# CMOS Delay-7 (H.7)
# Elmore Delay

20170117

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# References

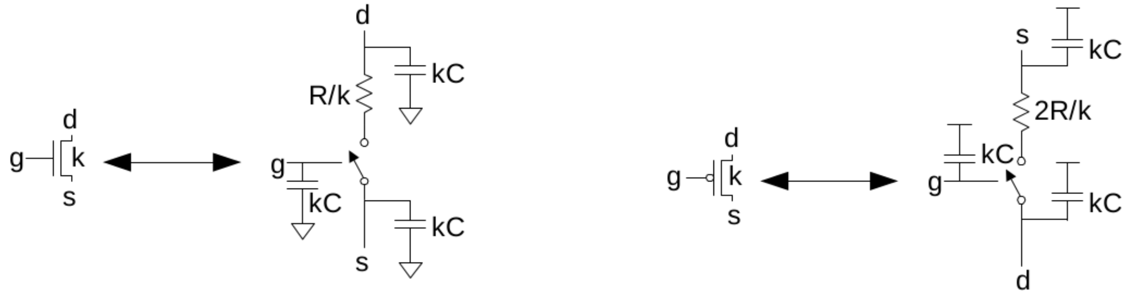
Some Figures from the following sites

[1] http://pages.hmc.edu/harris/cmosvlsi/4e/index.html
Weste & Harris Book Site

[2] en.wikipedia.org

# RC Delay Model

- Use equivalent circuits for MOS transistors
  - Ideal switch + capacitance and ON resistance
  - Unit nMOS has resistance R, capacitance C
  - Unit pMOS has resistance 2R, capacitance C
- Capacitance proportional to width
- Resistance inversely proportional to width

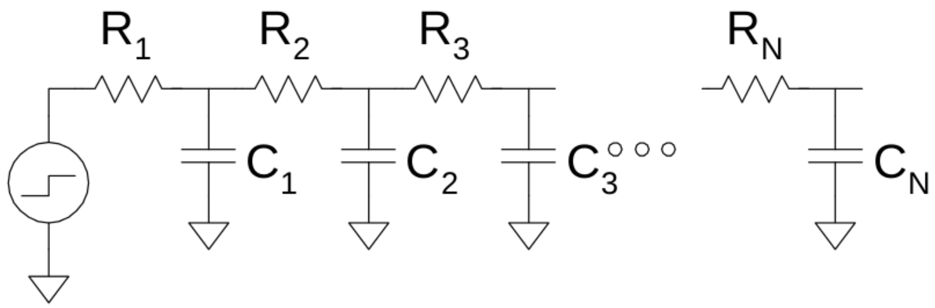


# Elmore Delay

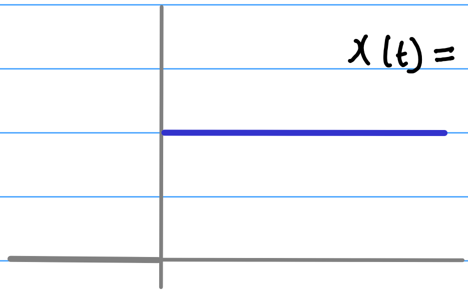
- ON transistors look like resistors
- Pullup or pulldown network modeled as *RC ladder*
- Elmore delay of RC ladder

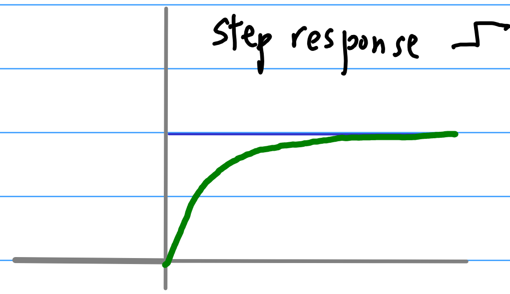
$$t_{pd} \approx \sum_{\text{nodes } i} R_{i-to-source} C_i$$

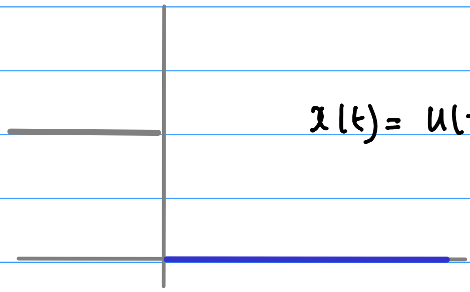
$$= R_1 C_1 + \left(R_1 + R_2\right) C_2 + \ldots + \left(R_1 + R_2 + \ldots + R_N\right) C_N$$

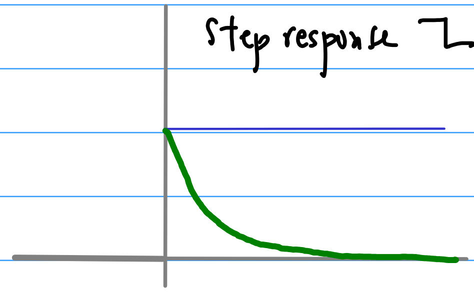


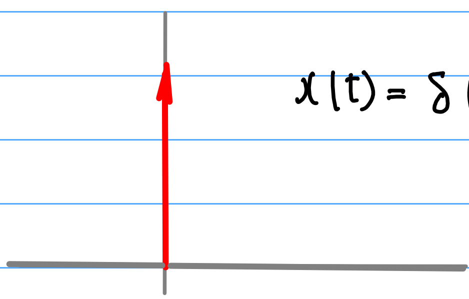
# Step Response & Impulse Response

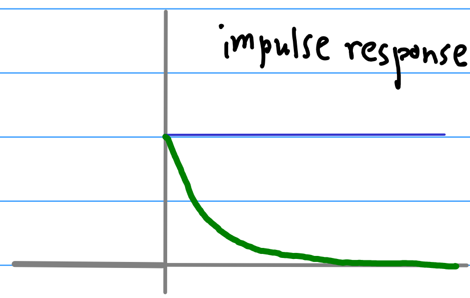


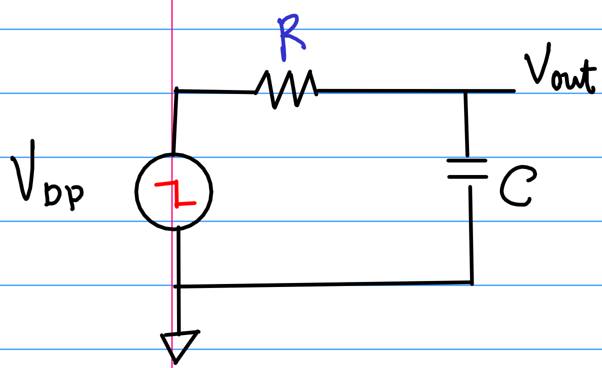



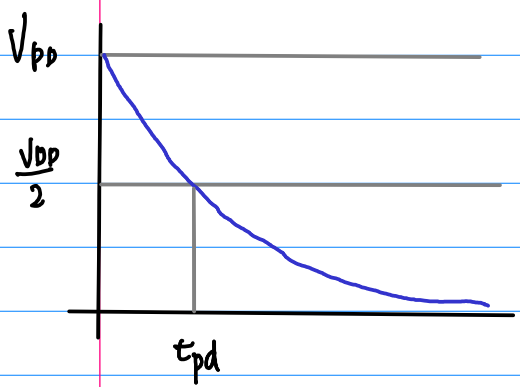



# Transient Response : 1st Order RC Systems




Step response to

$$\frac{\frac{1}{sC}}{R+\frac{1}{sC}} = \frac{1}{1+sRC}$$

$$H(s) = \frac{1}{1+sRC}$$

$$V_{out}(t) = V_{DD}\, e^{-t/\tau}$$

$$\tau = RC$$

$$\frac{V_{DD}}{2} = V_{DD}\, e^{-t/\tau}$$

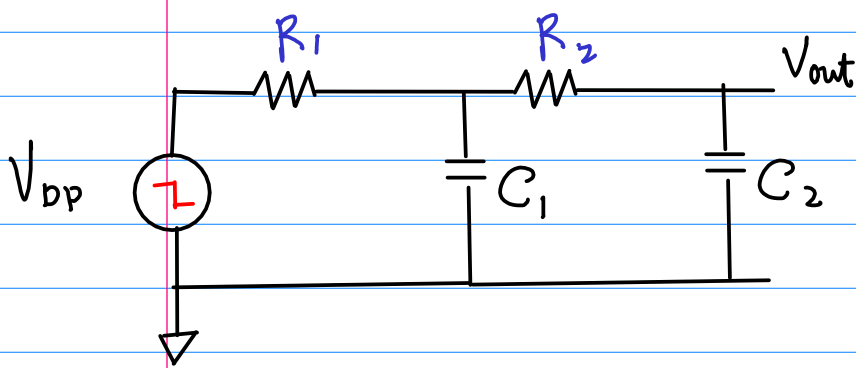
$$\frac{1}{2} = e^{-t/\tau}$$

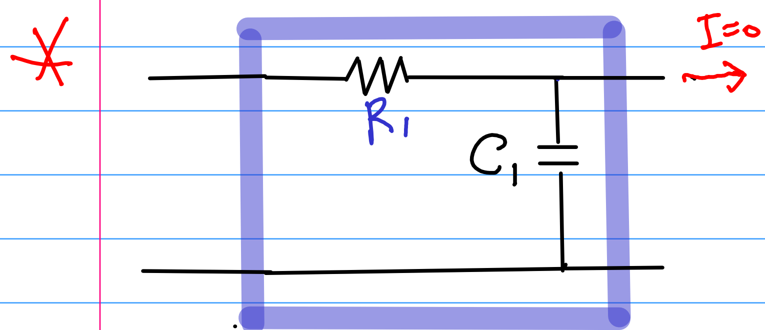
$$-t/\tau = \ln 2^{-1}$$

$$t = \ln 2\, \tau$$

$$t_{pd} = RC \ln 2$$
$$= R\ln 2 \cdot C$$
$$= R'C$$

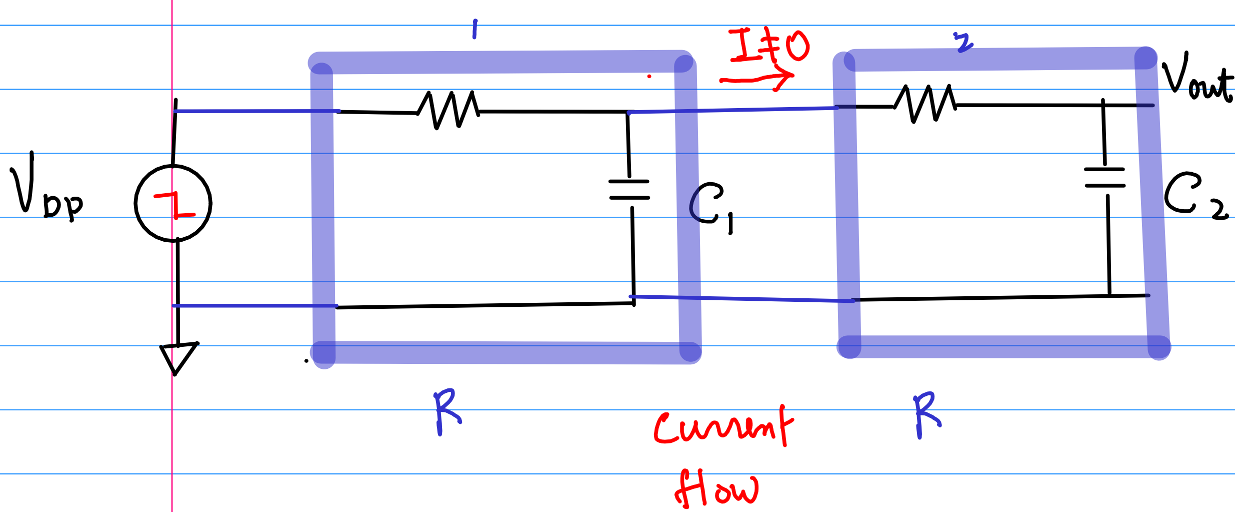
# Transient Response : $2^{nd}$ Order RC Systems



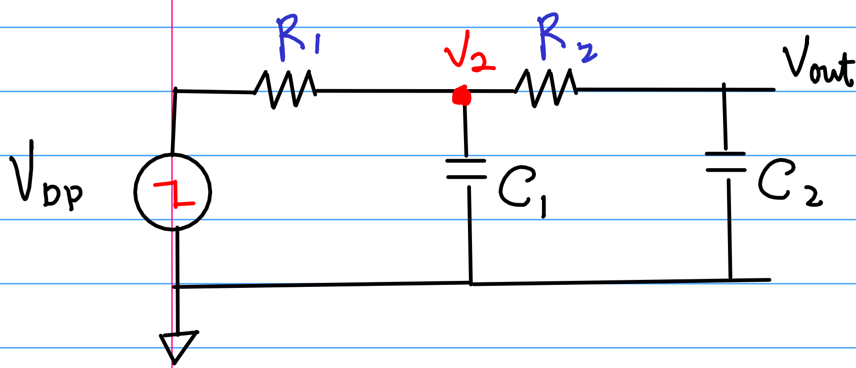

$$H(s) = \frac{1}{1+sRC}$$

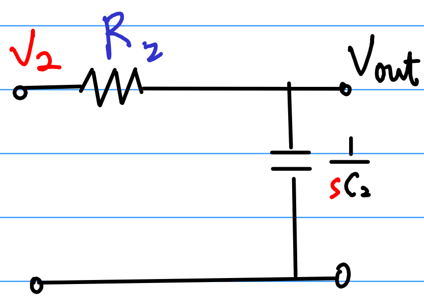
no current flow



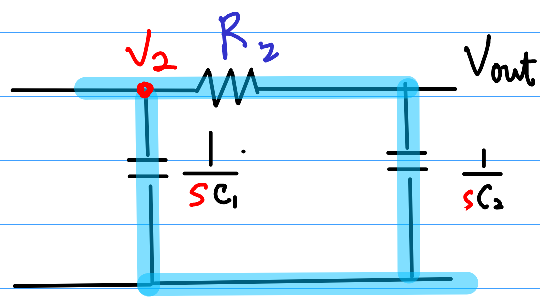

$$H(s) = \frac{1}{1+sR_1C_1} \cdot \frac{1}{1+sR_2C_2} = \frac{1}{1+s(R_1C_1+R_2C_2)+s^2R_1R_2C_1C_2}$$

# Voltage Divider

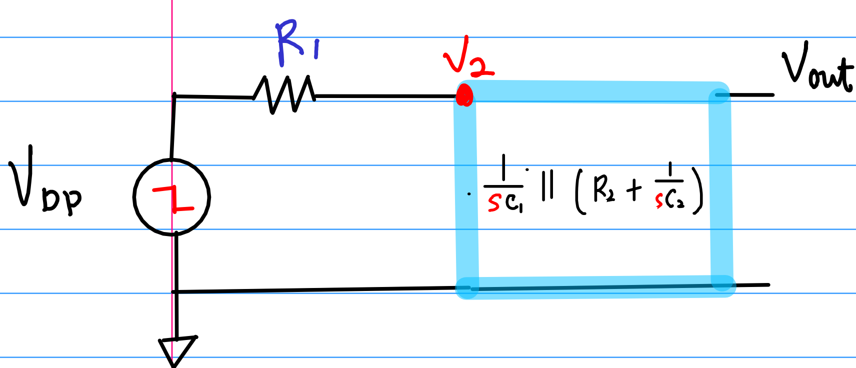



$$\boxed{V_{out}} = \frac{\frac{1}{sC_2}}{R_2 + \frac{1}{sC_2}} V_2$$



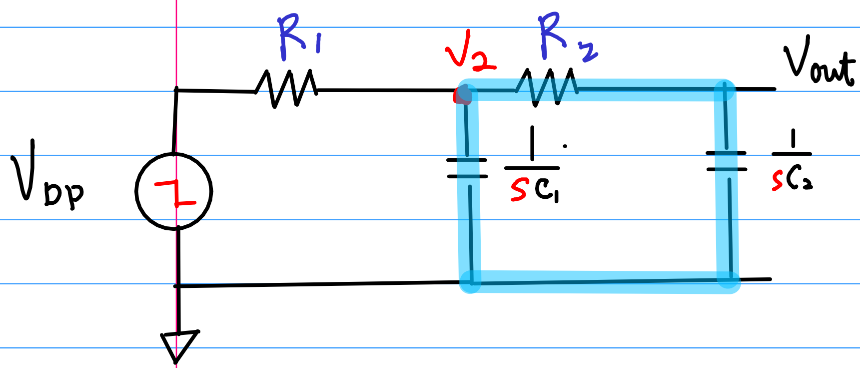

$$\boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}$$




$$V_2 = \frac{\boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}}{R_1 + \boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)}} V_{DD}$$

$$\Rightarrow \boxed{V_{out}} = \frac{1}{1 + sR_2C_2} V_2$$

# Transfer Function H(s)




$$\rightarrow \boxed{V_{out}} = \frac{1}{1+sR_2C_2}V_2$$

$$\boxed{\frac{1}{sC_1} \parallel \left(R_2 + \frac{1}{sC_2}\right)} = \frac{1}{sC_1 + \frac{1}{R_2 + \frac{1}{sC_2}}} = \frac{1}{sC_1 + \frac{sC_2}{sR_2C_2+1}}$$

$$= \frac{sR_2C_2+1}{s^2R_2C_1C_2 + sC_1 + sC_2} = \frac{sR_2C_2+1}{s^2R_2C_1C_2 + s(C_1+C_2)}$$

$$V_2 = \frac{\frac{sR_2C_2+1}{s^2R_2C_1C_2+s(C_1+C_2)}}{R_1 + \frac{sR_2C_2+1}{s^2R_2C_1C_2+s(C_1+C_2)}}V_{PP} = \frac{sR_2C_2+1}{s^2R_1R_2C_1C_2 + s(R_1C_1+R_1C_2+R_2C_2)+1}V_{PP}$$

$$\rightarrow \boxed{V_{out}} = \frac{1}{1+sR_2C_2}V_2 = \frac{1}{\cancel{(1+sR_2C_2)}}\,\frac{\cancel{(sR_2C_2+1)}}{s^2R_1R_2C_1C_2 + s(R_1C_1+R_1C_2+R_2C_2)+1}V_{PP}$$

$$H(s) = \frac{1}{1 + s\left[R_1C_1 + (R_1+R_2)C_2\right] + s^2R_1C_1R_2C_2}$$

# Quadratic Equations    the reciprocals of the roots

$$a s^2 + b s + c = 0$$

$$\boxed{\begin{aligned} s &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\ \frac{1}{s} &= \frac{2a}{-b \pm \sqrt{b^2-4ac}} \Rightarrow \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right] \end{aligned}}$$

$$\frac{1}{-b \pm \sqrt{b^2-4ac}} = \frac{-b \mp \sqrt{b^2-4ac}}{(-b)^2 - (b^2-4ac)} = \frac{-b \mp \sqrt{b^2-4ac}}{4ac}$$

$$\frac{2a}{-b \pm \sqrt{b^2-4ac}} = 2a \cdot \frac{-b \mp \sqrt{b^2-4ac}}{4ac} = \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right]$$

the reciprocals of the poles of H(s)

$$a s^2 + b s + c = 0$$

$$\frac{1}{s} = \frac{2a}{-b \pm \sqrt{b^2-4ac}} \rightarrow \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right]$$

$$1 + s\left[R_1C_1 + (R_1+R_2)C_2\right] + s^2 R_1C_1R_2C_2 = (1+s\tau_1)(1+s\tau_2) = 0$$

$$s = -\frac{1}{\tau_1}, -\frac{1}{\tau_2} \qquad \frac{1}{s} = \tau_1, \tau_2$$

$a = R_1C_1R_2C_2$

$b = \left[R_1C_1 + (R_1+R_2)C_2\right]$

$c = 1$

$$\frac{1}{s} = \frac{1}{2c}\left[-b \mp \sqrt{b^2-4ac}\right] \rightarrow$$

$$\frac{1}{s} = \frac{1}{2}\left[-\left[R_1C_1 + (R_1+R_2)C_2\right] \mp \sqrt{\left[R_1C_1 + (R_1+R_2)C_2\right]^2 - 4R_1C_1R_2C_2}\right]$$

$$\frac{1}{s} = -\frac{1}{2}\left[R_1C_1 + (R_1+R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R_1C_1R_2C_2}{\left[R_1C_1 + (R_1+R_2)C_2\right]^2}}\right]$$

Time constants $\tau_1$ & $\tau_2$

$$\boxed{\begin{gathered}1 + s\left[R_1C_1 + (R_1 + R_2)C_2\right] + s^2 R_1 C_1 R_2 C_2 = (1 + s\tau_1)(1 + s\tau_2) = 0 \\ s = -\frac{1}{\tau_1}, \; -\frac{1}{\tau_2}\end{gathered}}$$

$$\frac{1}{s} = -\frac{1}{2}\left[R_1C_1 + (R_1 + R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R_1C_1R_2C_2}{\left[R_1C_1 + (R_1 + R_2)C_2\right]^2}}\right]$$

$$\frac{R_2}{R_1} \equiv R'$$

$$\frac{C_2}{C_1} \equiv C'$$

$$\boxed{\begin{aligned}&\sqrt{1 - \frac{4R_1C_1R_2C_2}{\left[R_1C_1 + (R_1 + R_2)C_2\right]^2}} \\ &= \sqrt{1 - \frac{4\frac{R_2}{R_1}\frac{C_2}{C_1}}{\left[1 + \left(1 + \frac{R_2}{R_1}\right)\frac{C_2}{C_1}\right]^2}} \\ &= \sqrt{1 - \frac{4R'C'}{\left[1 + (1 + R')C'\right]^2}}\end{aligned}}$$

$$\boxed{\tau_1, \tau_2 = \frac{1}{2}\left[R_1C_1 + (R_1 + R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R'C'}{\left[1 + (1 + R')C'\right]^2}}\right]}$$

# Unit Step Response

$$H(s) = \frac{1}{1 + s\left[R_1C_1 + (R_1+R_2)C_2\right] + s^2 R_1 C_1 R_2 C_2}$$

$$= \frac{1}{(1+s\tau_1)(1+s\tau_2)} = \left[\frac{A}{(1+s\tau_1)} + \frac{B}{(1+s\tau_2)}\right]$$

$$\boxed{\begin{aligned} A &= \left.\frac{1}{(1+s\tau_2)}\right|_{s=-\frac{1}{\tau_1}} = \frac{1}{\left(1-\frac{\tau_2}{\tau_1}\right)} = \frac{\tau_1}{\tau_1-\tau_2} \\ B &= \left.\frac{1}{(1+s\tau_1)}\right|_{s=-\frac{1}{\tau_2}} = \frac{1}{\left(1-\frac{\tau_1}{\tau_2}\right)} = \frac{\tau_2}{\tau_2-\tau_1} \end{aligned}}$$

$$H(s) = \frac{1}{\tau_1-\tau_2}\left[\frac{\tau_1}{(1+s\tau_1)} - \frac{\tau_2}{(1+s\tau_2)}\right]$$

$$h(t) = \frac{1}{\tau_1-\tau_2}\left[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right]$$

Step response to

$$V_{out}(t) = \frac{1}{\tau_1-\tau_2}\left[\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}\right] V_{DD}$$

$$\boxed{\tau_1, \tau_2 = \frac{1}{2}\left[R_1C_1 + (R_1 + R_2)C_2\right]\left[1 \pm \sqrt{1 - \frac{4R'C'}{\left[1 + (1 + R')C'\right]^2}}\right]}$$

$$\tau = \tau_1 + \tau_2 = \left[R_1C_1 + (R_1 + R_2)C_2\right]$$

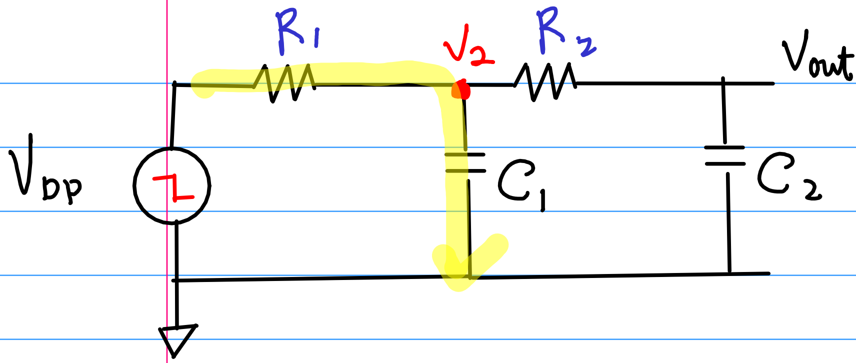
$R = R_1 = R_2$

$C = C_1 = C_2$

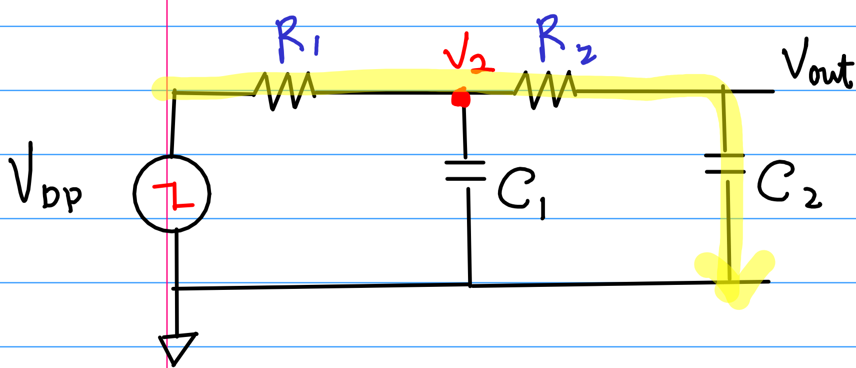
$\tau_1 = 2.6\,RC$

$\tau_2 = 0.4\,RC$

$\tau = 3.0\,RC$

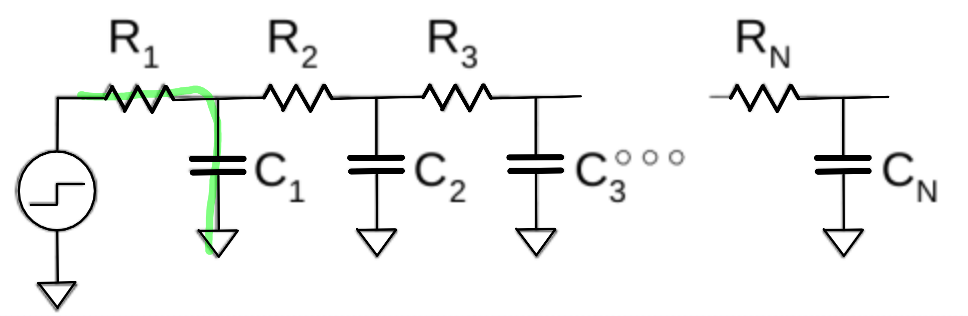



$$\tau_1 = R_1 C_1$$

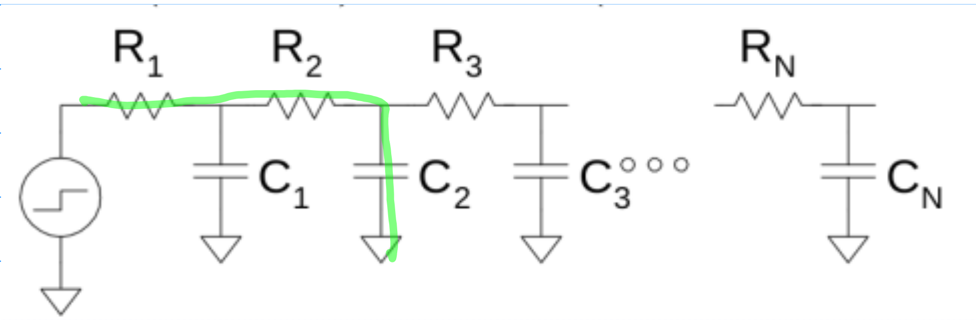



$$\tau_2 = (R_1 + R_2) C_2$$

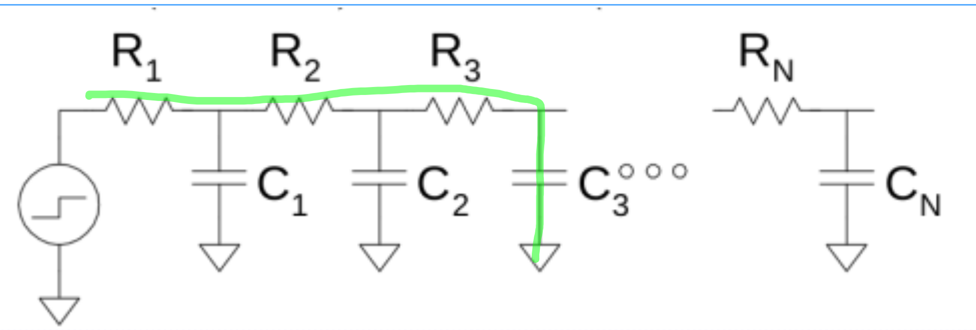
$$\tau = \tau_1 + \tau_2 = [R_1 C_1 + (R_1 + R_2) C_2]$$



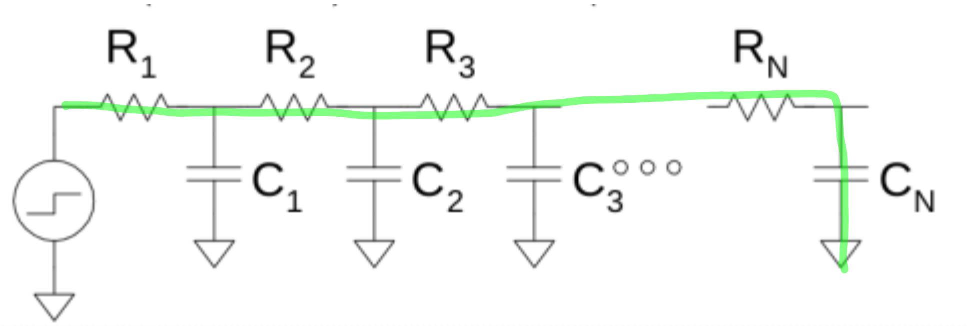

$R_1 C_1$




$(R_1 + R_2) C_2$

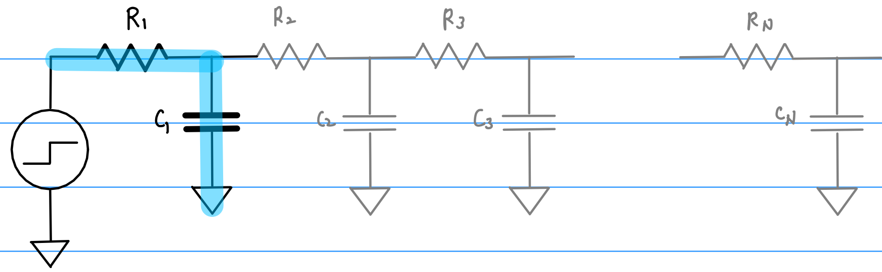



$(R_1 + R_2 + R_3) C_3$

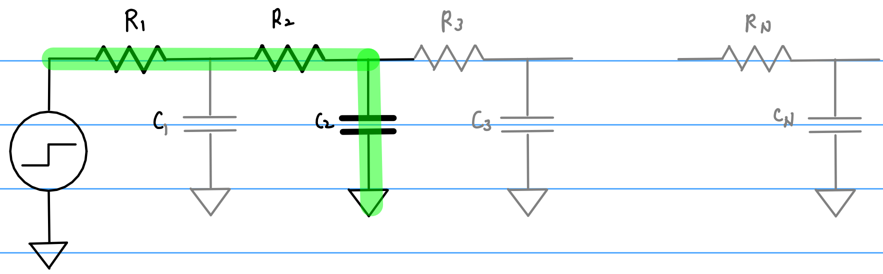



$(R_1 + R_2 + \cdots + R_N) C_N$

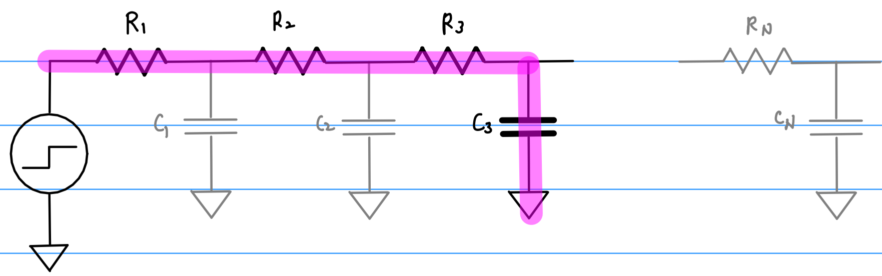
$$t_{pd} = R_1 C_1 + (R_1 + R_2) C_2 + \ldots + (R_1 + R_2 + \ldots + R_N) C_N$$



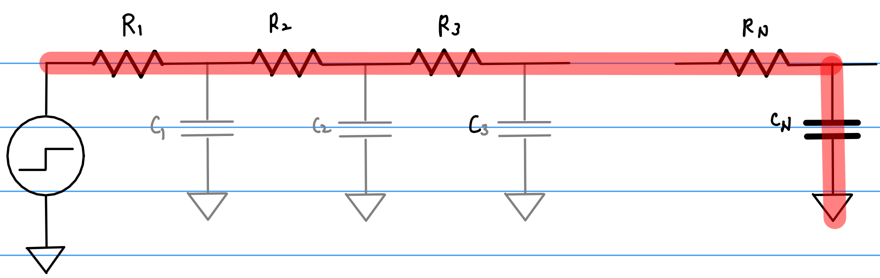

$(R_1)C_1$




$(R_1+R_2)C_2$



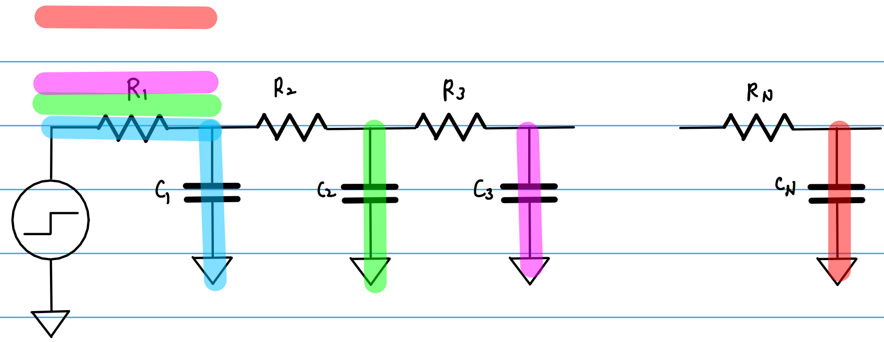

$(R_1+R_2+R_3)C_3$



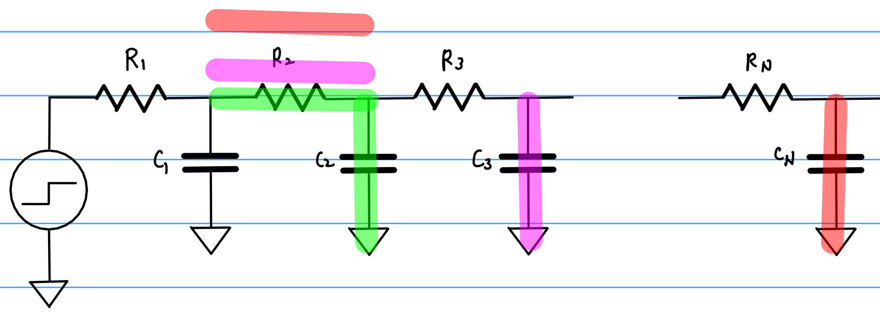

$(R_1+R_2+\cdots+R_N)C_N$

$$t_{pd} = (R_1)C_1 + (R_1+R_2)C_2 + (R_1+R_2+R_3)C_3 + \cdots + (R_1+R_2+\cdots+R_N)C_N$$

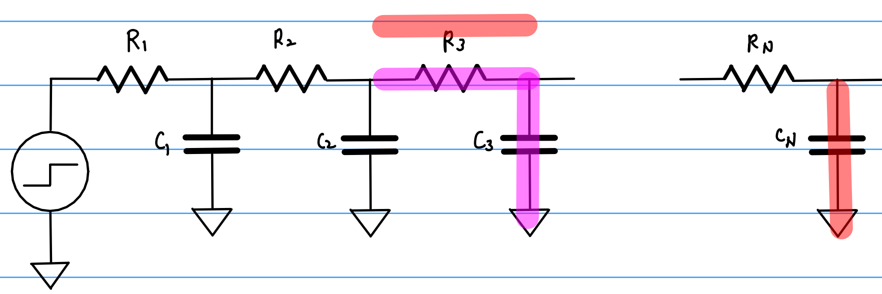
$$\boxed{t_{pd} = \sum_{i=1}^{n} \left( \sum_{j=1}^{i} R_j \right) C_i}$$



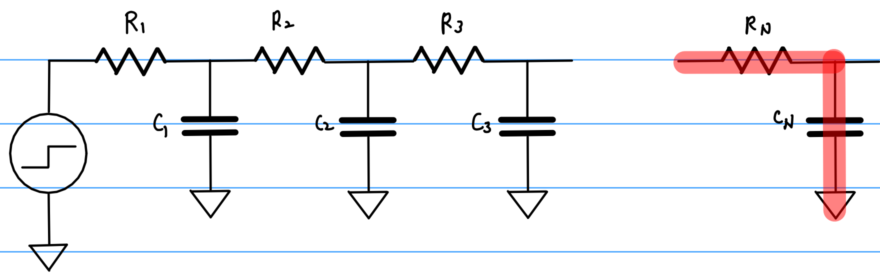

$(C_1 + C_2 + \cdots + C_N) R_1$




$(C_2 + C_3 + \cdots + C_N) R_2$

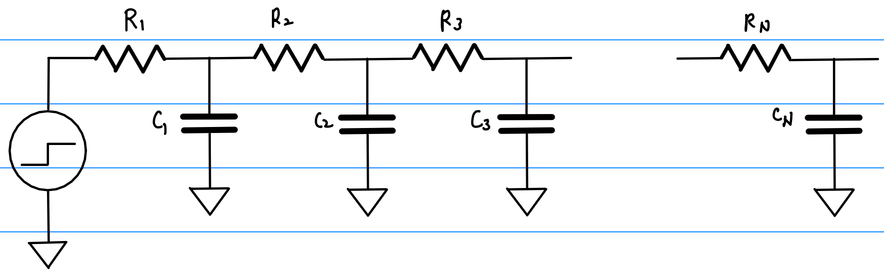


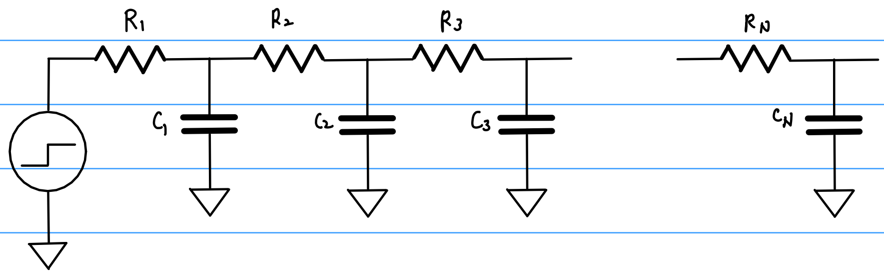

$(C_3 + \cdots + C_N) R_3$

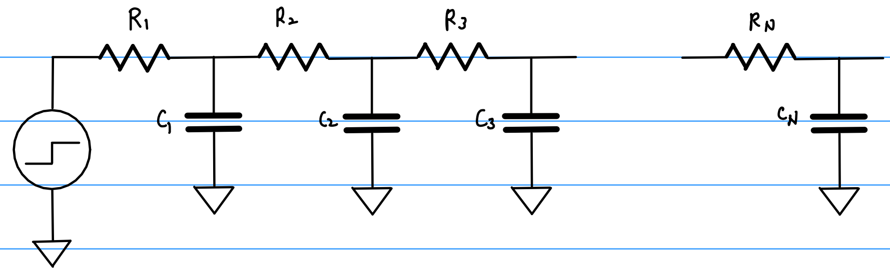



$$
\begin{aligned}
t_{pd} &= R_1 (C_1 + C_2 + \cdots + C_N) + R_2 (C_2 + C_3 + \cdots + C_N) + R_3 (C_3 + C_4 + \cdots + C_N) + \cdots + R_N C_N \\
&= (C_1 + C_2 + \cdots + C_N) R_1 + (C_2 + C_3 + \cdots + C_N) R_2 + (C_3 + C_4 + \cdots + C_N) R_3 + \cdots + C_N R_N
\end{aligned}
$$

$$\boxed{t_{pd} = \sum_{i=1}^{n} \left( \sum_{j=i}^{n} C_j \right) R_i}$$


R1
R2
R3
RN
C1
C2
C3
CN


R1
R2
R3
RN
C1
C2
C3
CN



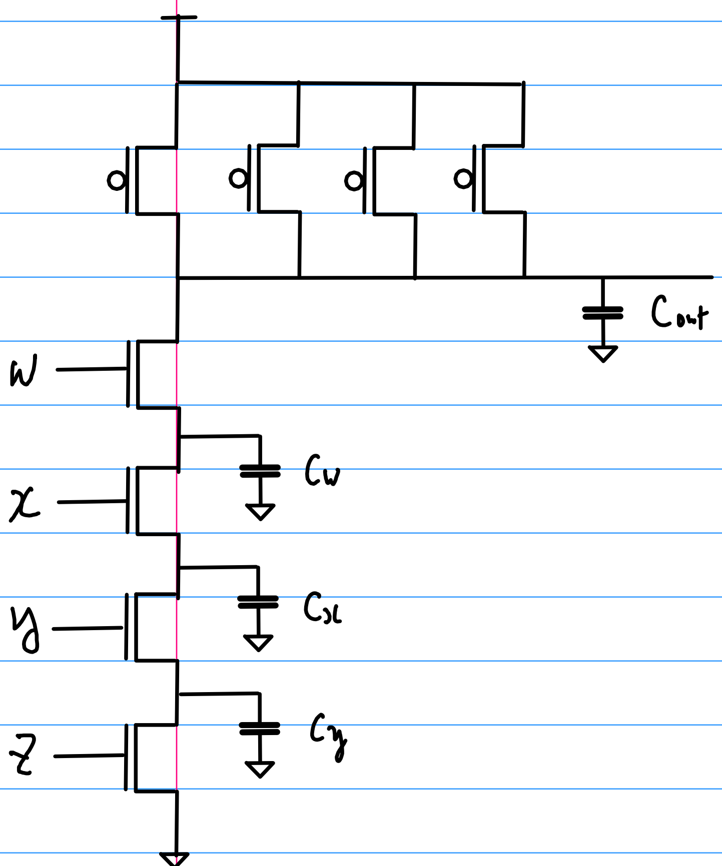

$$R_1 = R_2 = R_3 = \cdots = R_N = R$$

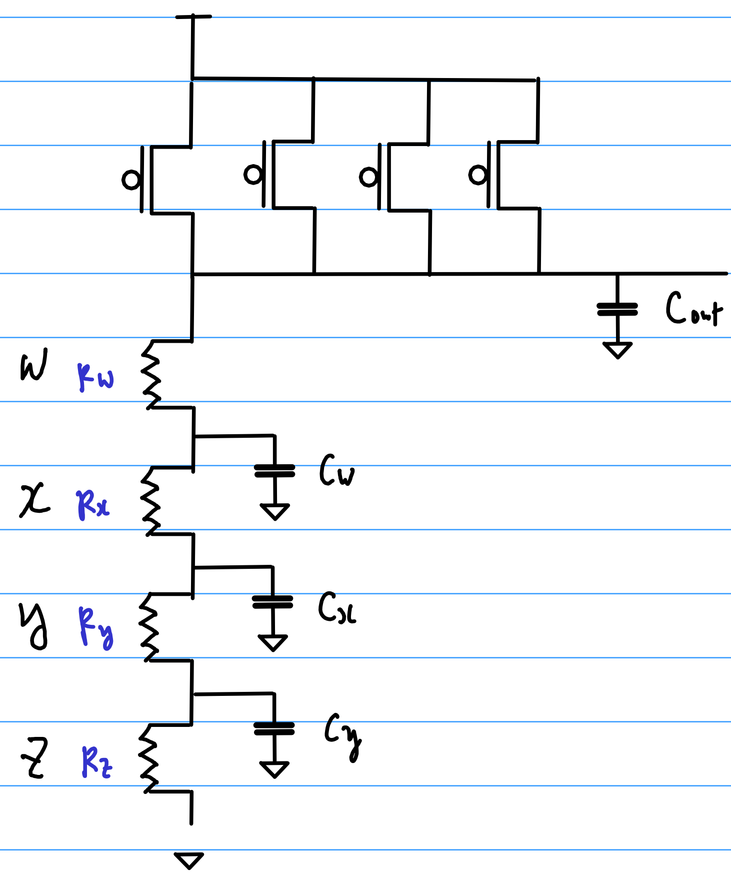
$$C_1 = C_2 = C_3 = \cdots = C_N = C$$

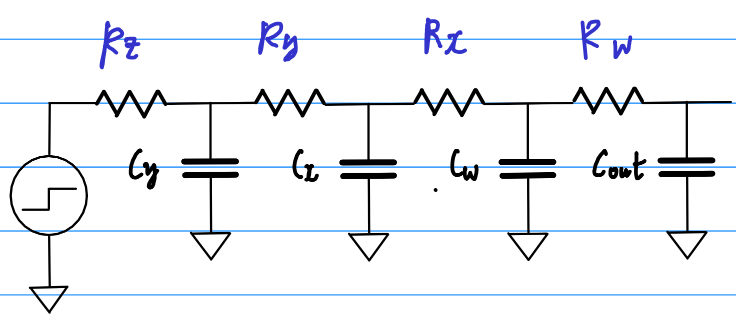
$$t_{pd} = nR \cdot C + (n-1)R \cdot C + \cdots + RC$$

$$= \frac{1}{2}n(n+1)RC \quad \propto n^2 RC$$

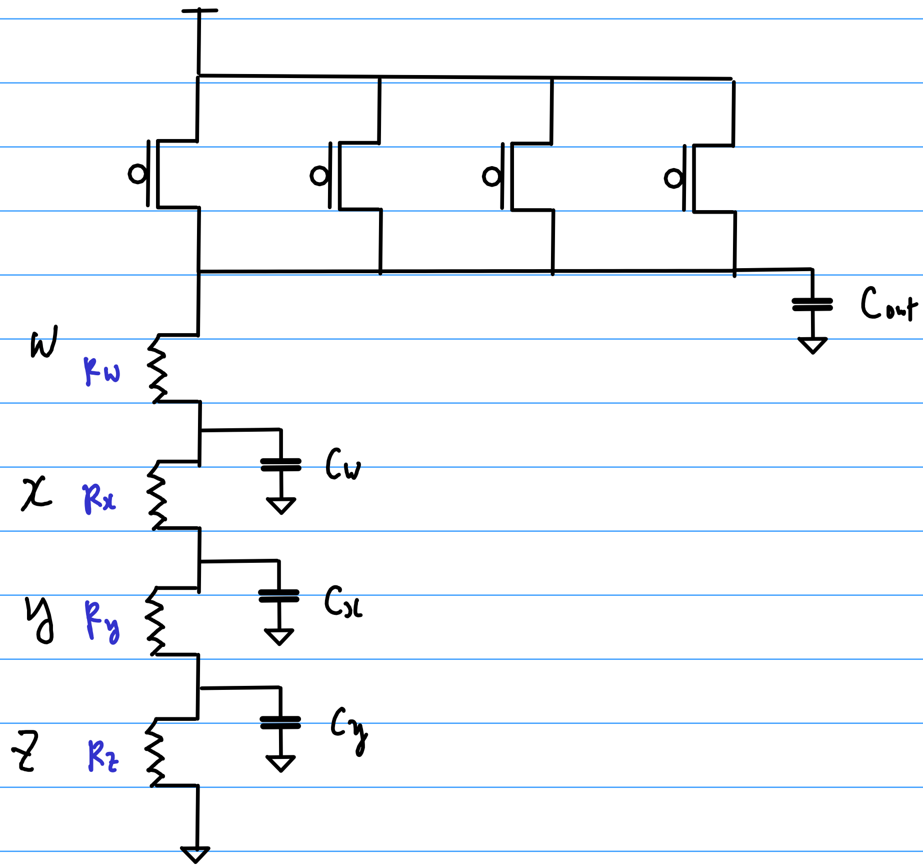
* STC (Single Time Constant) Circuit



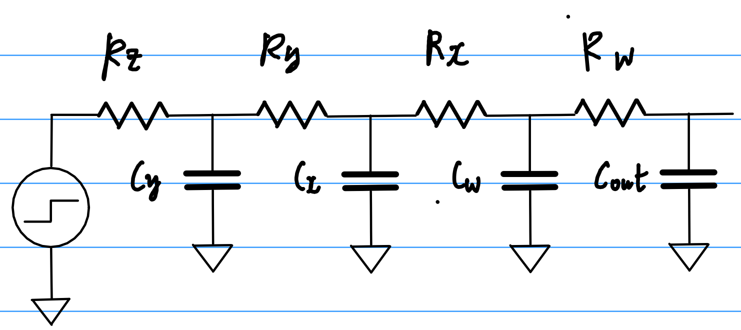




$$t_{pd} = C_y R_z + C_x(R_y + R_z) + C_w(R_x + R_y + R_z) + C_{out}(R_w + R_x + R_y + R_z)$$



W
Rw
x
Rx
y
Ry
Z
Rz
Cout
Cw
Cx
Cy

Rz
Ry
Rx
Rw
Cy
Cx
Cw
Cout

# RC-Tree Delay Model

the propagation delay at node $i$

Sum of all the time constants formed
by each capacitor $C_k$ and its associated resistance $R_{i,k}$

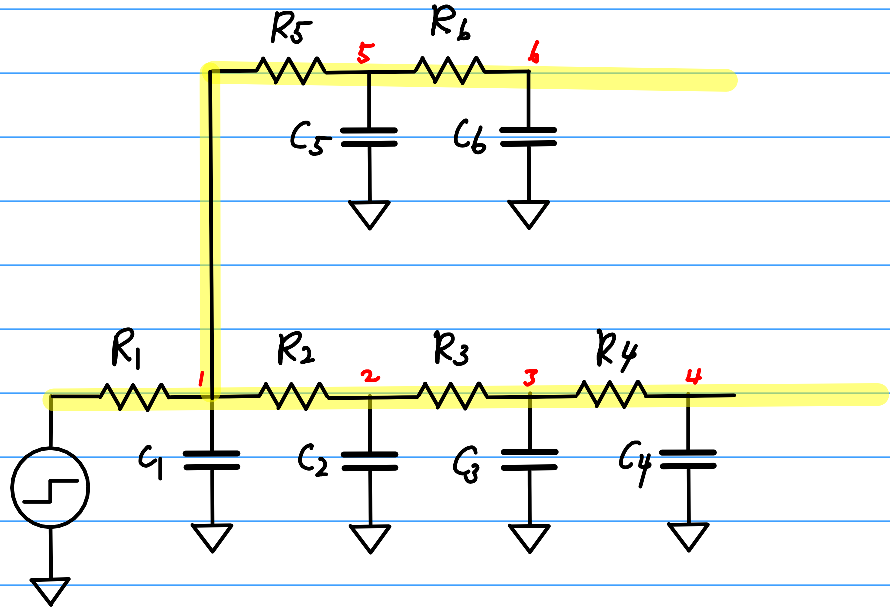
$$t_{pd\,i} = \sum_{k=1}^{n} C_k R_{i,k}$$

$$R_{i,k} = \sum R_j$$

$$\Rightarrow \quad R_j \in [\, path(i \to s) \cap path(k \to s) \,]$$

common path segment
from node $i$ to source node $s$
from node $k$ to source node $s$

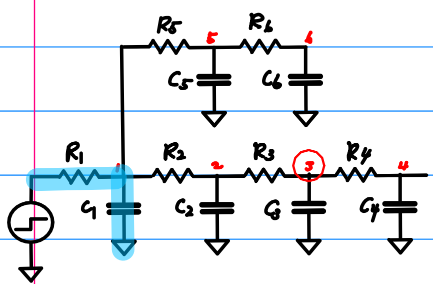
node $i$ : node of interest
node $k$ : $1 \leq k \leq n$ nodes to which capacitors are connected



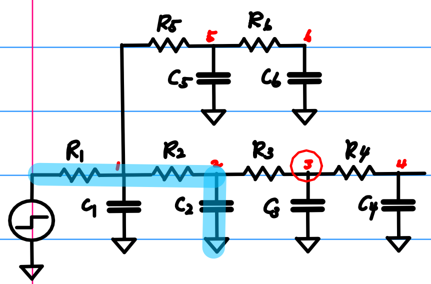

at node i

$$t_{pd\,i} = \sum_{k=1}^{n} C_k R_{i,k}$$

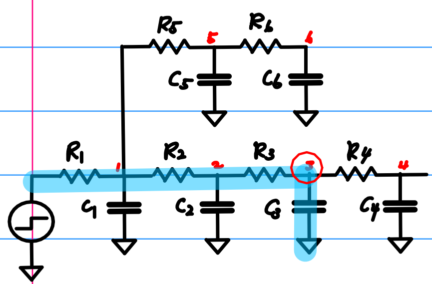
$$R_{i,k} = \sum R_j \Rightarrow R_j \in [\, path(i \to s) \cap path(k \to s) \,]$$



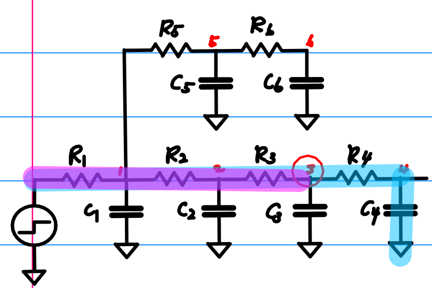

$R_1 C_1$



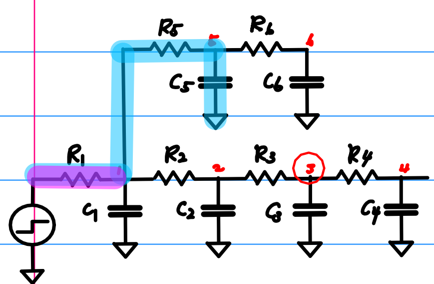

$(R_1 + R_2) C_2$



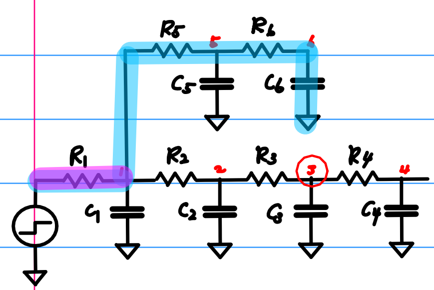

$(R_1 + R_2 + R_3) C_3$



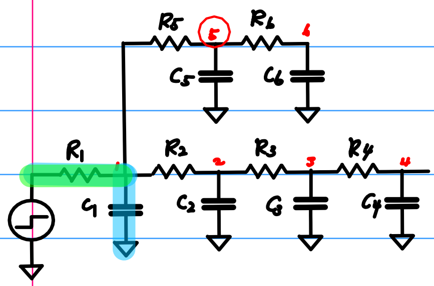

$(R_1 + R_2 + R_3) C_4$



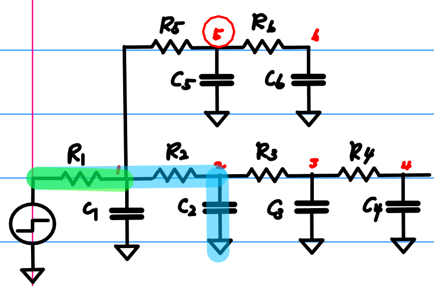

$R_1 C_5$



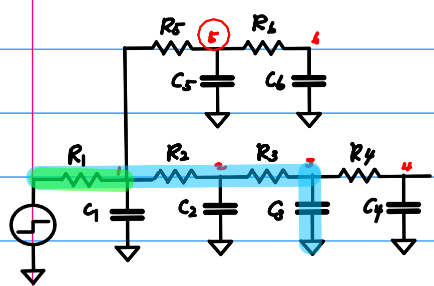

$R_1 C_6$



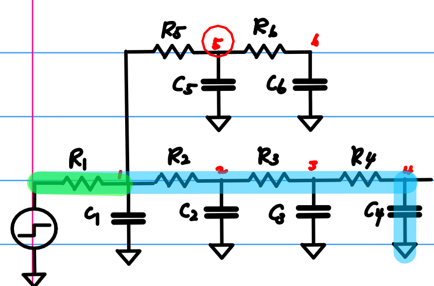

$R_1C_1$



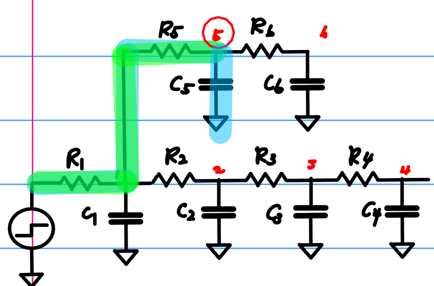

$R_1C_2$



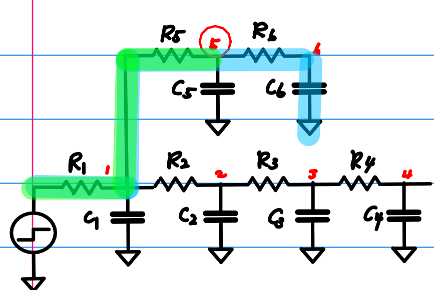

$R_1C_3$




$R_1C_4$




$(R_1+R_5)C_5$




$(R_1+R_5)C_6$

$$t_{pd3} = R_1C_1 + (R_1+R_2)C_2 + (R_1+R_2+R_3)C_3 + (R_1+R_2+R_3)C_4 + R_1C_5 + R_1C_6$$

$$t_{pd5} = R_1C_1 + R_1C_2 + R_1C_3 + R_1C_4 + (R_1+R_5)C_5 + (R_1+R_5)C_6$$