Fourier Integrals (3A)

- Continuous Time Fourier Transform

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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CTFT of a Rect(t/T) function (1)

Continuous Time Fourier Transform Aperiodic Continuous Time Signal

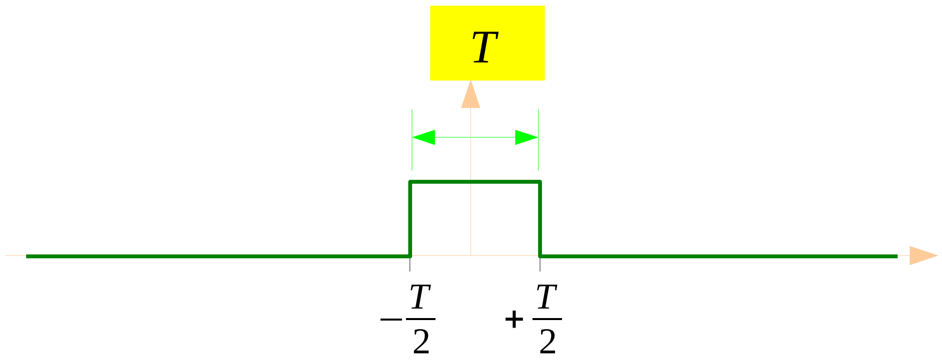
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt$$

$$= \left[\frac{-1}{j\omega} e^{-j\omega t}\right]_{-T/2}^{+T/2} = -\frac{e^{-j\omega T/2} - e^{+j\omega T/2}}{j\omega}$$

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

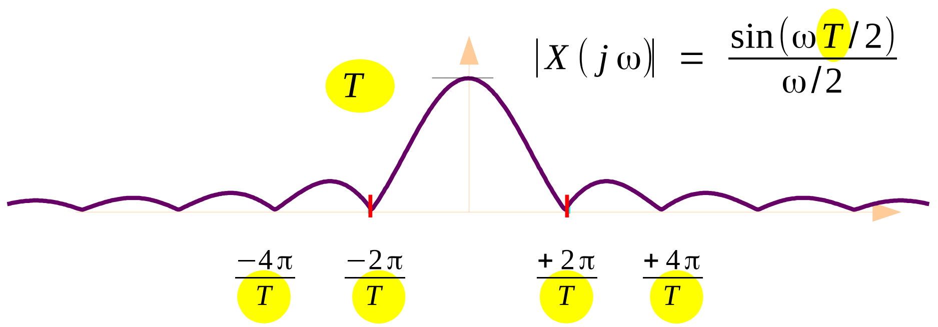
$$X(j0) = \lim_{\omega \to 0} \frac{\sin(\omega T/2)}{\omega/2} = \lim_{\omega \to 0} \frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$$



$$\sin(\omega T/2) = 0 \quad \Rightarrow \quad \omega T/2 = \pi n$$

$$\Rightarrow \quad \omega = \frac{2\pi}{T} n$$

$$\Rightarrow \quad \omega = \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \pm\frac{6\pi}{T}, \cdots$$



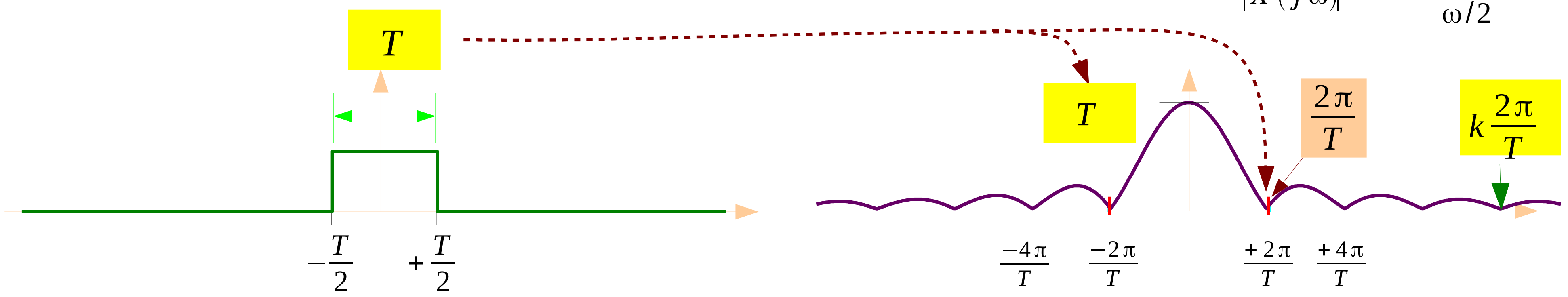
CTFT of a Rect(t/T) function (2)

Continuous Time Fourier <u>Transform</u> Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2}$$

$$|X(j\omega)| = \frac{\sin(\omega T/2)}{\omega/2}$$



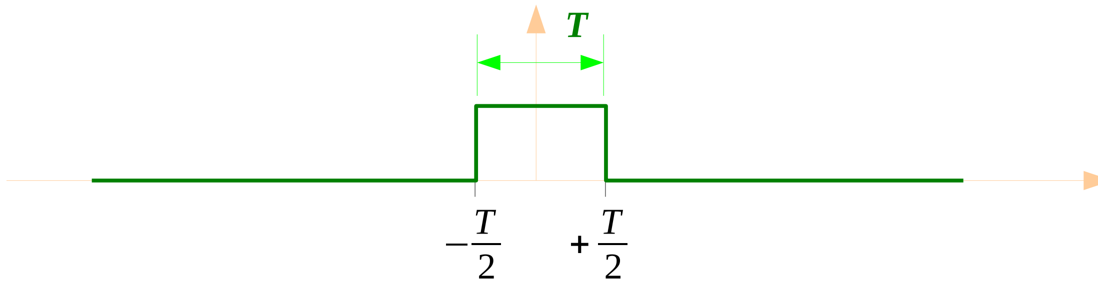
CTFT and CTFS

Continuous Time Fourier Transform

Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

$$X(j\omega) = \frac{\sin(\omega \boldsymbol{T}/2)}{\omega/2}$$

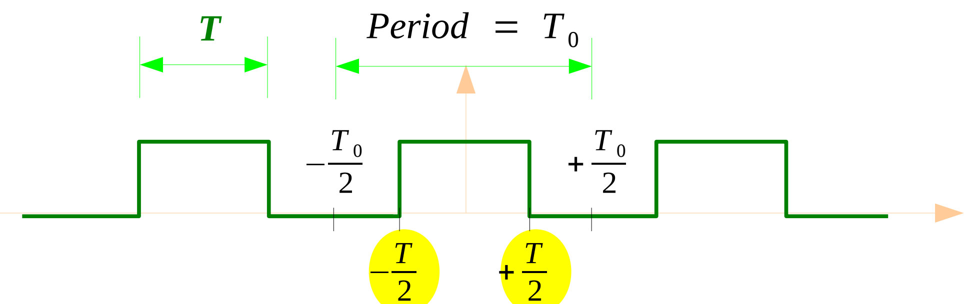


Continuous Time Fourier Series

Periodic Continuous Time Signal

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Longleftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

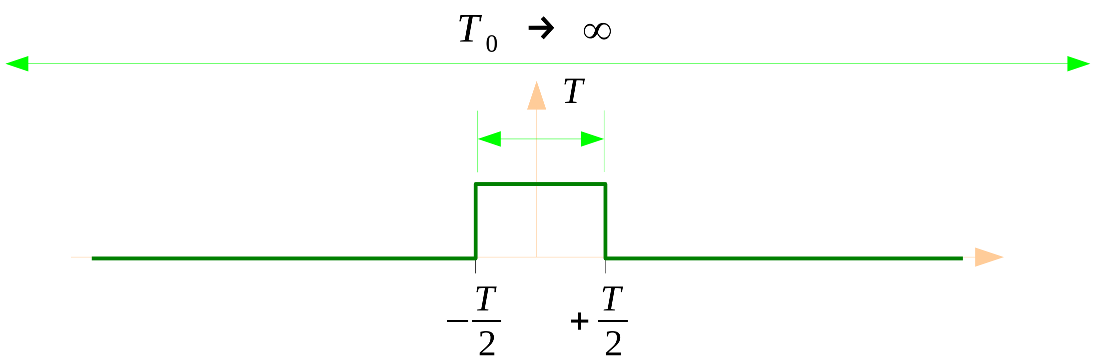
$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 \boldsymbol{T}/2)}{k\omega_0/2}$$



CTFT ← CTFS

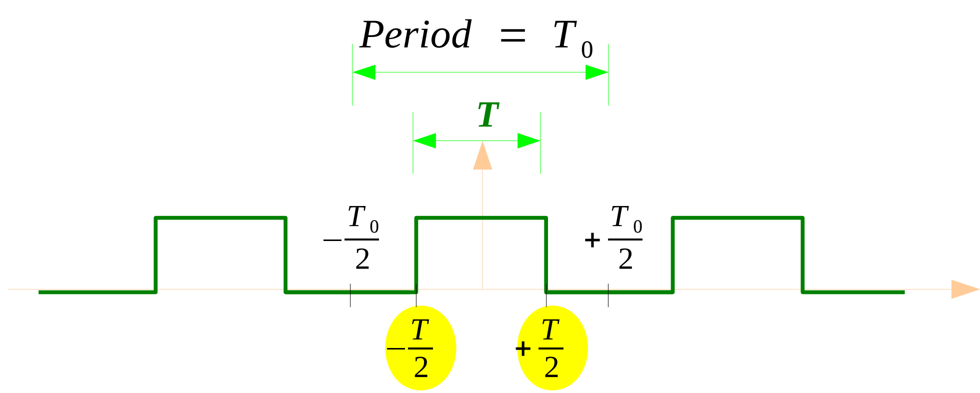
Aperiodic Continuous Time Signal

Continuous Time Fourier <u>Transform</u>



Periodic Continuous Time Signal

Continuous Time Fourier <u>Series</u>



$$x(t)$$

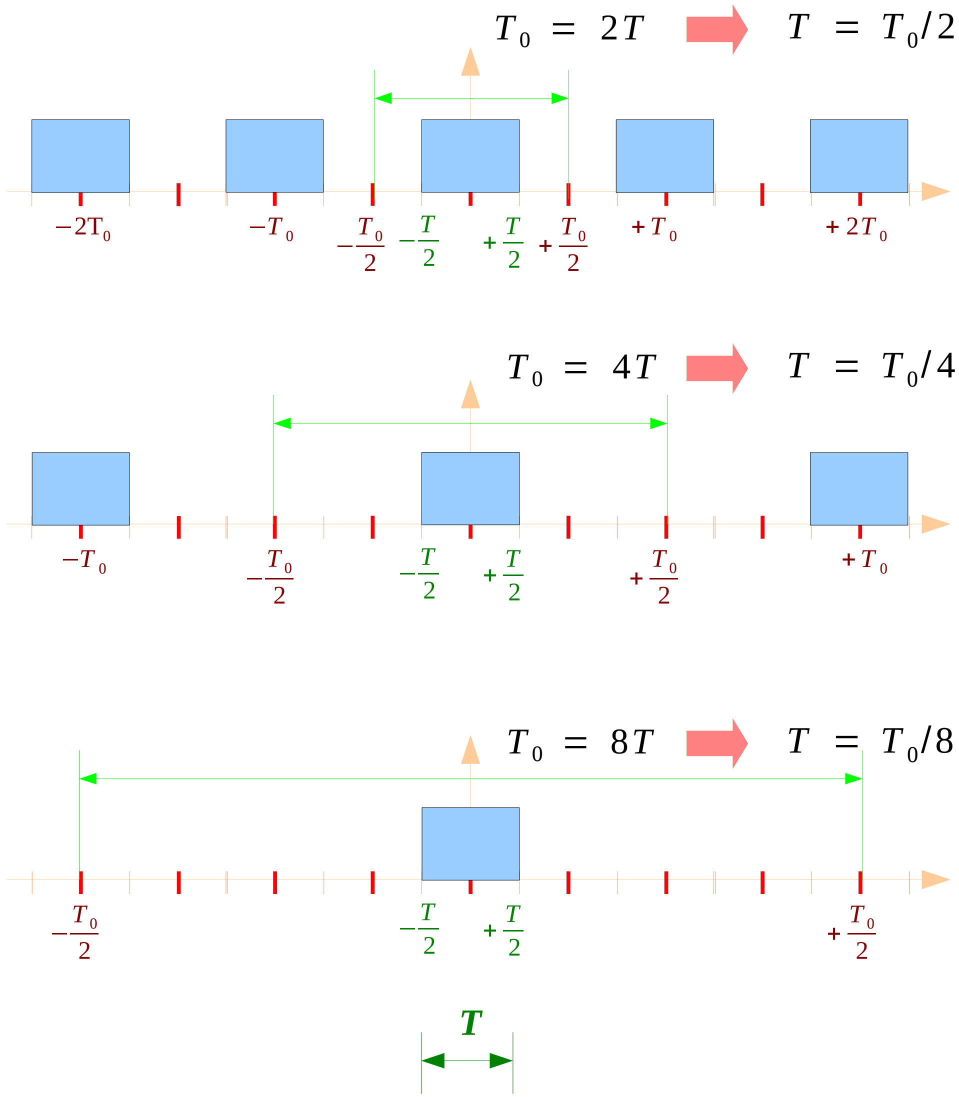
As $T_0 \rightarrow \infty$,

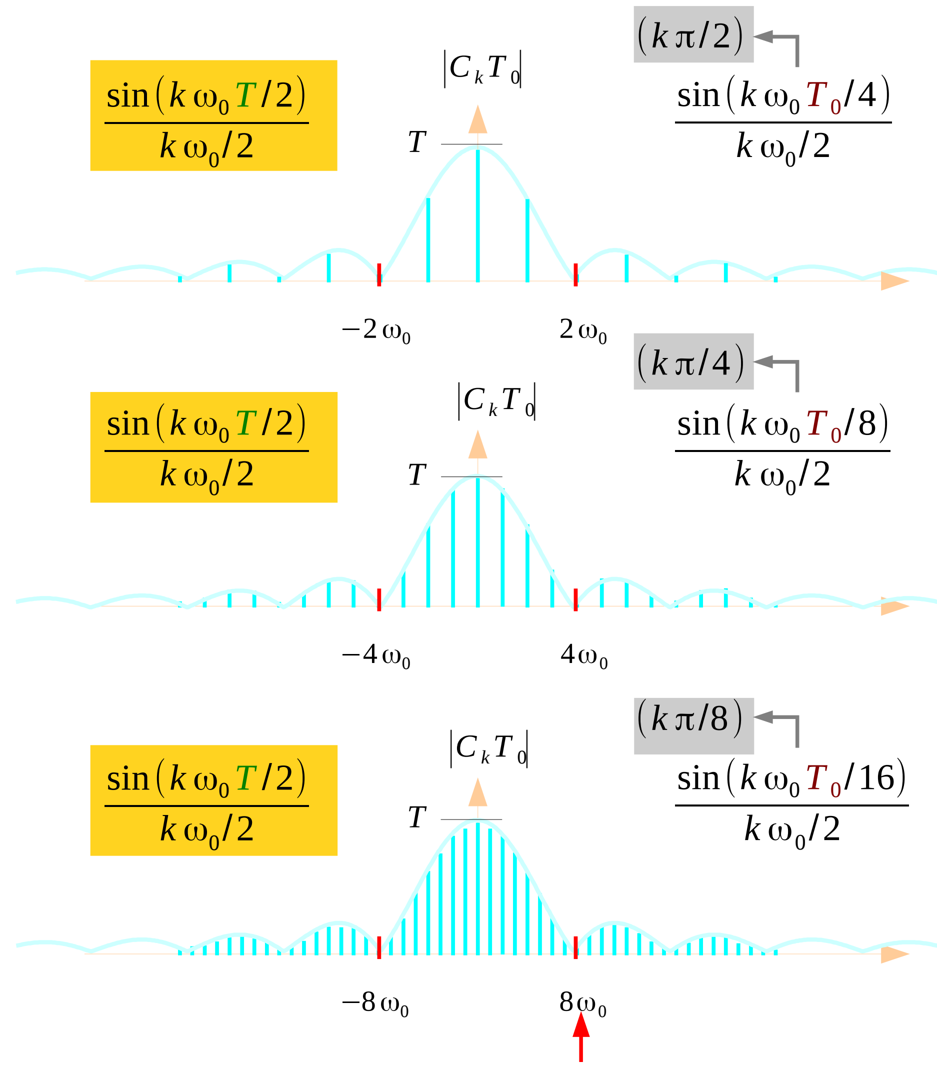
$$x_{T_0}(t) \rightarrow x(t)$$

$$\omega_0 = \frac{2\pi}{T_0} \rightarrow 0$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{+\infty} x(t - nT_0)$$

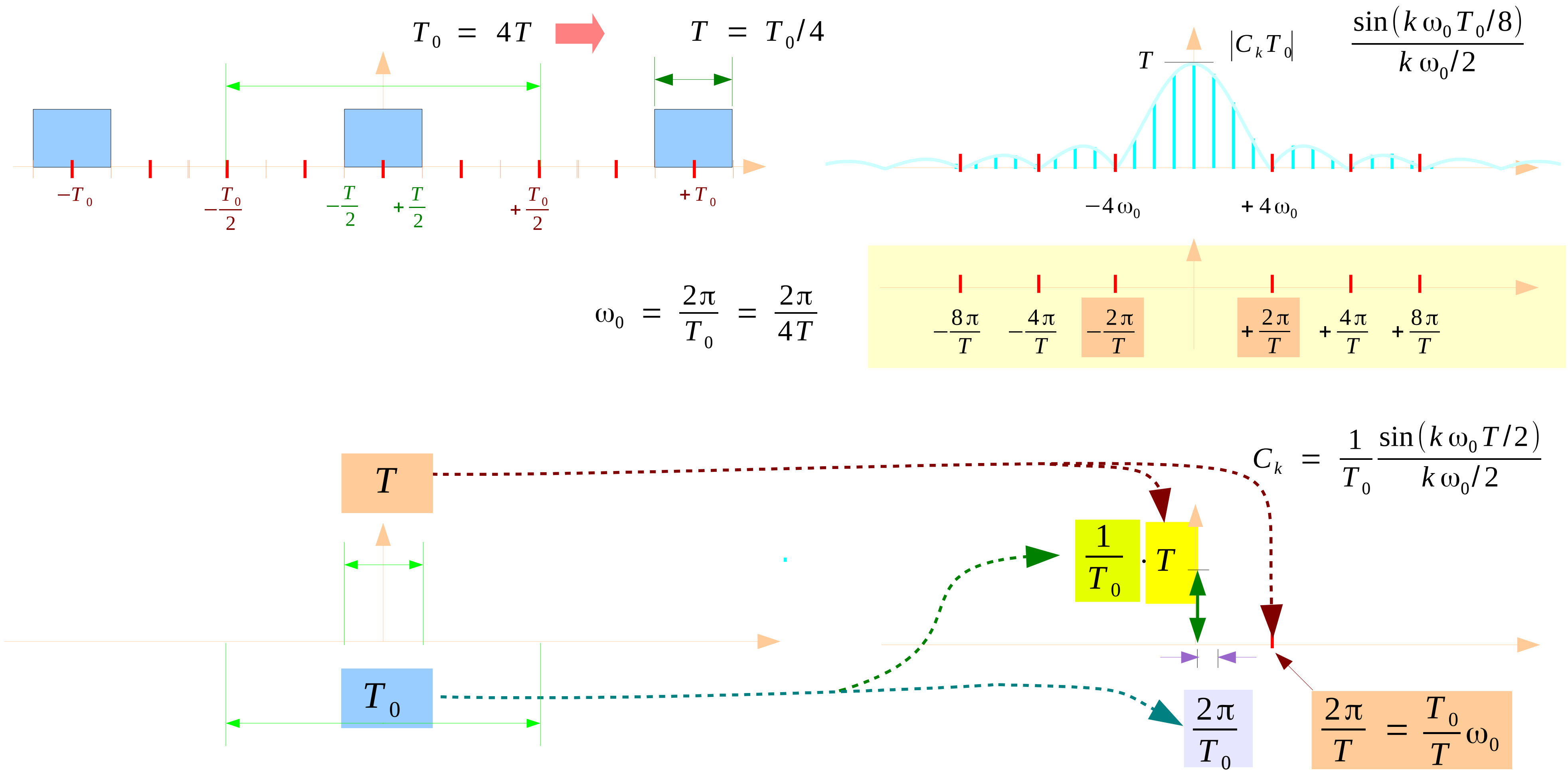
CTFT and CTFS as $T_0 \to \infty$ (1)



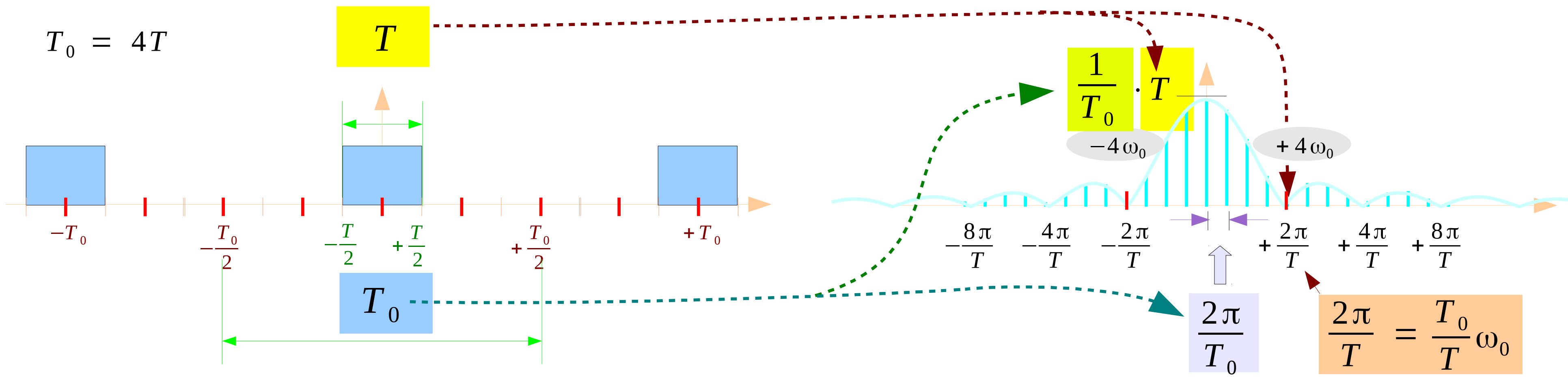


$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{8T} \quad \Rightarrow \quad \frac{T_0}{T}\omega_0 = \frac{2\pi}{T}$$

CTFT and CTFS as $T_0 \to \infty$ (2)



CTFT of a Rect(t/T) function (3)



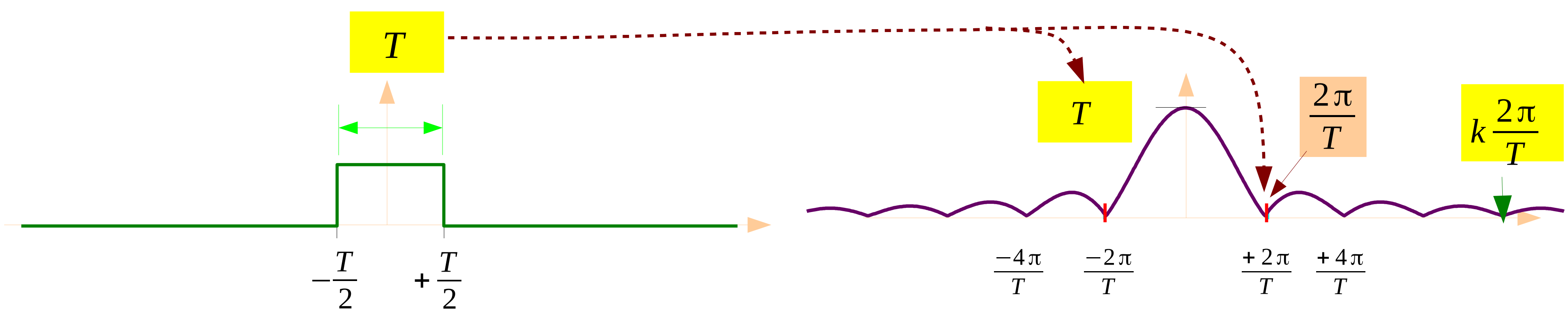
$$C_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$C_k = \frac{1}{T_0}\frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$



Continuous Time Fourier Series

$$C_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt \quad \Leftrightarrow \quad x(t) = \sum_{k=-\infty}^{\infty} C_k\, e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0}\int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \sum_{n=0}^{\infty} C_k\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=0}^{\infty} C_k T_0\, e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$T_0 \to \infty \quad \Rightarrow \quad C_k T_0 \to X(j\omega) \qquad x_{T_0} \to x(t) \qquad \omega_0 = \frac{2\pi}{T_0} \to d\omega$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Other Convention

Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Other Convention

Continuous Time Fourier Transform {unitary, angular frequency}

$$X(j\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Continuous Time Fourier Transform {non-unitary, angular frequency}

$$X(j\omega) = 1 \cdot \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

Fourier Integral (1)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right)$$

$$a_0 = \frac{1}{p}\int_{-p}^{+p} f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{+p} f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{+p} f(x)\sin\frac{n\pi}{p}x\,dx$$

$$n = 1,\ 2,\ 3,\ \ldots$$

$$f(x) = \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \frac{1}{p}\sum_{n=1}^{+\infty}\left[\left(\int_{-p}^{+p} f(t)\cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{+p} f(t)\sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right]$$

$$\alpha_n = \frac{n\pi}{p} \qquad \Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{p}$$

$$f(x) = \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{+\infty}\left[\left(\int_{-p}^{+p} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{+p} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha$$

$$p \to \infty \qquad \Delta\alpha \to 0$$

$$\lim_{\Delta\alpha\to 0} F(\alpha_n)\Delta\alpha_n = \int_0^{\infty} F(\alpha)\,d\alpha$$

$$f(x) = \frac{1}{\pi}\sum_{n=1}^{+\infty}\left[\left(\int_{-p}^{+p} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-p}^{+p} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha$$

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003