- Spectrum Plots of the CTFT of a Shifted Rectangular Pulse

CTFT of a Rectangular Pulse (1)

Continuous Time Fourier Transform **CTFT**

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$





$$x(t) = \boldsymbol{rect}\left(\frac{t}{T}\right)$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = T \cdot \boldsymbol{sinc}(f T)$$

$$X(j0) = T$$

CTFT of a Rectangular Pulse (2)

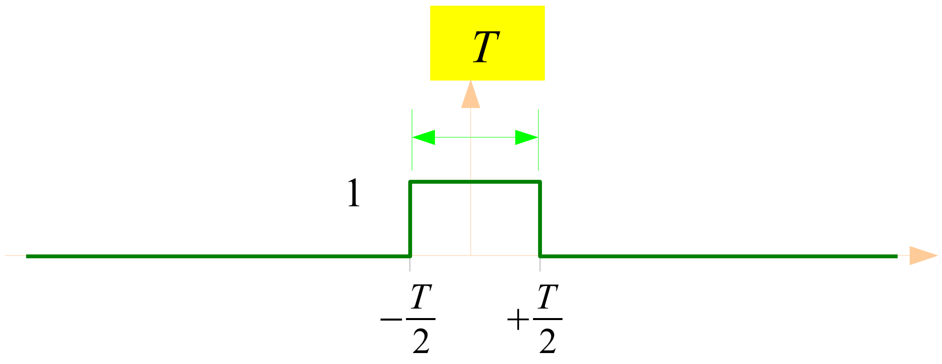
Continuous Time Fourier Transform Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt$$

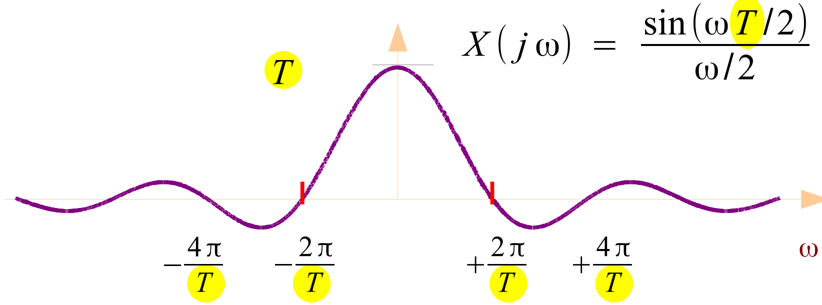
$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_{-T/2}^{+T/2} = -\left(\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega}\right)$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

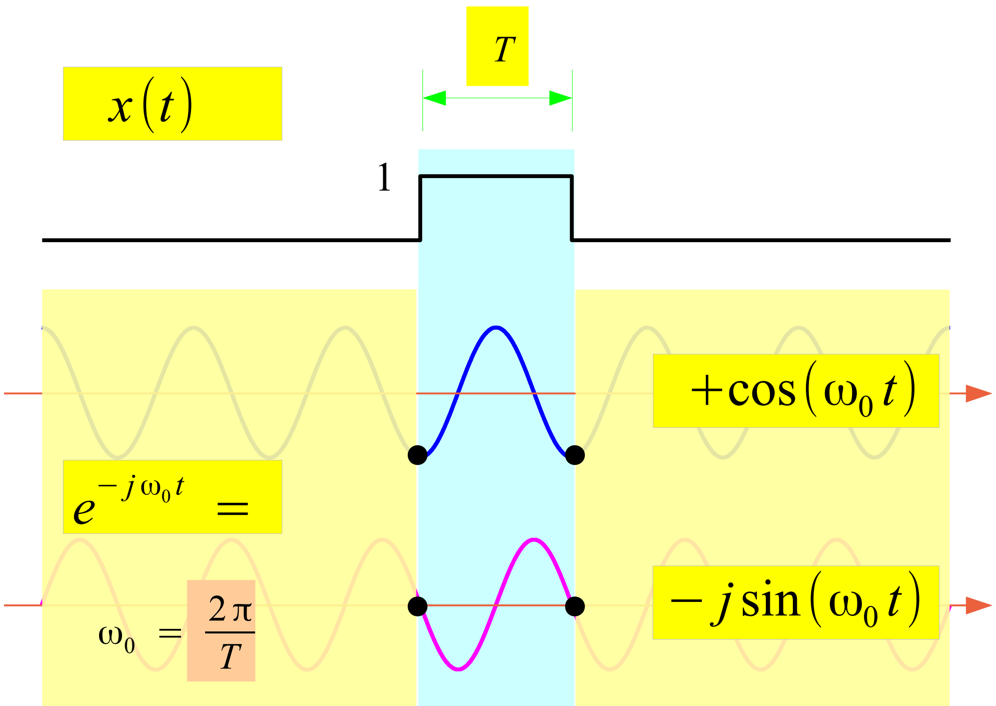


$$X(j\,0) = \lim_{\omega\to 0} \frac{\sin(\omega T/2)}{\omega/2}$$

$$= \lim_{\omega\to 0} \frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$$



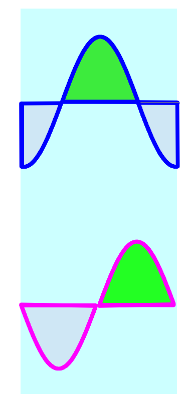
Zero Crossings of $T\,sinc(f\,T)$ (1)



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{+\infty} x(t)\cos\omega_0 t\, dt = 0 \qquad \omega_0 = \frac{2\pi}{T}$$

$$\int_{-\infty}^{+\infty} x(t)\sin\omega_0 t\, dt = 0 \qquad \omega_0 = \frac{2\pi}{T}$$

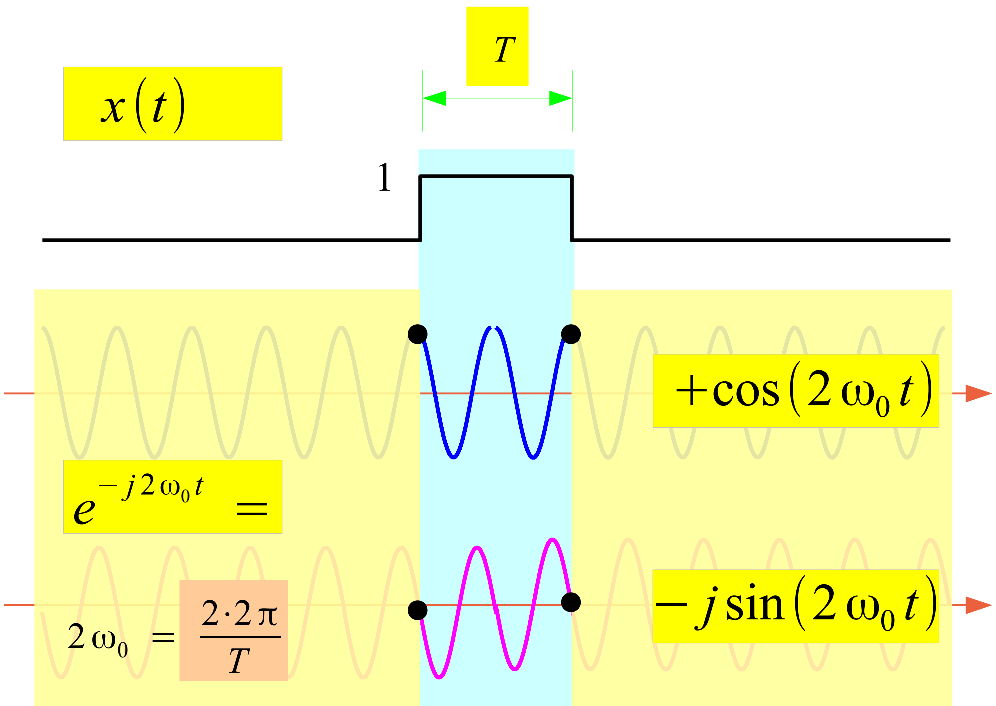


$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = \boxed{T \cdot sinc(f\,T)}$$

Zeros at

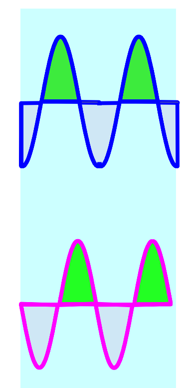
$$X\left(j\,k\frac{2\pi}{T}\right) = 0 \quad \Leftarrow \quad \omega = k\frac{2\pi}{T}$$

Zero Crossings of $T\,\mathit{sinc}(f\,T)$ (2)



$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\int_{-\infty}^{+\infty} x(t) \cos 2\omega_0 t\, dt = 0 \qquad 2\omega_0 = \frac{2\cdot 2\pi}{T}$$



$$\int_{-\infty}^{+\infty} x(t) \sin 2\omega_0 t\, dt = 0 \qquad 2\omega_0 = \frac{2\cdot 2\pi}{T}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = \boxed{T \cdot \mathit{sinc}(f\,T)}$$

Zeros at

$$X\left(j k \frac{2\pi}{T}\right) = 0 \quad \Leftarrow \quad \omega = k\frac{2\pi}{T}$$

Zero Crossings of $T\,sinc(f\,T)$ (3)

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = T\frac{\sin(\omega T/2)}{\omega T/2} = T\frac{\sin(\pi f T)}{\pi f T}$$

Normalized Sinc function

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

$$X(f) = T \cdot sinc(f\,T)$$

Zeros at $t = \pm 1, \pm 2, \cdots$

Zeros at $f\,T = \pm 1, \pm 2, \cdots$

$$f = \pm\frac{1}{T}, \pm\frac{2}{T}, \cdots$$

Unnormalized Sinc function

$$sinc(x) = \frac{\sin(x)}{x}$$

$$X(j\omega) = T \cdot sinc(\omega T/2)$$

Zeros at $x = \pm\pi, \pm 2\pi, \cdots$

Zeros at $\omega T/2 = \pm\pi, \pm 2\pi, \cdots$

$$\omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \cdots$$

Zero Crossings of $T\,sinc(f\,T)$ (4)

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = T\frac{\sin(\omega T/2)}{\omega T/2} = T\frac{\sin(\pi f T)}{\pi f T}$$

Normalized Sinc function

$$sinc(t) = \frac{\sin(\pi t)}{\pi t}$$

$$X(f) = T \cdot sinc(f\,T)$$

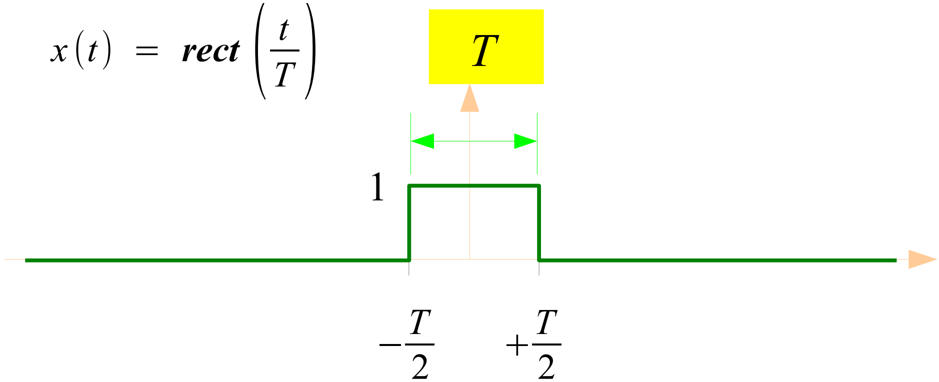
Zeros at $t = \pm 1,\ \pm 2,\ \cdots$

Zeros at $f\,T = \pm 1,\ \pm 2,\ \cdots$

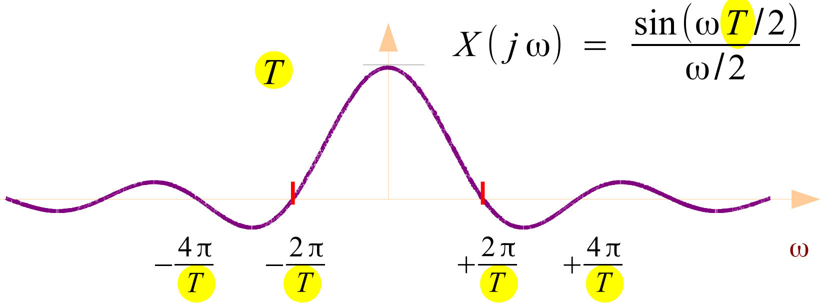
$$f = \pm\frac{1}{T},\ \pm\frac{2}{T},\ \cdots$$

$$\omega = \pm\frac{2\pi}{T},\ \pm\frac{4\pi}{T},\ \cdots$$

$$x(t) = rect\left(\frac{t}{T}\right)$$



$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



Summary : CTFS of a Rectangular Pulse

Continuous Time Fourier Transform — Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2} = T\frac{\sin(\omega T/2)}{\omega T/2} = T\frac{\sin(\pi f T)}{\pi f T}$$

$$X(f) = T \cdot \boldsymbol{sinc}(f\,T)$$

$$X(j\omega) = T \cdot \boldsymbol{sinc}(\omega T/2)$$

Zeros at

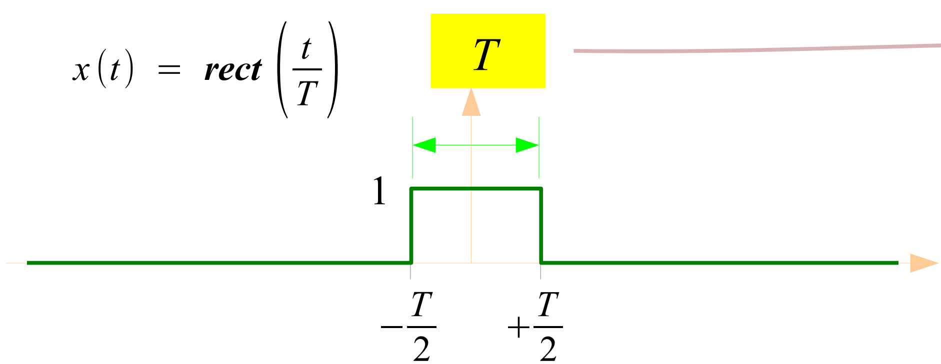
$$f = \pm\frac{1}{T},\ \pm\frac{2}{T},\ \cdots$$

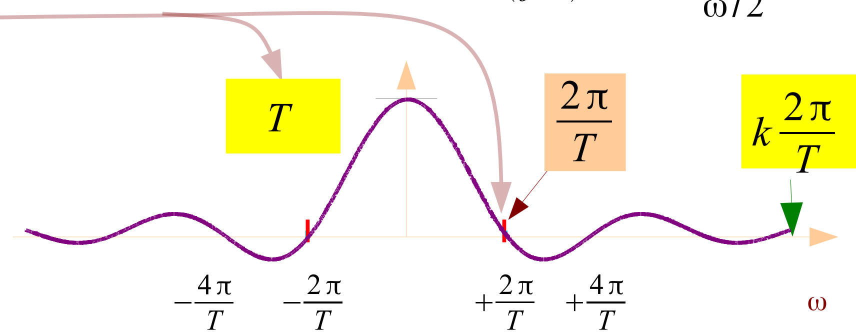
Zeros at

$$\omega = \pm\frac{2\pi}{T},\ \pm\frac{4\pi}{T},\ \cdots$$

$$x(t) = \boldsymbol{rect}\left(\frac{t}{T}\right)$$

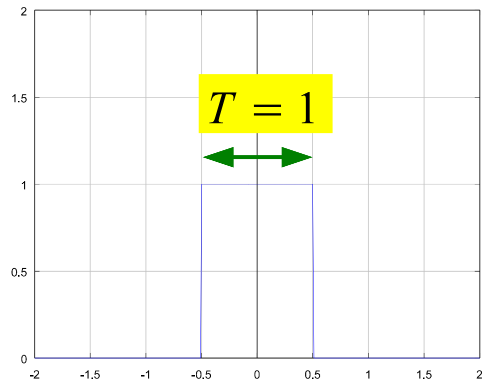
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

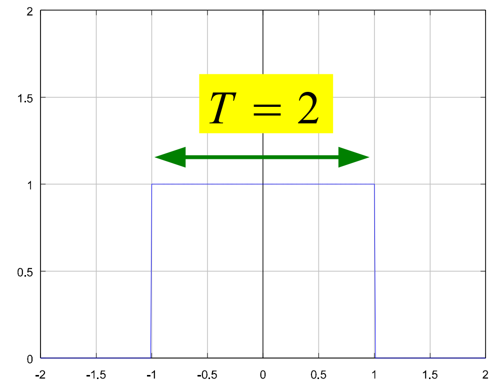


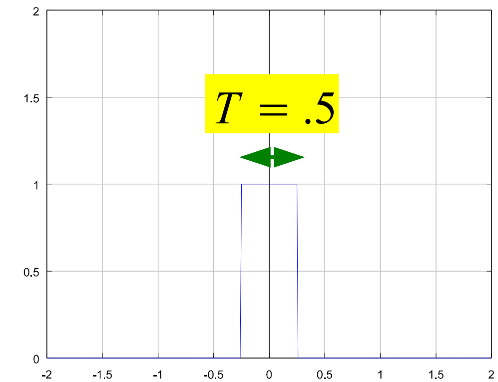


CTFS Pairs of Rect(t/T) - (1)

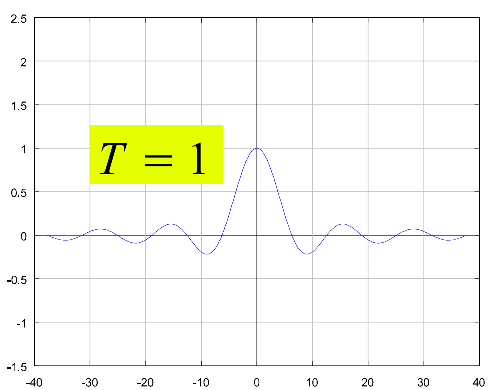
$$x(t) = \boldsymbol{rect}(t) \qquad x(t) = \boldsymbol{rect}\left(\frac{t}{2}\right) \qquad x(t) = \boldsymbol{rect}(2t)$$

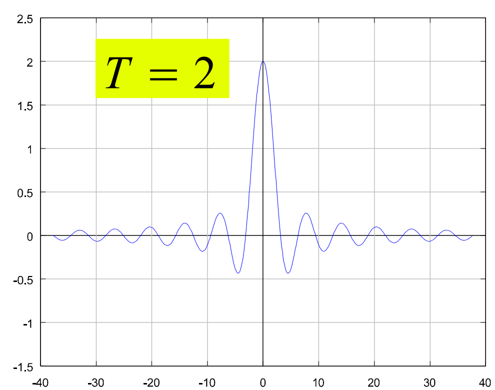


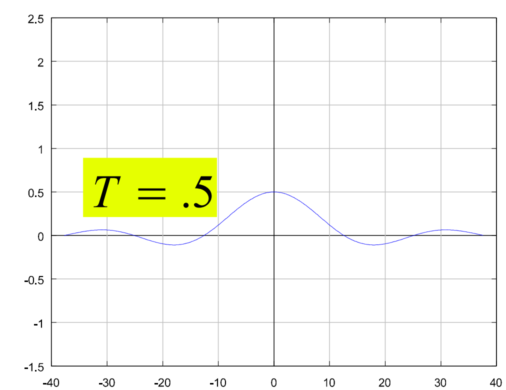




$$X(j\omega) = \boldsymbol{sinc}(\omega) = \frac{\sin(\omega/2)}{\omega/2} \qquad X(j\omega) = 2\,\boldsymbol{sinc}(\omega) = \frac{\sin(\omega)}{\omega/4} \qquad X(j\omega) = \frac{1}{2}\boldsymbol{sinc}\left(\frac{\omega}{2}\right) = \frac{\sin(\omega/4)}{\omega}$$





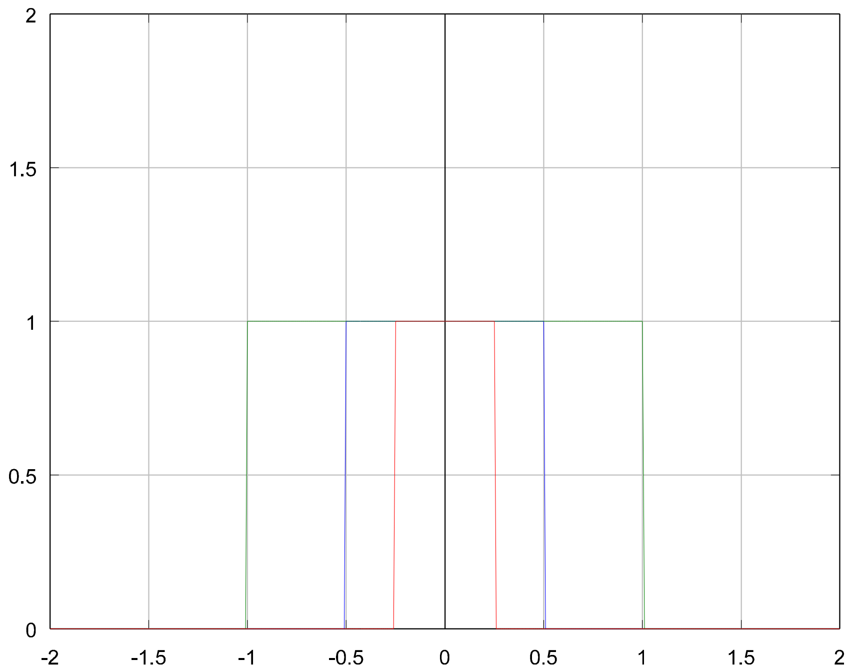


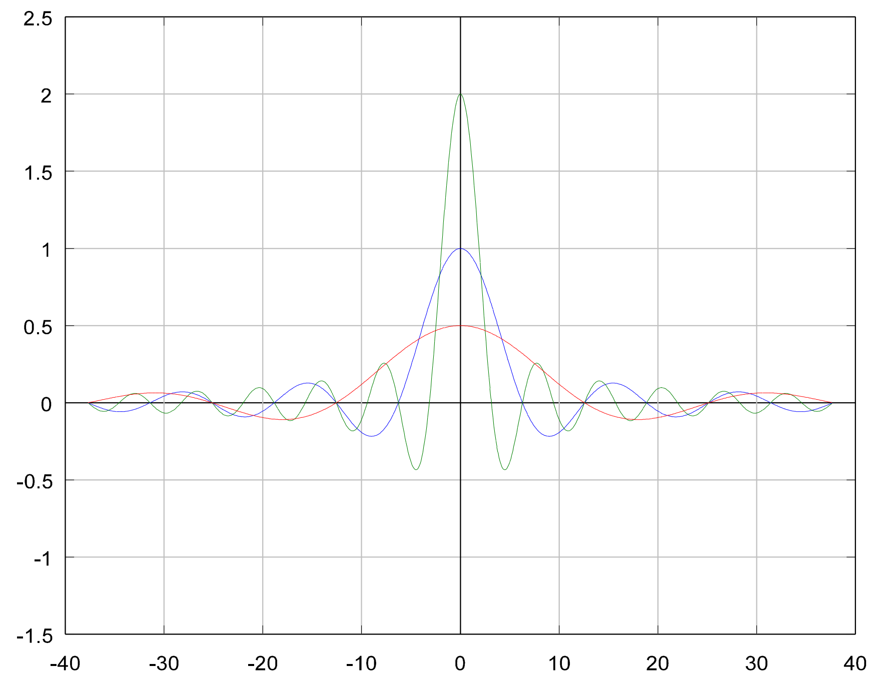
$$zeros\ at\ \ X\left(jk\frac{2\pi}{1}\right) = 0 \qquad zeros\ at\ \ X\left(jk\frac{2\pi}{2}\right) = 0 \qquad zeros\ at\ \ X\left(jk\frac{2\pi}{1/2}\right) = 0$$

CTFS Pairs of Rect(t/T) - (2)

$$x(t) = \boldsymbol{rect}\left(\frac{t}{T}\right)$$

$$X(j\omega) = T\,\boldsymbol{sinc}(T\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

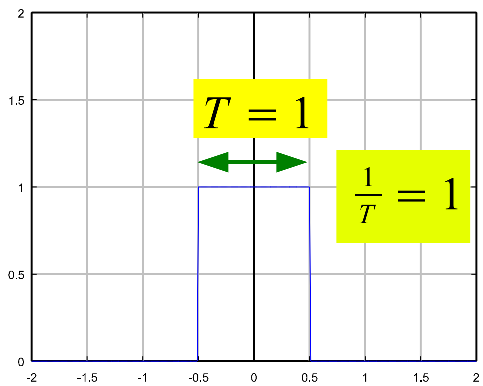




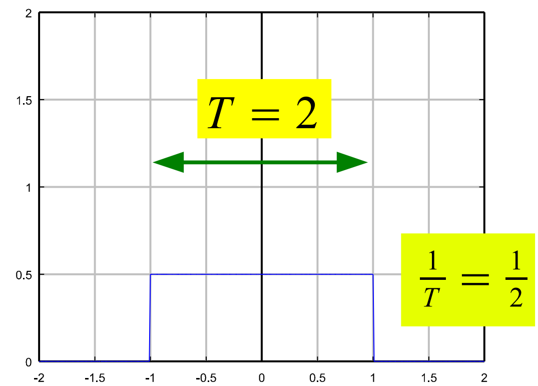
$$zeros\ at\quad X\left(jk\frac{2\pi}{T}\right) = 0$$

CTFS Pairs of (1/T)Rect(t/T) - (1)

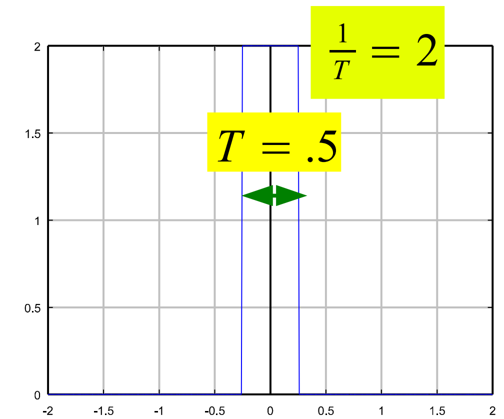
$$x(t) = \textbf{rect}(t)$$



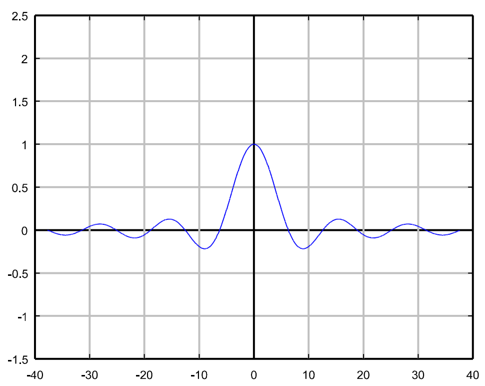
$$x(t) = \frac{1}{2}\textbf{rect}\left(\frac{t}{2}\right)$$



$$x(t) = 2\,\textbf{rect}(2t)$$

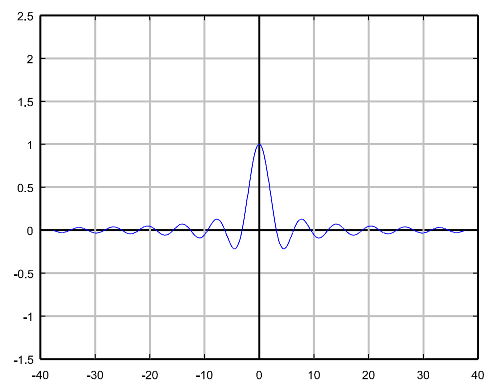


$$X(j\omega) = \textbf{sinc}(\omega) = \frac{\sin(\omega/2)}{\omega/2}$$



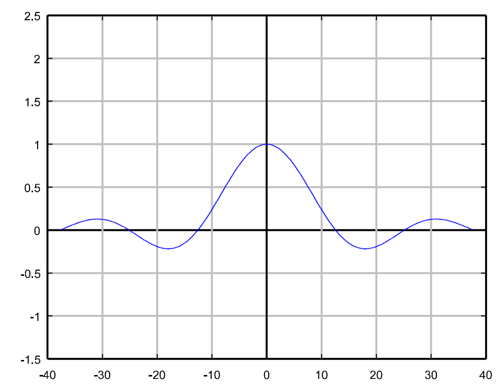
$$zeros\ at\quad X\left(jk\frac{2\pi}{1}\right) = 0$$

$$X(j\omega) = \textbf{sinc}(2\omega) = \frac{\sin(\omega)}{\omega/2}$$



$$zeros\ at\quad X\left(jk\frac{2\pi}{2}\right) = 0$$

$$X(j\omega) = \textbf{sinc}\left(\frac{\omega}{2}\right) = \frac{\sin(\omega/4)}{\omega/2}$$

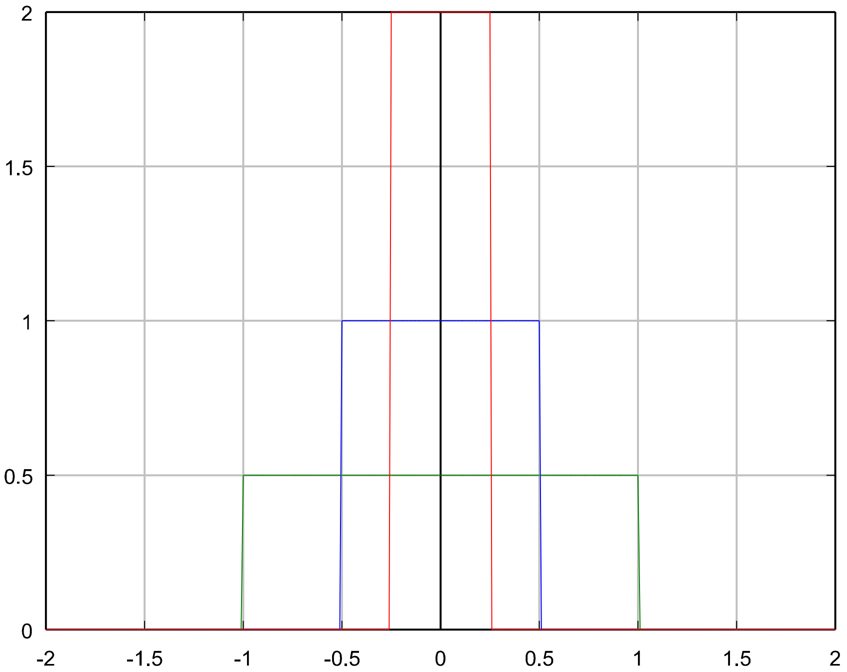


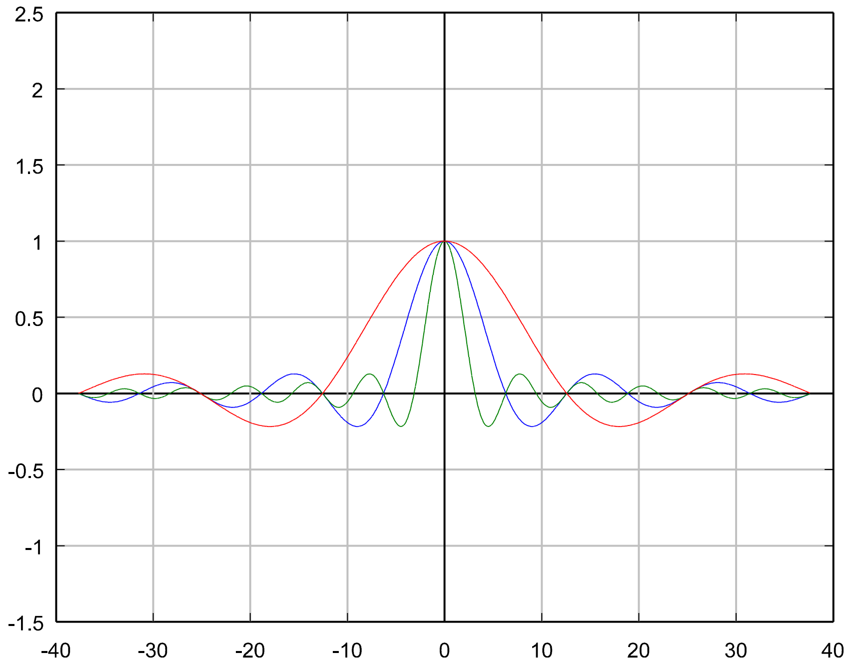
$$zeros\ at\quad X\left(jk\frac{2\pi}{1/2}\right) = 0$$

CTFS Pairs of (1/T)Rect(t/T) - (2)

$$x(t) = \left(\frac{1}{T}\right) \boldsymbol{rect}\left(\frac{t}{T}\right)$$

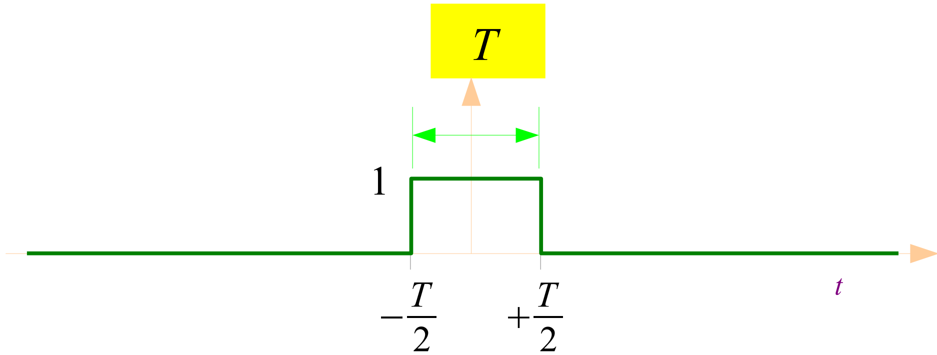
$$X(j\omega) = T\,\boldsymbol{sinc}(T\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

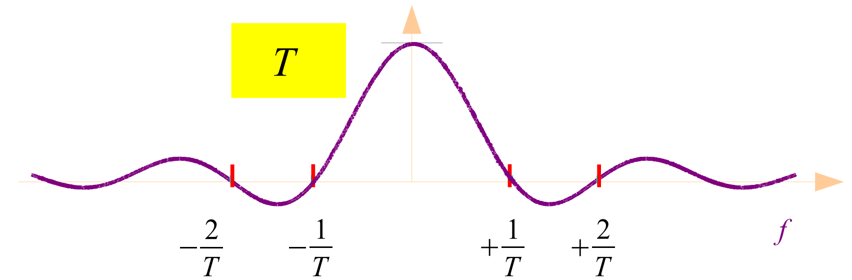




$$zeros\ at\quad X\left(jk\frac{2\pi}{T}\right) = 0$$

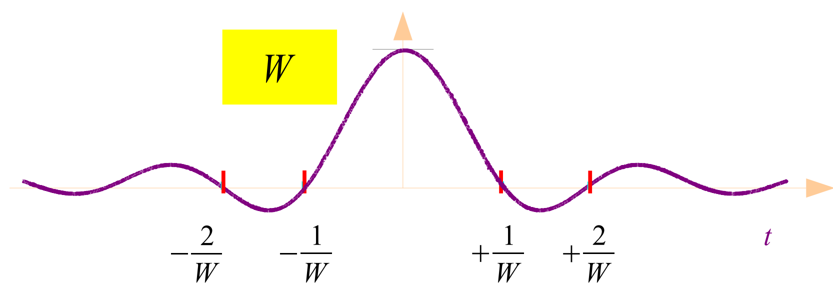
Duality (1)

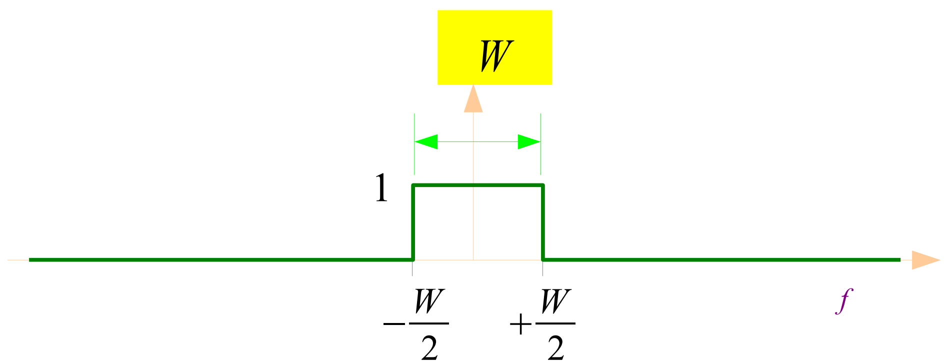




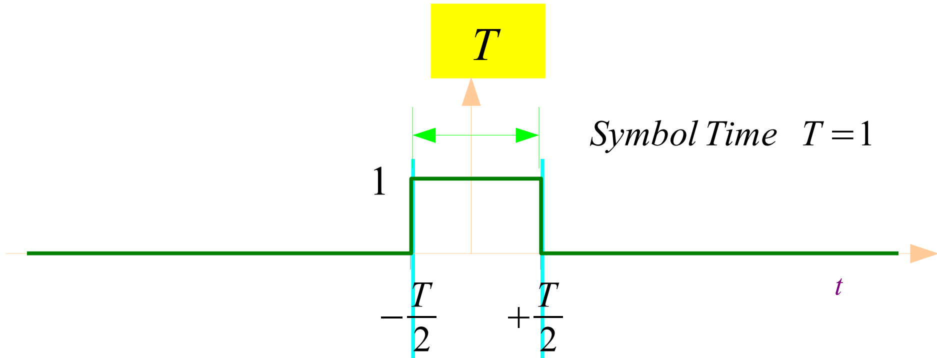
$$x(t) = \boldsymbol{rect}\left(\frac{t}{T}\right) \quad \Longleftrightarrow \quad X(f) = T\,\boldsymbol{sinc}(f\,T)$$

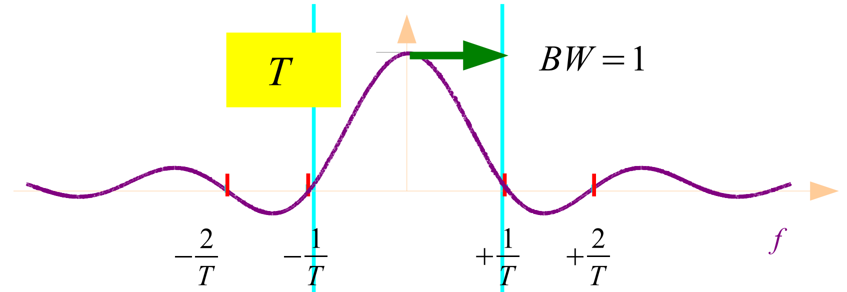
$$X(t) = W\,\boldsymbol{sinc}(t\,W) \quad \Longleftrightarrow \quad x(t) = \boldsymbol{rect}\left(\frac{f}{W}\right)$$





Duality (2)

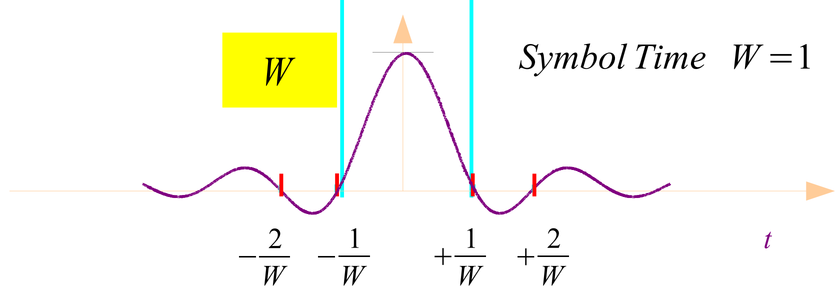


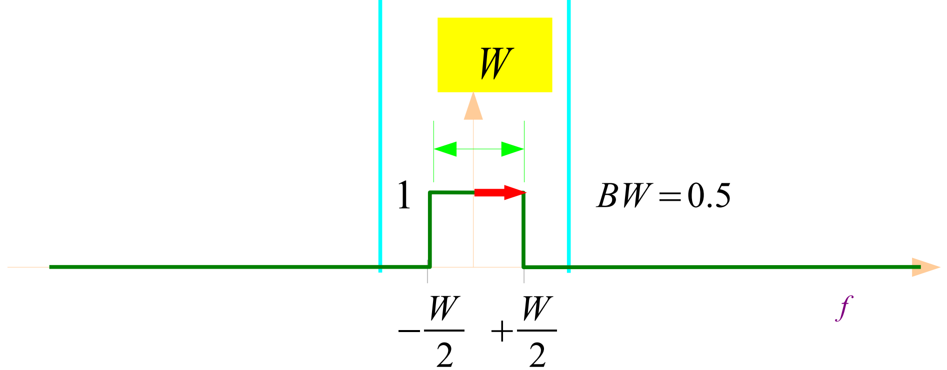


$$x(t) = \textbf{rect}\left(\frac{t}{T}\right) \quad \Longleftrightarrow \quad X(f) = T\,\textbf{sinc}(f\,T)$$

T= W =1

$$X(t) = W\,\textbf{sinc}(t\,W) \quad \Longleftrightarrow \quad x(t) = \textbf{rect}\left(\frac{f}{W}\right)$$

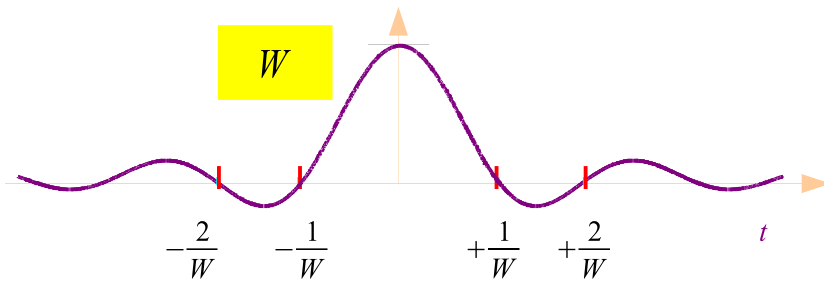


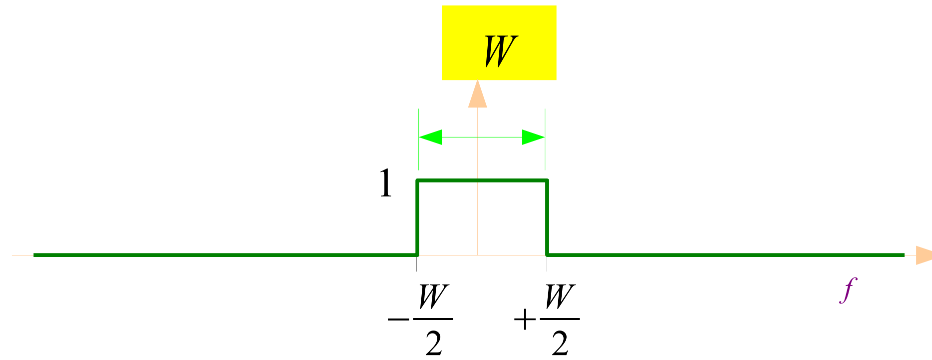


Duality (3)

$$X(t) = W\,\mathbf{sinc}(t\,W) \quad \Longleftrightarrow \quad x(t) = \mathbf{rect}\left(\frac{f}{W}\right)$$

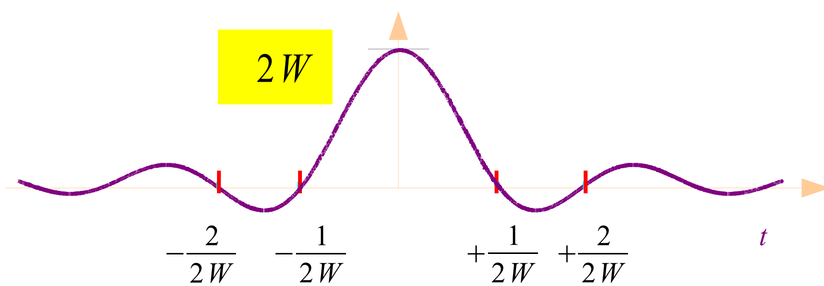
W

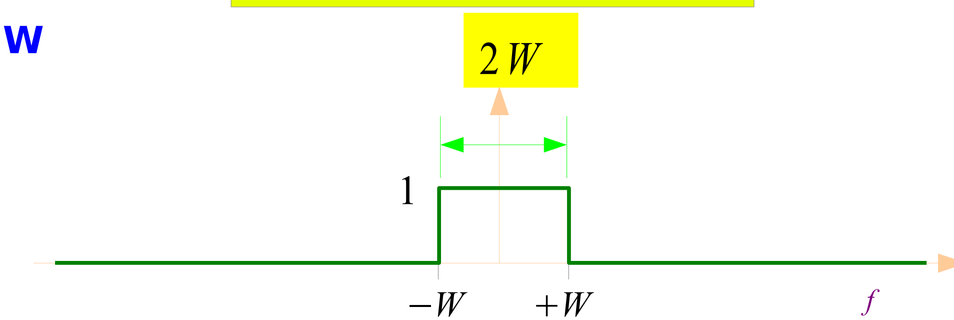




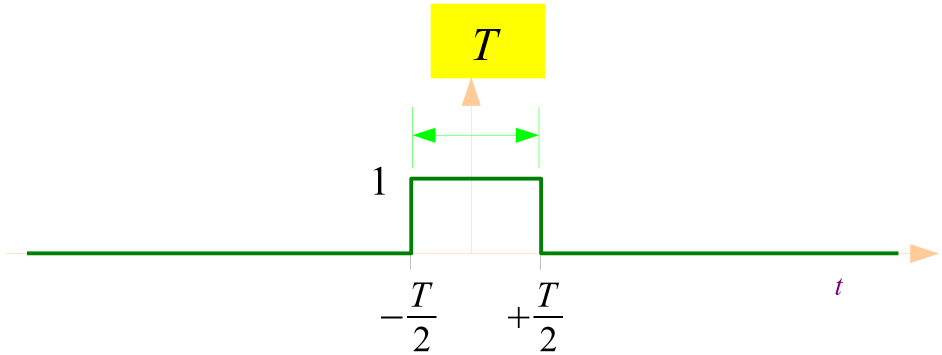
$$2\,W\,\mathbf{sinc}(2\,W\,t) \quad \Longleftrightarrow \quad \mathbf{rect}\left(\frac{f}{2\,W}\right)$$

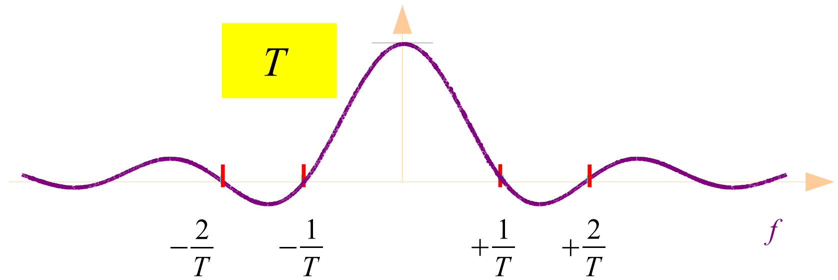
2W





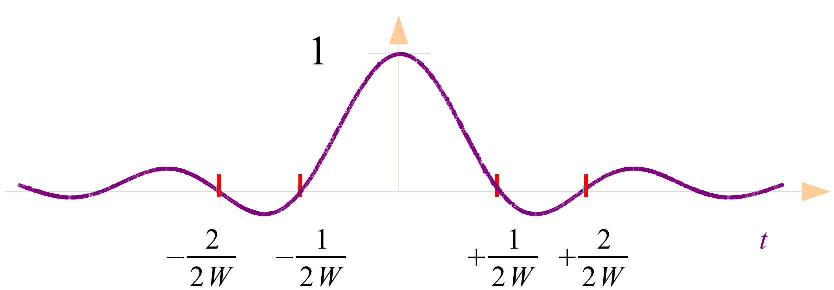
Duality (4)

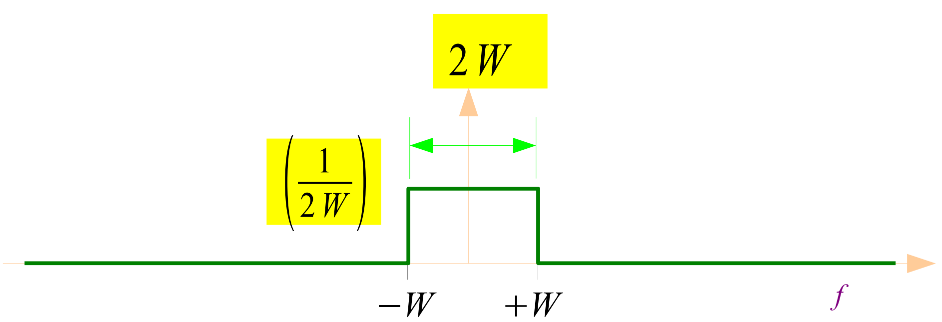




$$rect\left(\frac{t}{T}\right) \quad \overset{T}{\Longleftrightarrow} \quad T\,sinc(f\,T)$$

$$sinc(2\,W\,t) \quad \overset{2W}{\Longleftrightarrow} \quad \left(\frac{1}{2\,W}\right) rect\left(\frac{f}{2\,W}\right)$$





- CTFT of a Rectangular Pulse
- CTFT of a Shifted Rectangular Pulse
- Spectrum Plots of the CTFT of a Rectangular Pulse
- Spectrum Plots of the CTFT of a Shifted Rectangular Pulse

Shifted Rect(t/T) CTFT (1)

Continuous Time Fourier Transform — Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

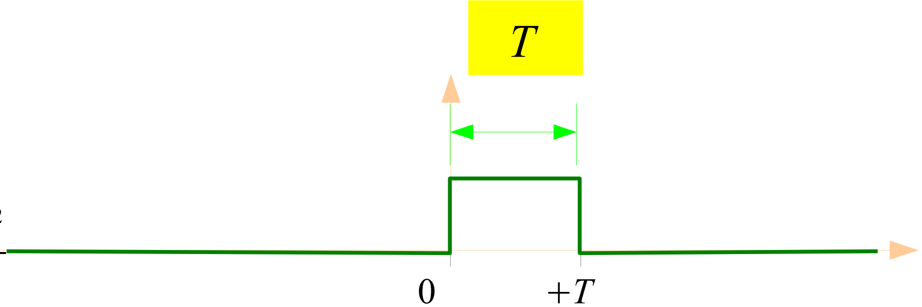
$$X(j\omega) = \int_0^{+T} e^{-j\omega t}\, dt = \left[\frac{-1}{j\omega}e^{-j\omega t}\right]_{-T/2}^{+T/2}$$

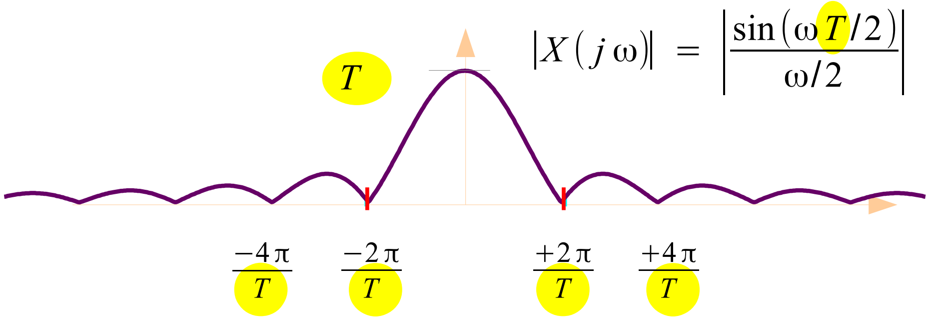
$$= -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega} = e^{-j\omega T/2}\cdot\frac{e^{+j\omega T/2} - e^{-j\omega T/2}}{j\omega}$$

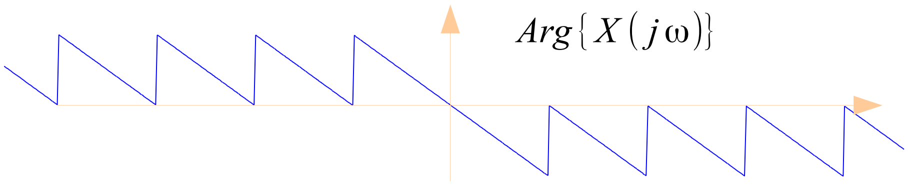
$$= \frac{\sin(\omega T/2)}{\omega/2}\cdot \boxed{e^{-j\omega T/2}}$$

$$|X(j\omega)| = \left|\frac{\sin(\omega T/2)}{\omega/2}\right|$$

$$Arg\{X(j\omega)\} = Arg\left\{\frac{\sin(\omega T/2)}{\omega/2}\right\} - \omega T/2$$

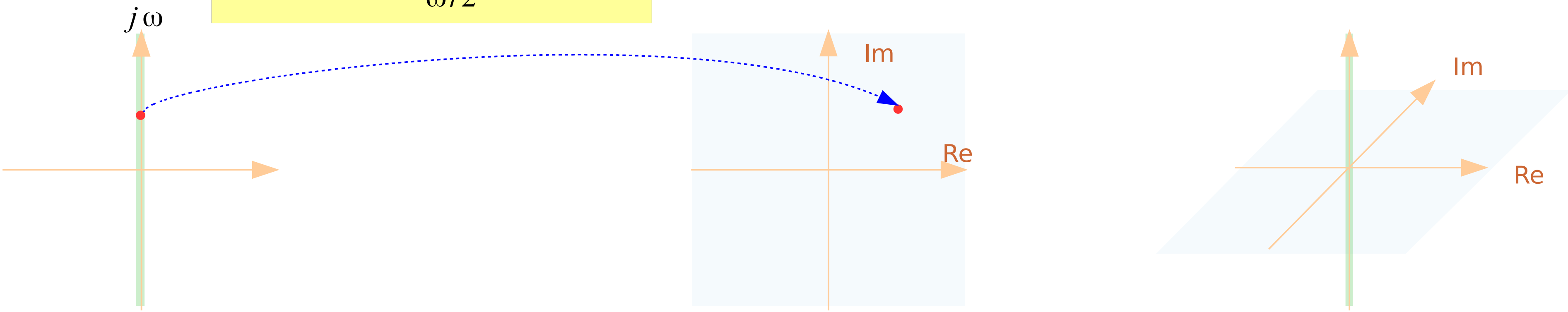




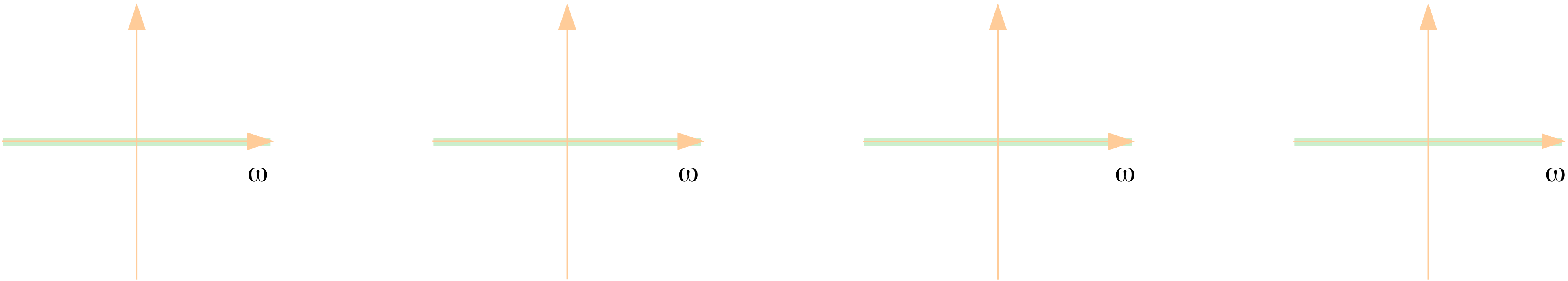


Complex Function Plotting

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} \cdot e^{-j\omega T/2}$$



$\Re\{X(j\omega)\}$ $\Im\{X(j\omega)\}$ $|X(j\omega)|$ $Arg\{X(j\omega)\}$



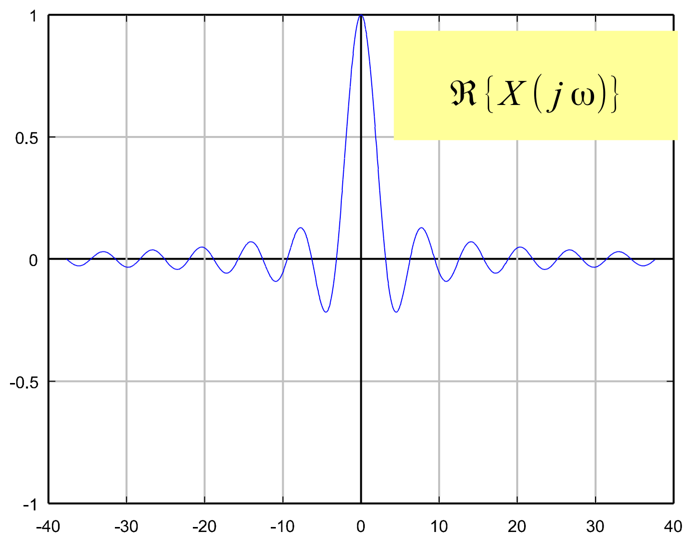
$$\frac{\sin(\omega T/2)}{\omega/2} \cdot \cos(\omega T/2)$$

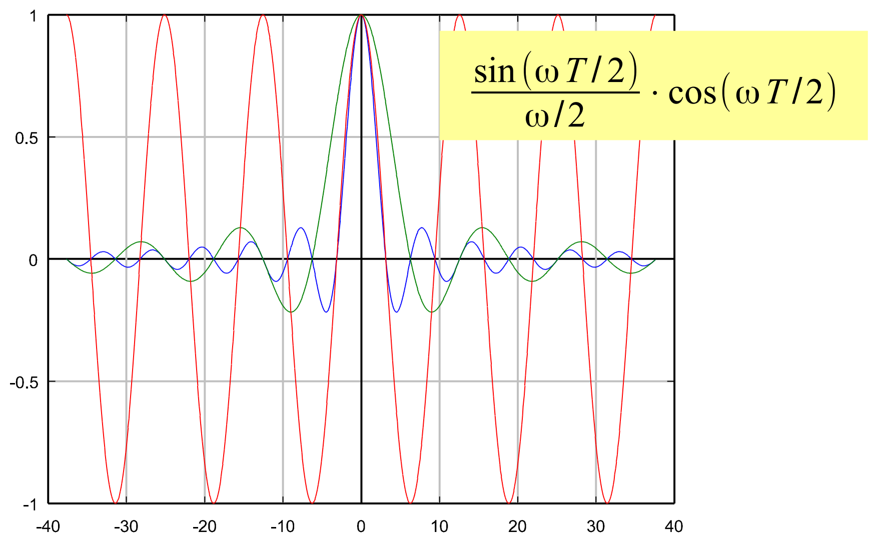
$$-\frac{\sin(\omega T/2)}{\omega/2} \cdot \sin(\omega T/2)$$

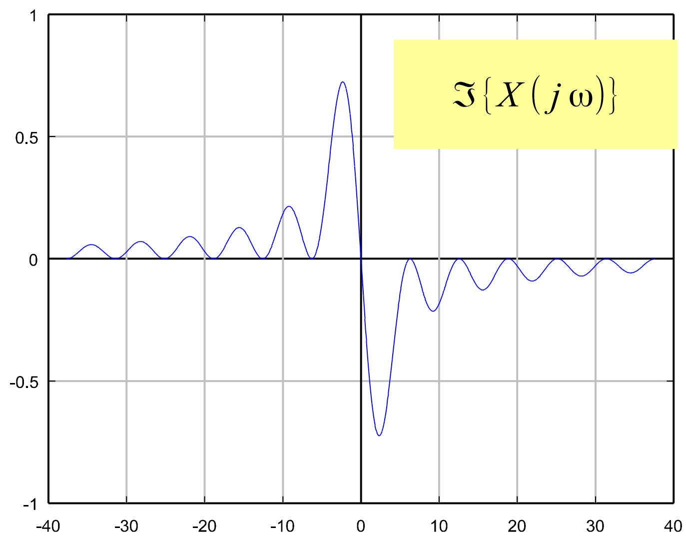
$$\left|\frac{\sin(\omega T/2)}{\omega/2}\right|$$

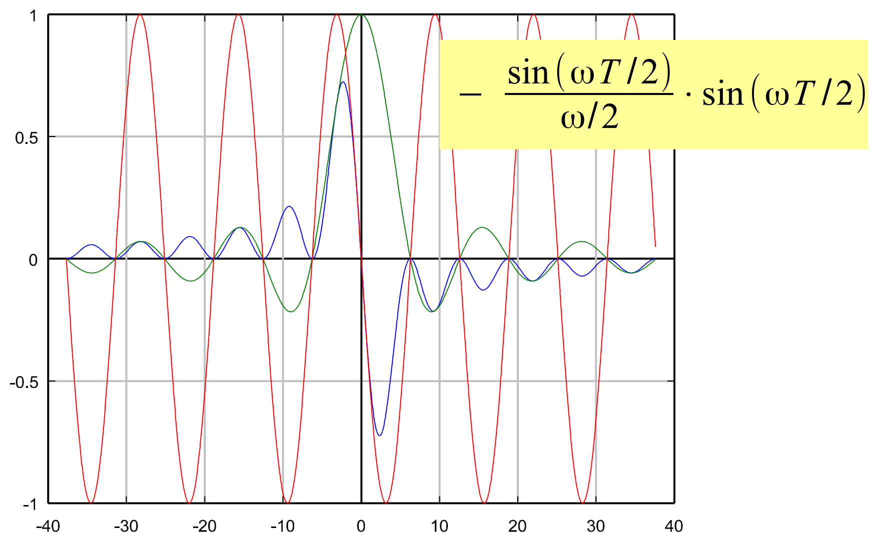
$$Arg\left\{\frac{\sin(\omega T/2)}{\omega/2}\right\} - \omega T/2$$

Re and Im of X(jω)

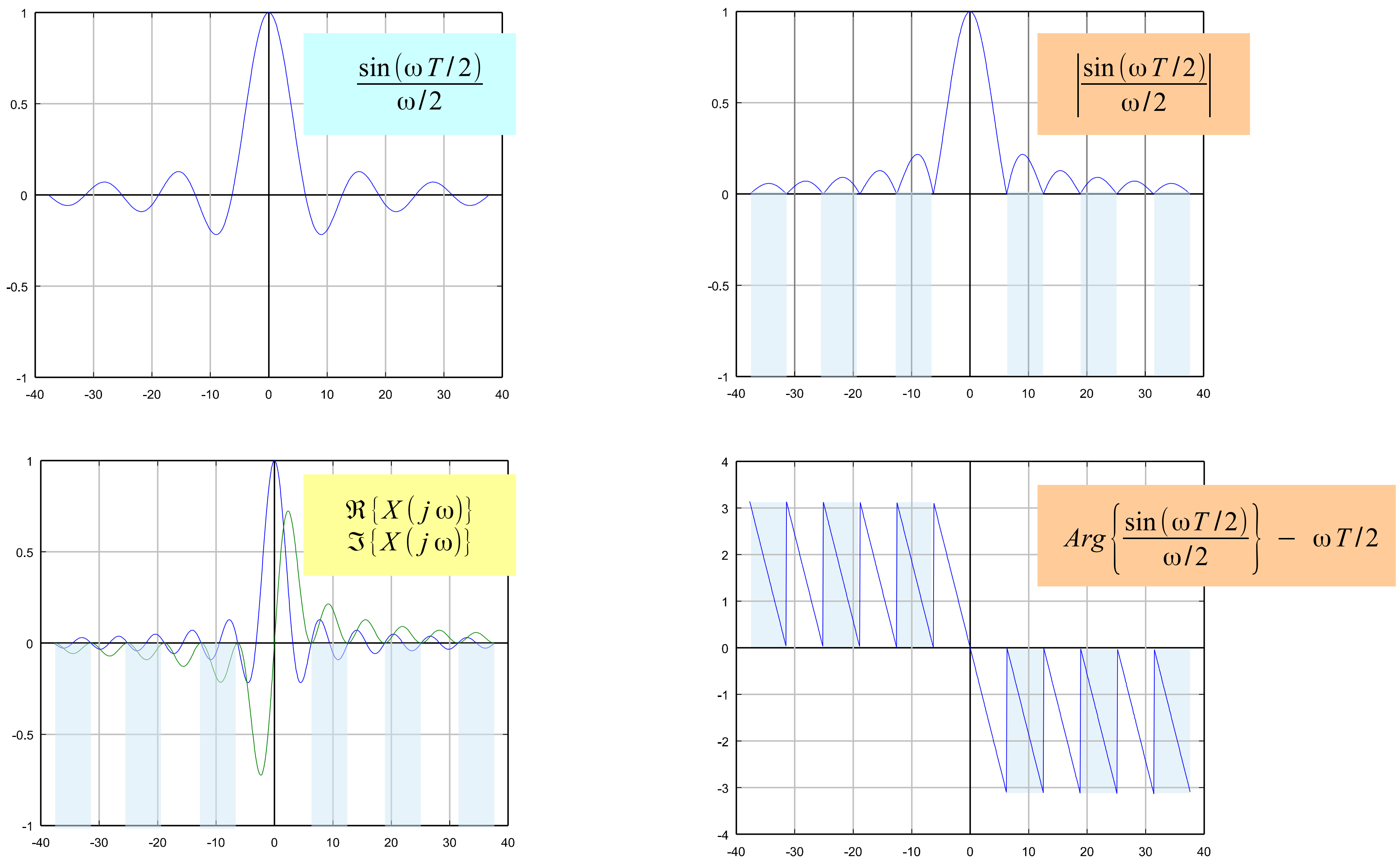




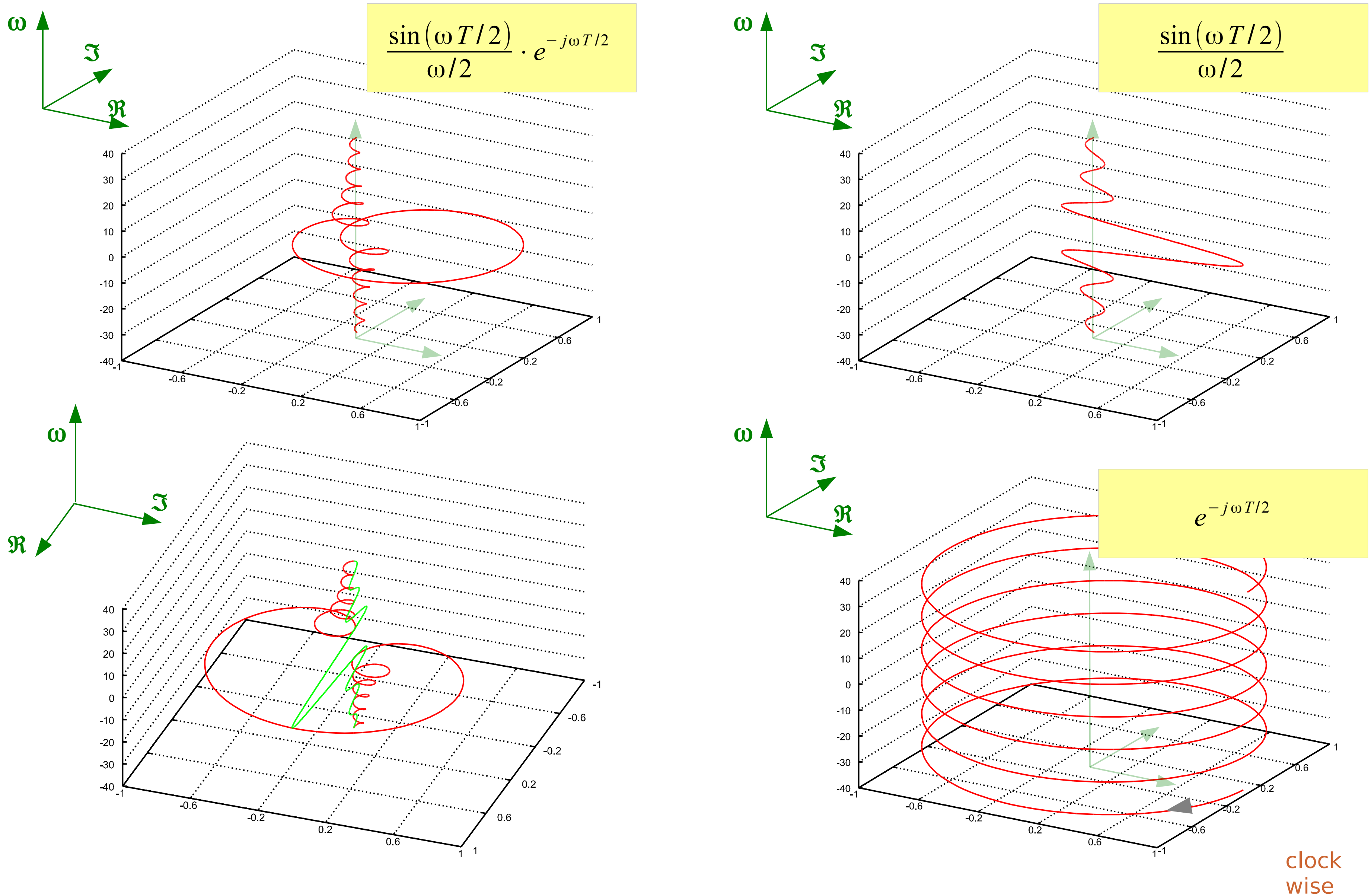




Abs and Arg of X(jω)



3-D Plots of X(jω)



Spectrum of the CTFS of a Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

Spectrum $X(j\omega) = |X(j\omega)|\, arg(X(j\omega))$

Magnitude Spectrum $|X(j\omega)|$

Phase Spectrum $arg(X(j\omega))$

Spectrum of the CTFS of a Real Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{+j\omega t}\, dt$$

$$X^*(j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{+j\omega t}\, dt$$

a real signal $x(t) = x^*(t)$ ➡

magnitude: an <u>even</u> function

$$|X(-j\omega)| = |X(j\omega)|$$

phase: an <u>odd</u> function

$$arg(X(-j\omega)) = -arg(X(j\omega))$$

a real even signal ➡ $x(t) = x^*(t)$

➡ $X(-j\omega) = X^*(j\omega)$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{+j\omega t}\, dt = X^*(j\omega)$$

Spectrum of the CTFS of a Real Even Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t} dt$$

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{+j\omega t} dt$$

$$X^*(j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{+j\omega t} dt$$

a real <u>even</u> signal $x(t) = x^*(t) = x(-t)$

➡ **a real <u>even</u> spectrum**

$$X(j\omega) = X^*(j\omega) = X(-j\omega)$$

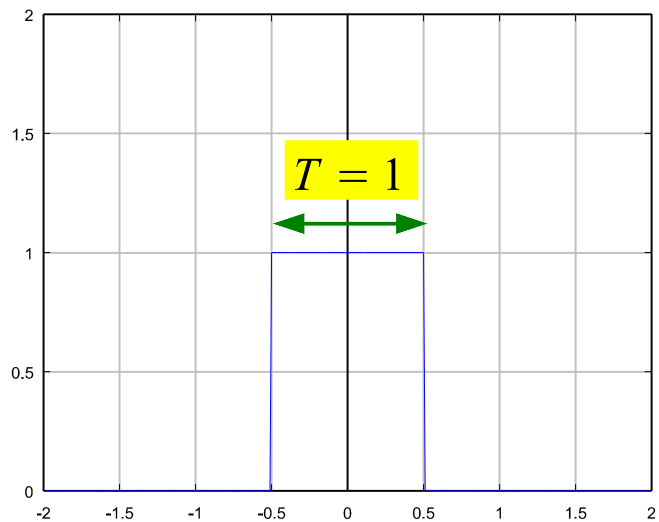
a real <u>even</u> signal $x(t) = x^*(t) = x(-t)$

$$X^*(j\omega) = \int_{-\infty}^{+\infty} x^*(t)\, e^{+j\omega t} dt = \int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)} dt = X(j\omega)$$

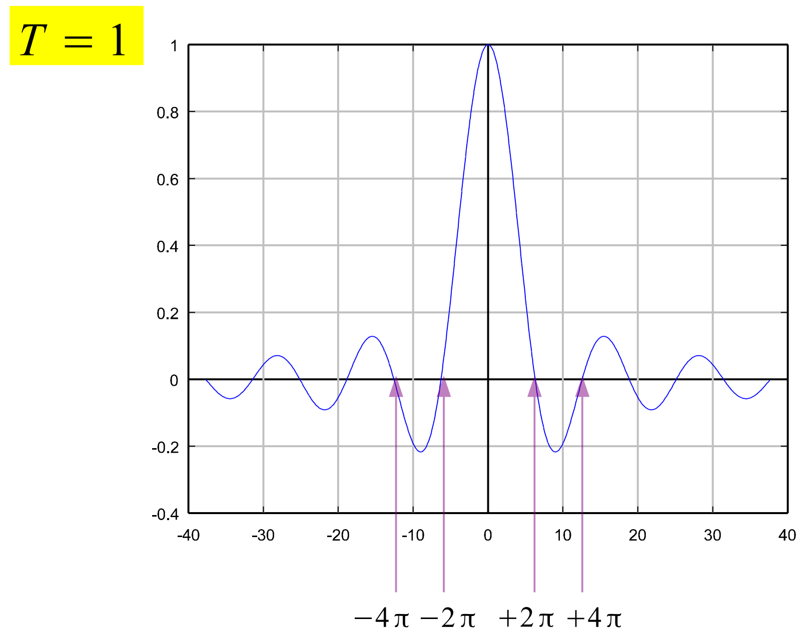
$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{+j\omega t} dt = \int_{-\infty}^{+\infty} x(-t)\, e^{-j\omega(-t)} dt = X(j\omega)$$

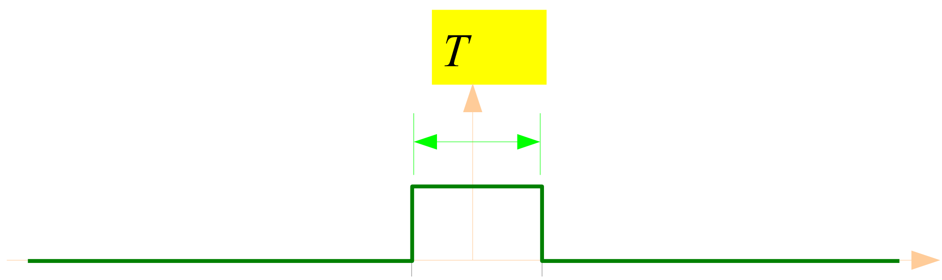
* CTFT of a Rectangular Pulse
* CTFT of a Shifted Rectangular Pulse
* Spectrum Plots of the CTFT of a Rectangular Pulse
* Spectrum Plots of the CTFT of a Shifted Rectangular Pulse

Spectrum of a Rectangular Pulse

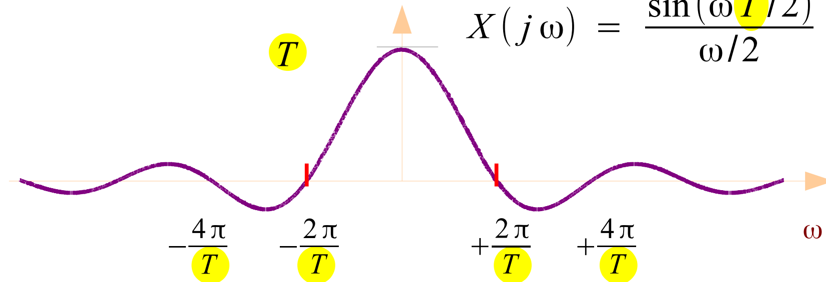


$$X(j\omega) = \frac{\sin(\omega/2)}{\omega/2}$$

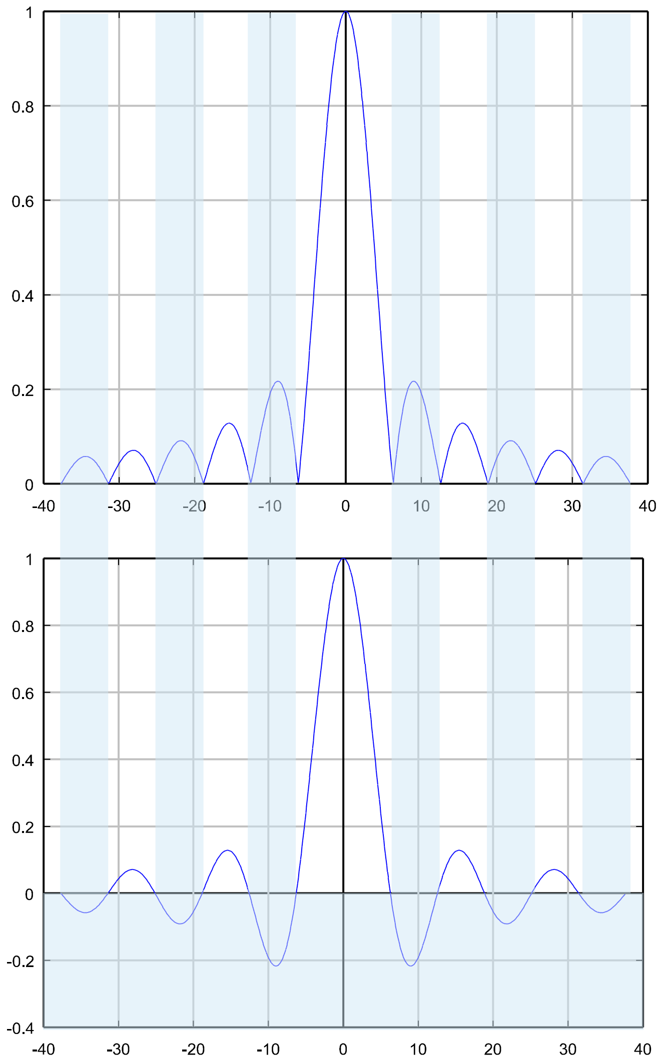




$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



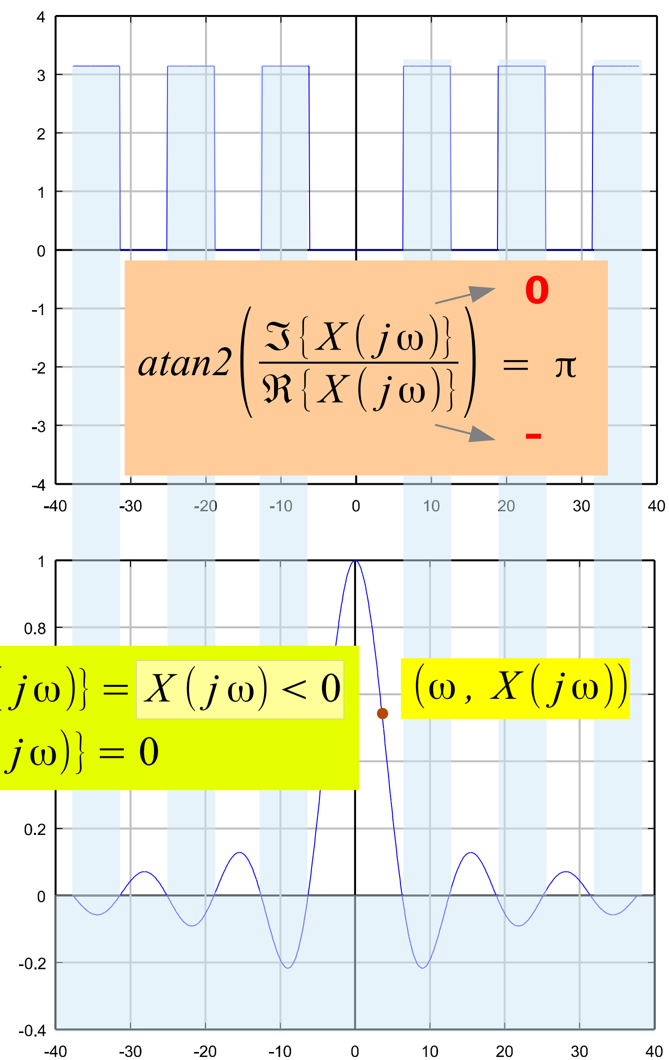
Magnitude Spectrum of a Rectangular Pulse



$$|X(j\omega)| = \left|\frac{\sin(\omega/2)}{\omega/2}\right|$$

$$X(j\omega) = \frac{\sin(\omega/2)}{\omega/2}$$

Phase Spectrum of a Rectangular Pulse



$$atan2\left(\frac{\Im\{X(j\omega)\}}{\Re\{X(j\omega)\}}\right) = \pi$$

0

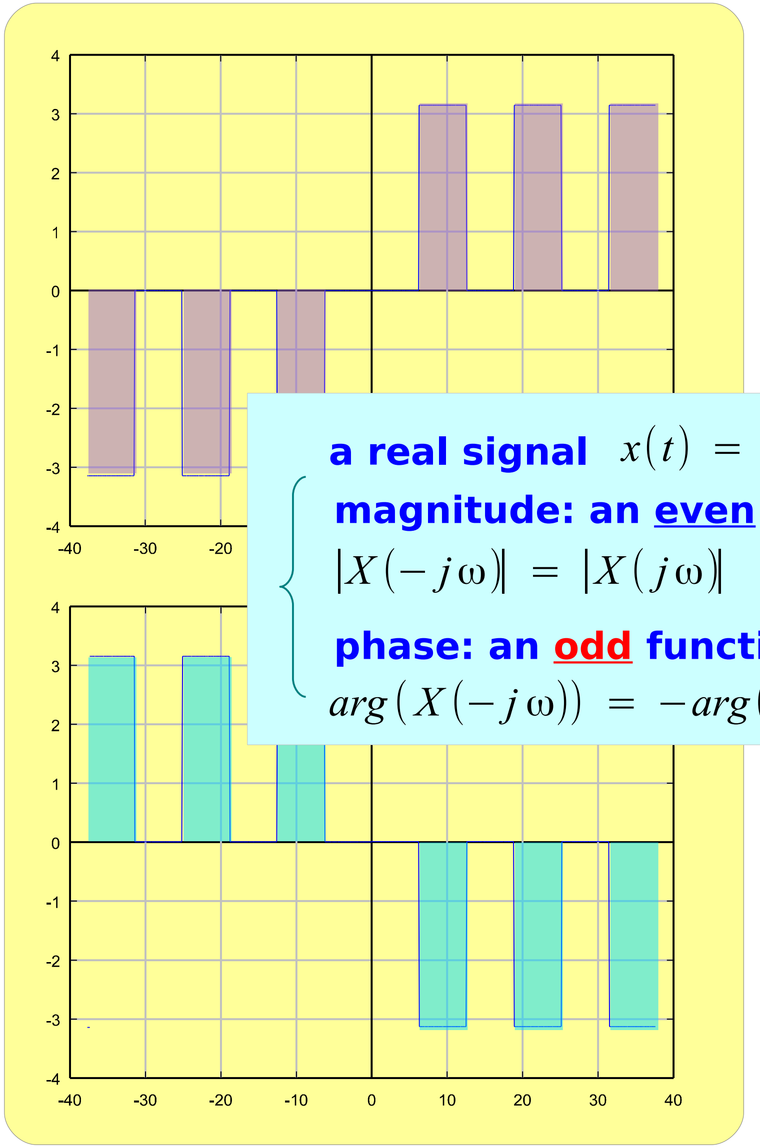
-

$$\Re\{X(j\omega)\} = X(j\omega) < 0$$

$$\Im\{X(j\omega)\} = 0$$

$$(\omega, X(j\omega))$$

OR



a real signal $x(t) = x^*(t)$

magnitude: an even function

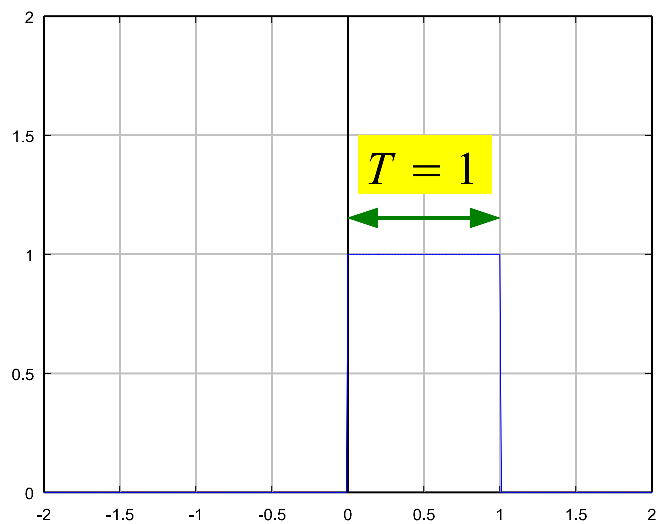
$$|X(-j\omega)| = |X(j\omega)|$$

phase: an odd function

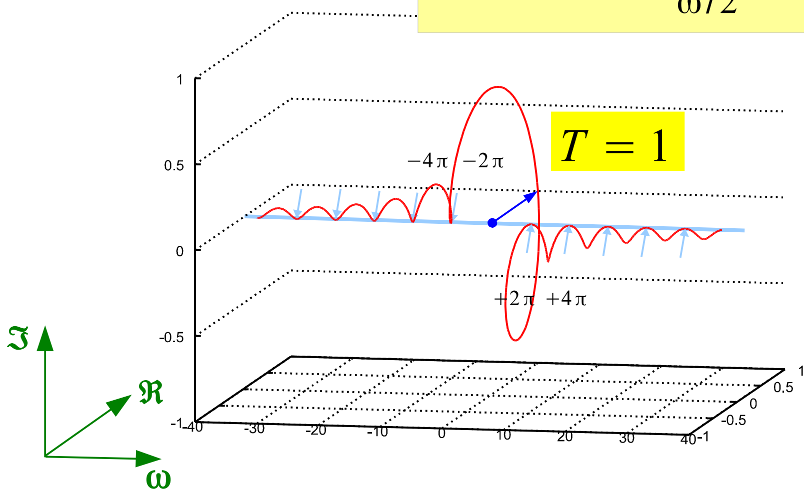
$$arg(X(-j\omega)) = -arg(X(j\omega))$$

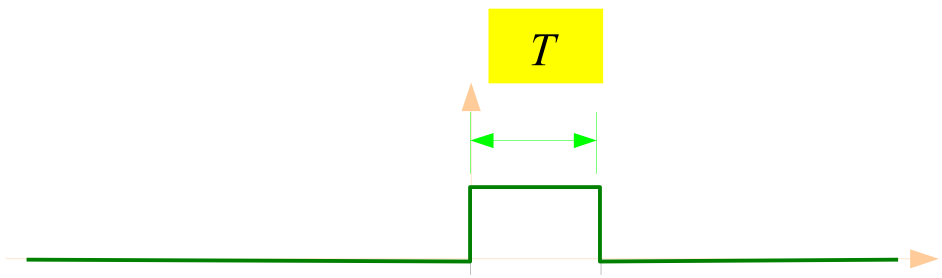
- CTFT of a Rectangular Pulse
- CTFT of a Shifted Rectangular Pulse
- Spectrum Plots of the CTFT of a Rectangular Pulse
- Spectrum Plots of the CTFT of a Shifted Rectangular Pulse

Spectrum of a Shifted Rectangular Pulse

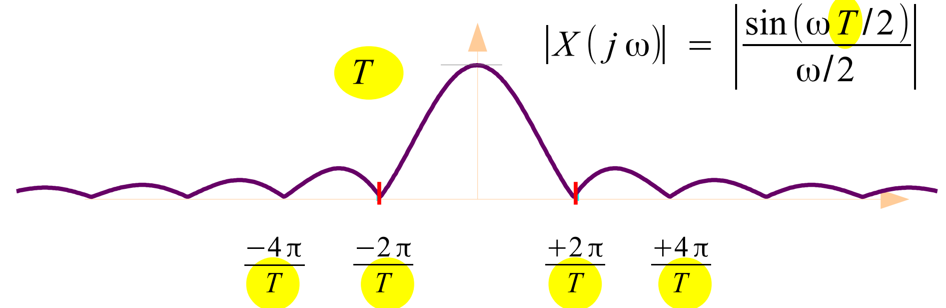


$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} \cdot e^{-j\omega T/2}$$

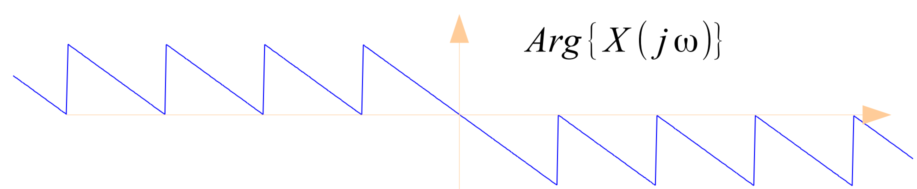




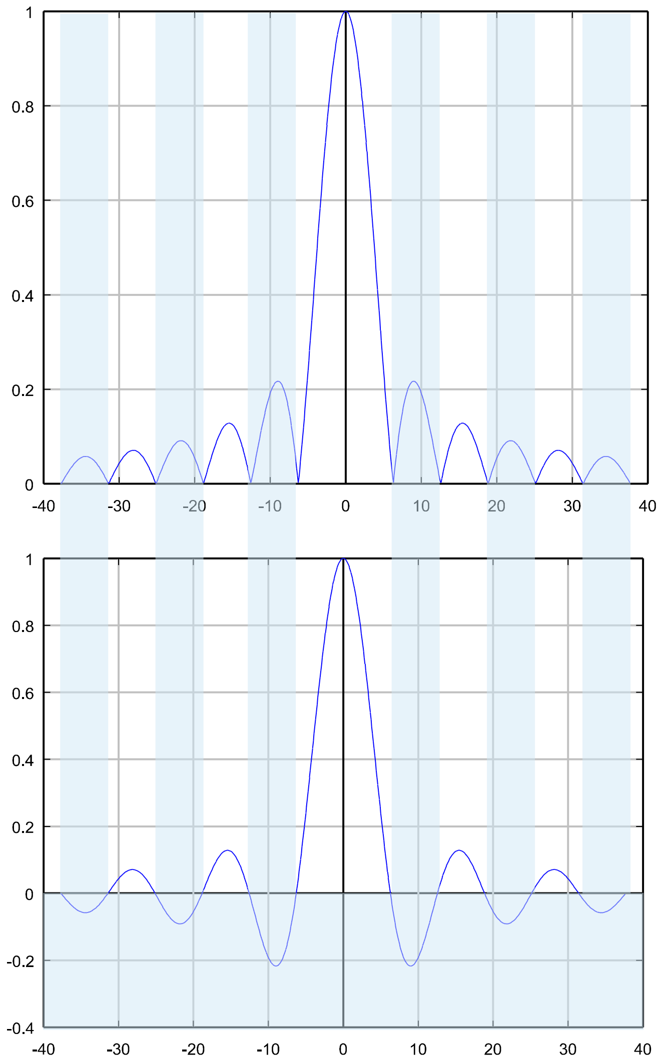
$$|X(j\omega)| = \left|\frac{\sin(\omega T/2)}{\omega/2}\right|$$



$Arg\{X(j\omega)\}$



Magnitude Spectrum of a Shifted Rectangular Pulse



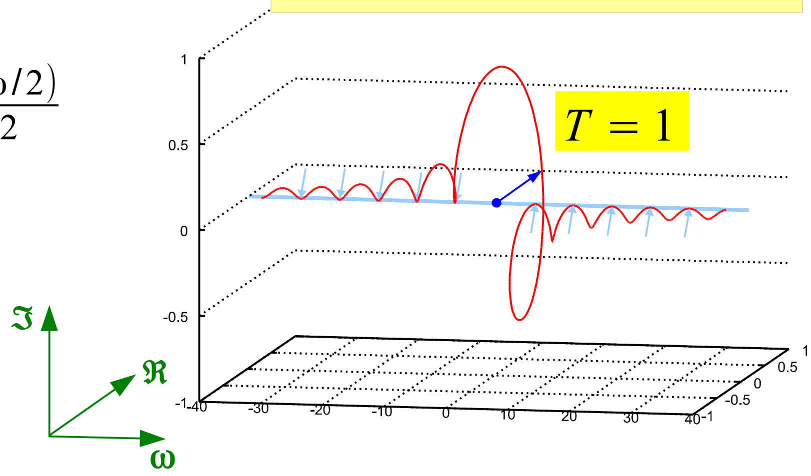
$$|X(j\omega)| = \left|\frac{\sin(\omega/2)}{\omega/2}\right| \cdot \cancel{\left|e^{-j\omega T/2}\right|}$$

$$|X(j\omega)| = \left|\frac{\sin(\omega/2)}{\omega/2}\right|$$

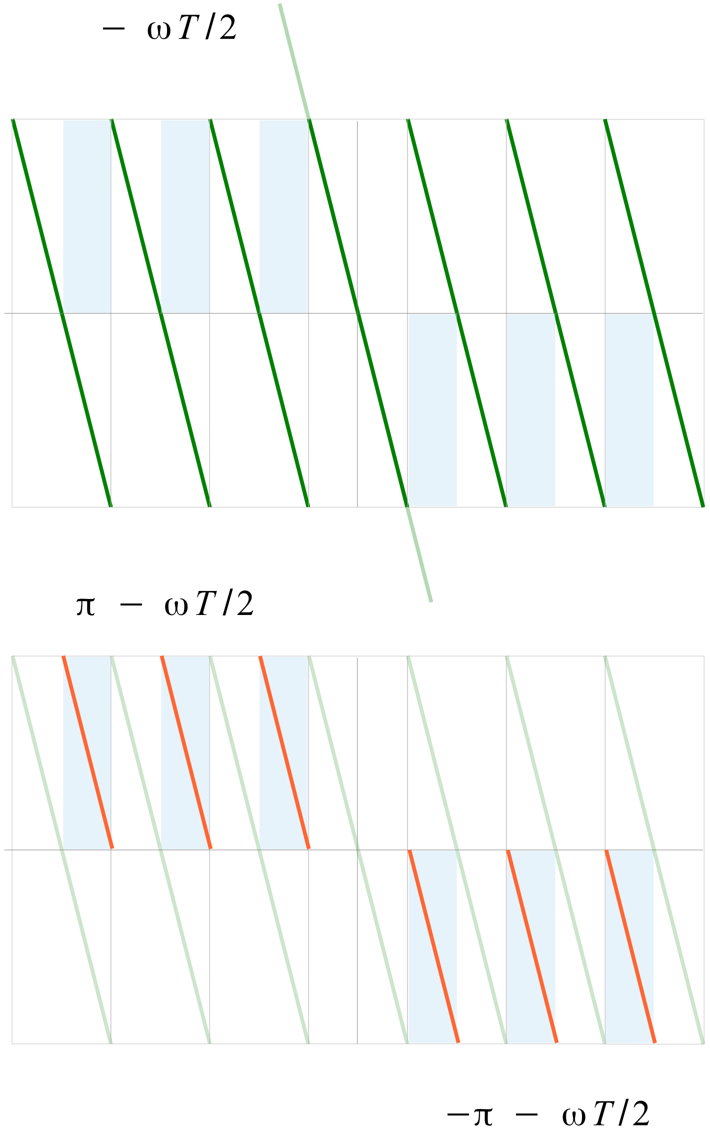
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} \cdot e^{-j\omega T/2}$$

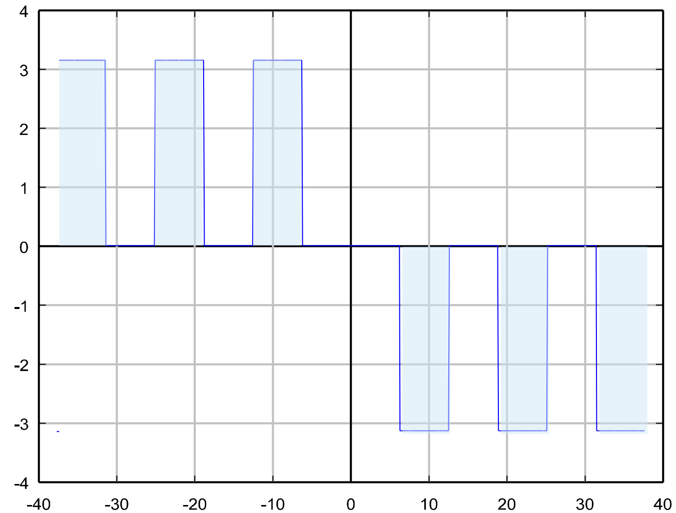
$$\frac{\sin(\omega/2)}{\omega/2}$$

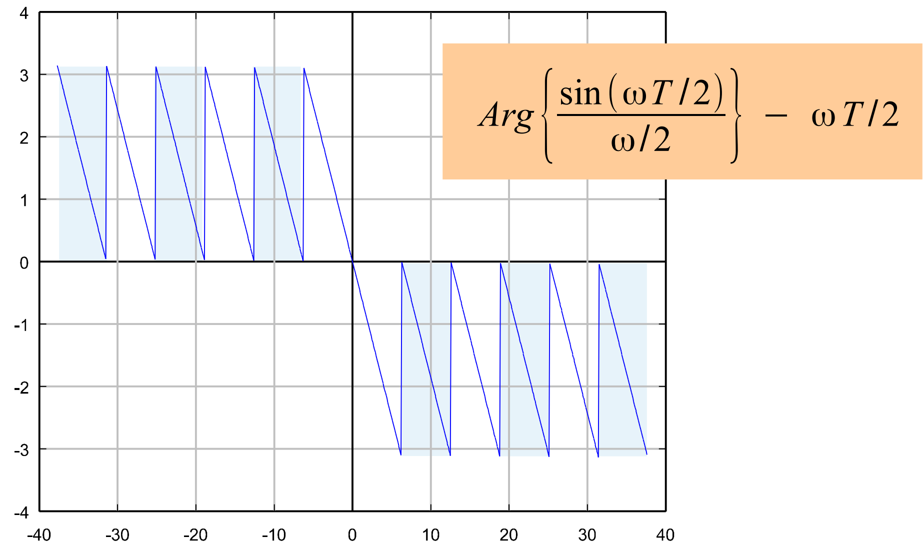
$T = 1$



Phase Spectrum of a Shifted Rectangular Pulse







References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] G. Beale, http://teal.gmu.edu/~gbeale/ece_220/fourier_series_02.html
[4] C. Langton, http://www.complextoreal.com/chapters/fft1.pdf