Relations (3A)

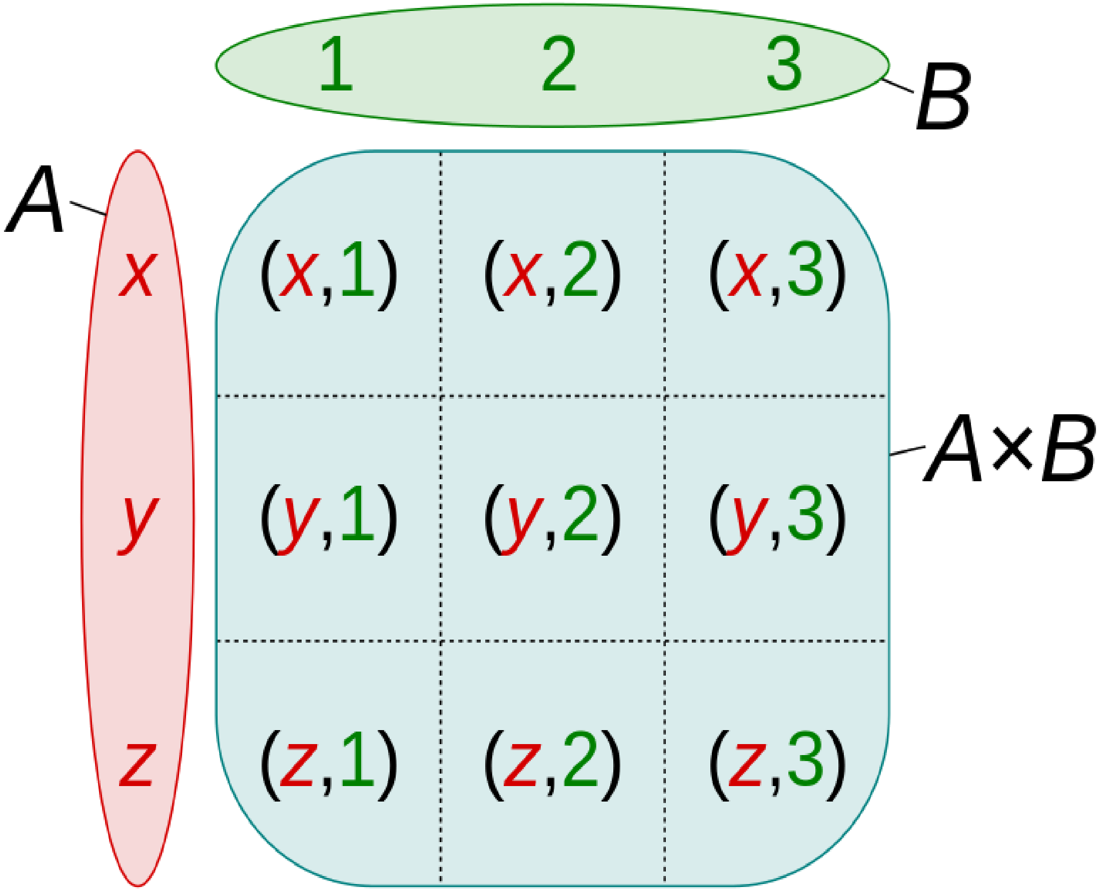
Copyright (c) 2015 – 2018 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice and Octave.

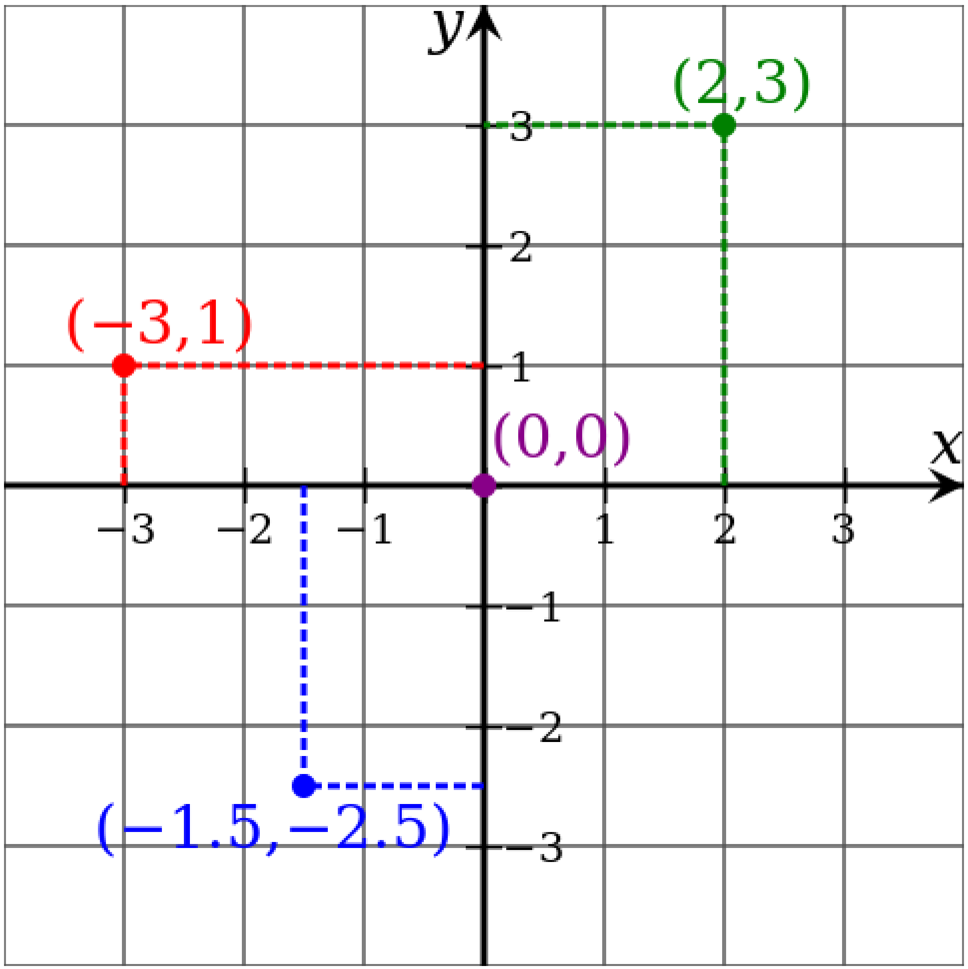
Cartesian Product



Cartesian product A × B
of the sets A = { x , y , z }
and B = { 1 , 2 , 3 }

https://en.wikipedia.org/wiki/Cartesian_product

Cartesian Coordinates



https://en.wikipedia.org/wiki/Cartesian_product

Cartesian coordinates of example points

Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

https://en.wikipedia.org/wiki/Cartesian_product

Cartesian Product

	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

	1	2	3	4	5
1	1**R**1	1**R**2	1**R**3	1**R**4	1**R**5
2	2**R**1	2**R**2	2**R**3	2**R**4	2**R**5
3	3**R**1	3**R**2	3**R**3	3**R**4	3**R**5
4	1**R**1	4**R**2	4**R**3	4**R**4	4**R**5
5	5**R**1	5**R**2	5**R**3	5**R**4	5**R**5

https://en.wikipedia.org/wiki/Cartesian_product

Types of Relations (1)

$$x \; R \; x$$

$$x \; R \; y \Leftrightarrow y \; R \; x$$

$$x \; R \; y \wedge y \; R \; z \Rightarrow x \; R \; z$$

- Reflexive Relation
- Symmetric Relation
- Transitive Relation

https://en.wikipedia.org/wiki/Reflexive_relation

Definitions of Relations

Reflexive: for all x in X it holds that x**Rx**.

Symmetric: for all x and y in X it holds that if x**R**y then y**R**x.

Antisymmetric: for all x and y in X, if x**R**y and y**R**x then x = y.

Transitive: for all x, y and z in X it holds that if x**R**y and y**R**z then x**R**z.

https://en.wikipedia.org/wiki/Reflexive_relation

More Definitions of Relations

A relation **R** on a set ***A*** is called **reflexive**
if $(a, a) \in$ **R** for every element $a \in$ ***A***.

A relation **R** on a set ***A*** is called **symmetric**
if $(b, a) \in$ **R** whenever $(a, b) \in$ **R**, for all $a, b \in$ ***A***

A relation **R** on a set ***A*** such that for all a, b in **R**,
if $(a, b) \in$ **R** and $(b, a) \in$ **R**, then $a = b$ is called **anti-symmetric**.

A relation **R** on a set ***A*** is called **transitive**
if whenever $(a, b) \in$ **R** and $(b, c) \in$ **R**,
then $(a, c) \in$ **R**, for all $a, b, c \in$ ***A***

https://en.wikipedia.org/wiki/Reflexive_relation

Relation Examples (1)

$x \geq y$

	1	2	3	4	5
1	(1,1)				
2	(2,1)	(2,2)			
3	(3,1)	(3,2)	(3,3)		
4	(4,1)	(4,2)	(4,3)	(4,4)	
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)

$x > y$

	1	2	3	4	5
1					
2	(2,1)				
3	(3,1)	(3,2)			
4	(4,1)	(4,2)	(4,3)		
5	(5,1)	(5,2)	(5,3)	(5,4)	

https://en.wikipedia.org/wiki/Cartesian_product

Relation Examples (2)

$$x = y$$

	1	2	3	4	5
1	(1,1)				
2		(2,2)			
3			(3,3)		
4				(4,4)	
5					(5,5)

$$x = y + 1$$

	1	2	3	4	5
1					
2	(2,1)				
3		(3,2)			
4			(4,3)		
5				(5,4)	

https://en.wikipedia.org/wiki/Cartesian_product

Relation Examples (3)

$$x + y = 4$$

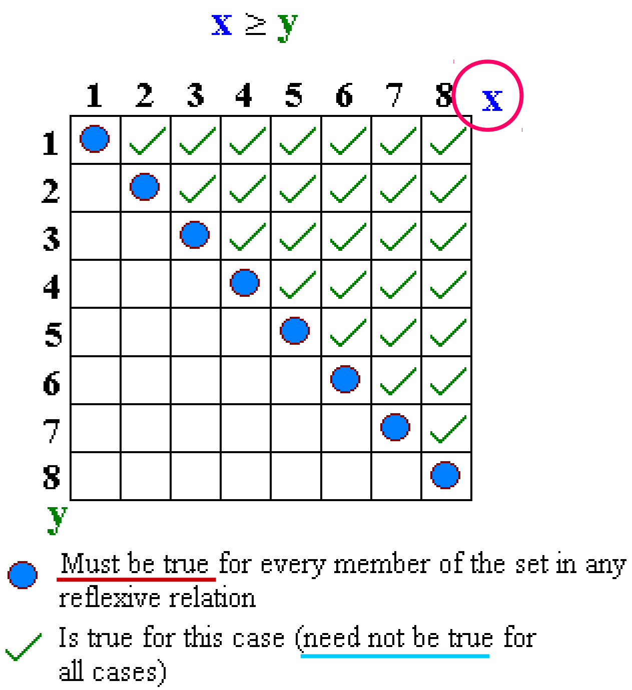
	1	2	3	4	5
1			(1,3)		
2		(2,2)			
3	(3,1)				
4					
5					

$$x + y \leq 4$$

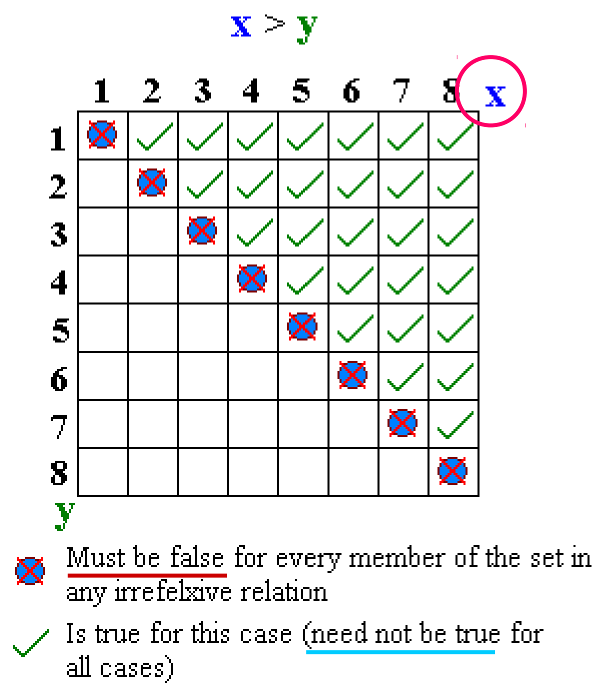
	1	2	3	4	5
1	(1,1)	(1,2)	(1,3)		
2	(2,1)	(2,2)			
3	(3,1)				
4					
5					

https://en.wikipedia.org/wiki/Cartesian_product

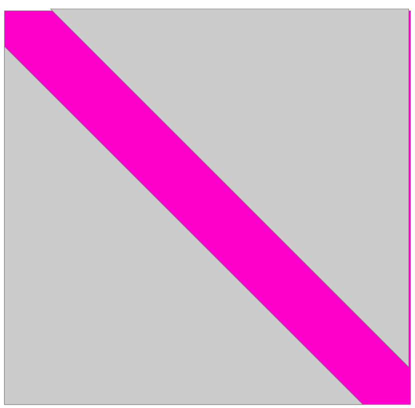
Reflexive Relation Examples



Reflexive Relation

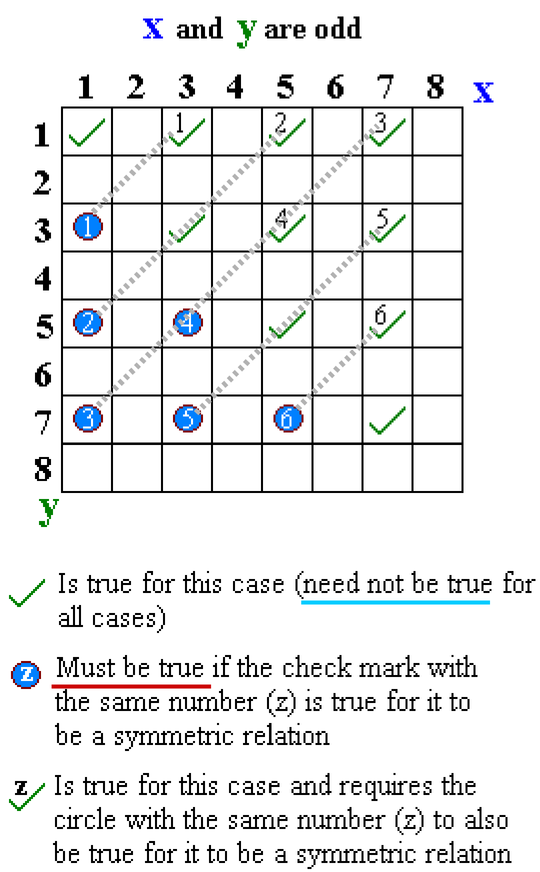


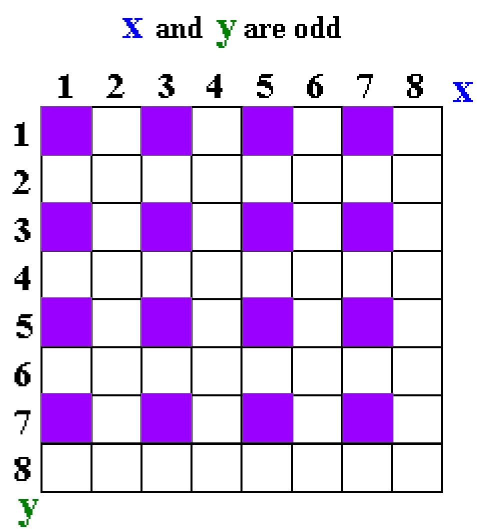
Irreflexive Relation

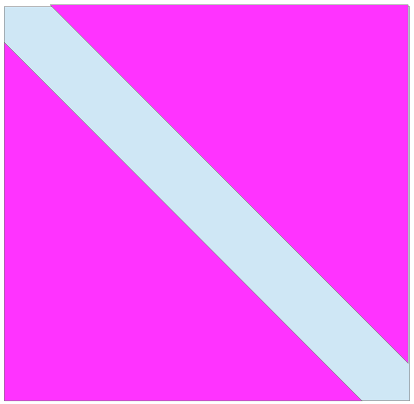


https://en.wikipedia.org/wiki/Reflexive_relation

Symmetric Relation Examples



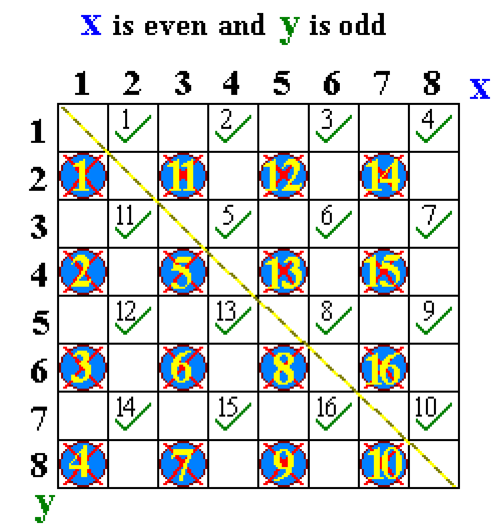




Symmetric Relation

https://en.wikipedia.org/wiki/Cartesian_product

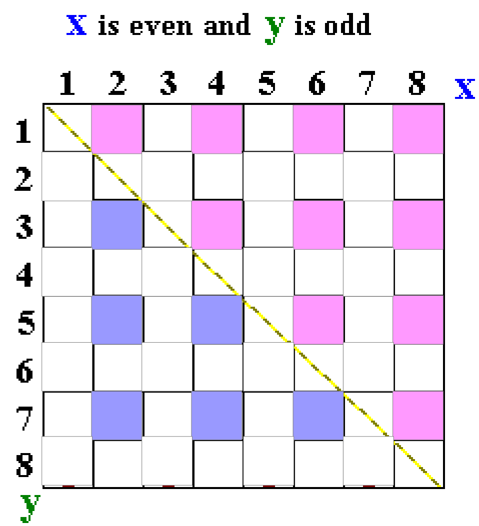
Anti-Symmetric Relation Examples

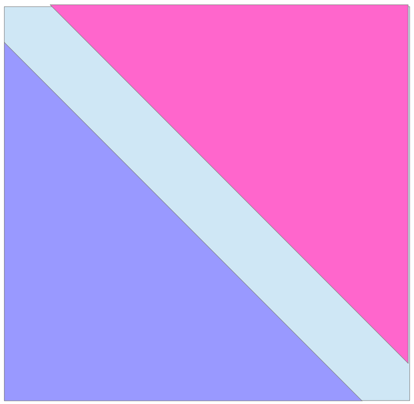


Is true for this case (need not be true for all cases)

Must be false if the check mark with the same number (z) is true for it to be an antisymmetric relation

Is true for this case and requires the circle with the same number (z) to be false for it to be a symmetric relation

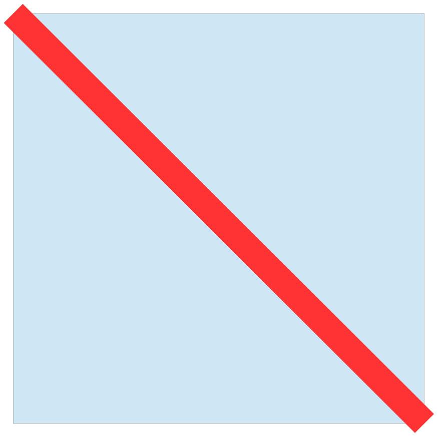




https://en.wikipedia.org/wiki/Cartesian_product

Reflexive Relation

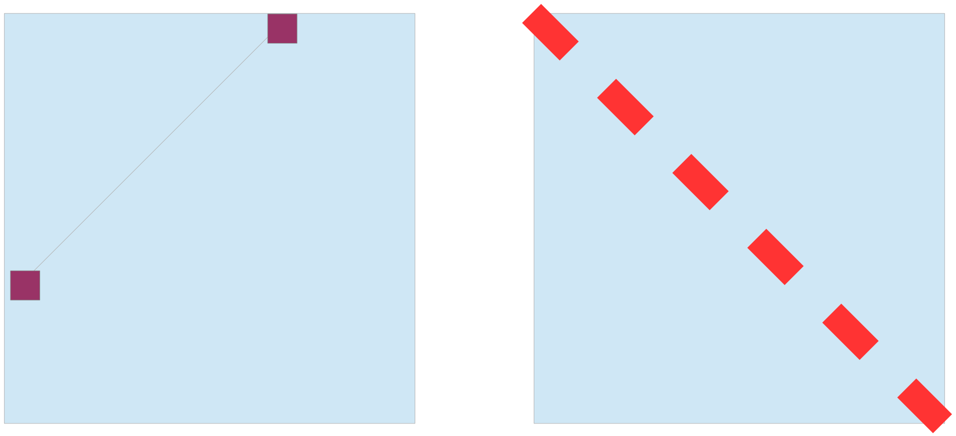
$$\forall x \quad (x, x) \in R$$



All diagonal relations
<u>must</u> exist

Symmetric Relation

$$\forall x, \forall y \; \left[\; (x, y) \in R \rightarrow (y, x) \in R \; \right]$$



no relation is mandatory

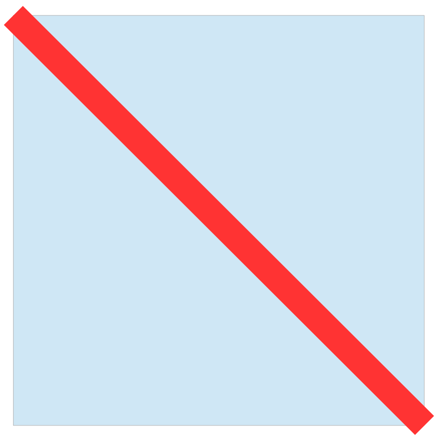
but for any relation,
its symmetric relation must exist
including diagonal relations

symmetric

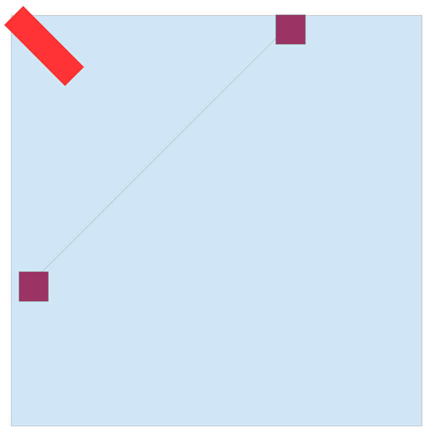
Symmetric Relation Examples

$$\forall x, \forall y \; [\; (x, y) \in R \;\rightarrow\; (y, x) \in R \;]$$

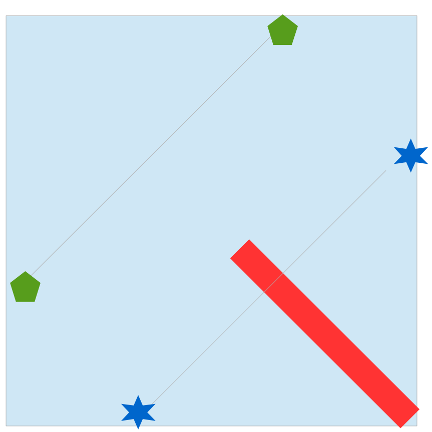
symmetric

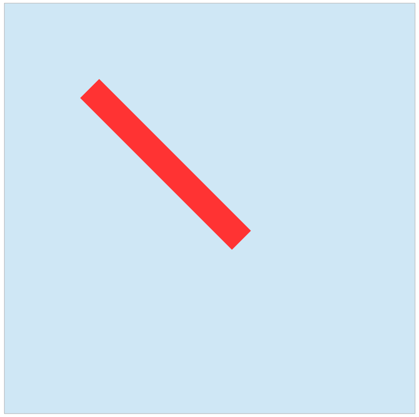


symmetric

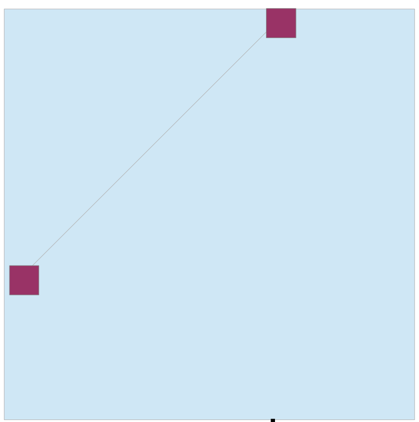


symmetric

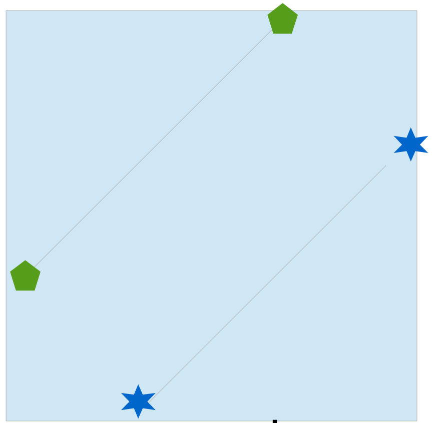




symmetric



symmetric



symmetric

Not Symmetric Relation

$$\neg\{\ \forall x, \forall y\ [\ (x,\ y) \in R\ ] \rightarrow [\ (y,\ x) \in R\ ]\ \}$$

$$\exists x, \exists y\ \neg\{\ [\ (x,\ y) \in R\ ] \rightarrow [\ (y,\ x) \in R\ ]\ \}$$

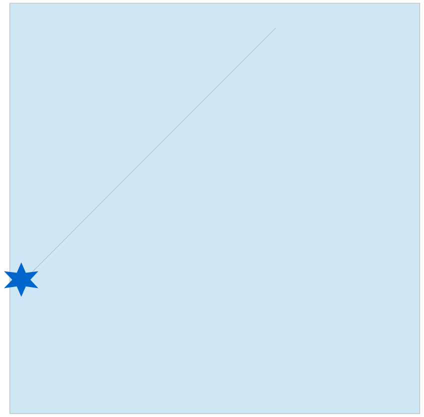
$$\exists x, \exists y\ \neg\{\ \neg[\ (x,\ y) \in R\ ] \vee [\ (y,\ x) \in R\ ]\ \}$$

$$\exists x, \exists y\ [\ (x,\ y) \in R\ ] \wedge \neg[\ (y,\ x) \in R\ ]$$

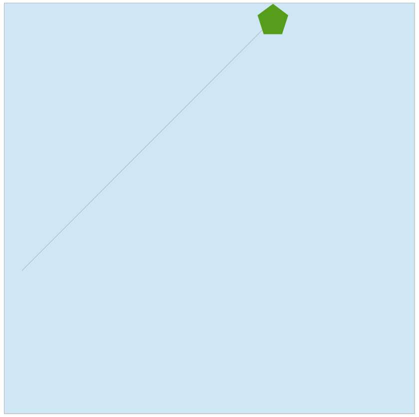
$$\exists x, \exists y\ [\ (x,\ y) \in R\ ] \wedge [\ (y,\ x) \notin R\ ]$$

counter example

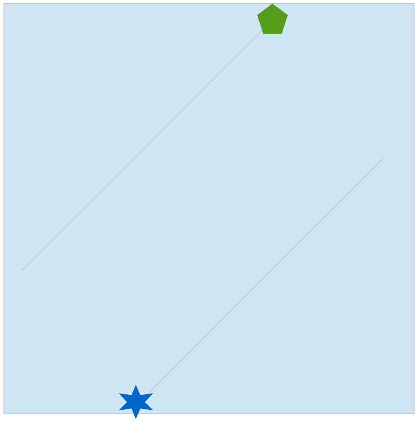
not symmetric



not symmetric

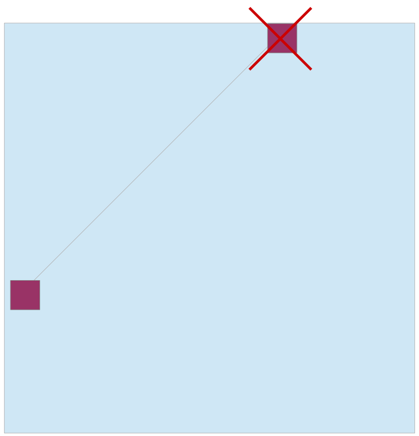


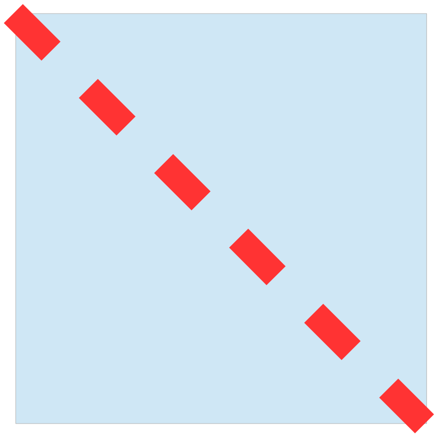
not symmetric



Anti-symmetric Relation

$$\forall x, \forall y \ \left[\left((x, y) \in R \ \wedge \ (y, x) \in R \right) \rightarrow \ x = y \right]$$





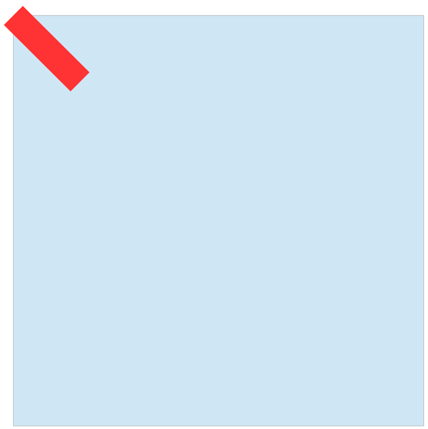
no relation is mandatory

but for any relation,
its symmetric relation must NOT exist
excluding diagonal relations

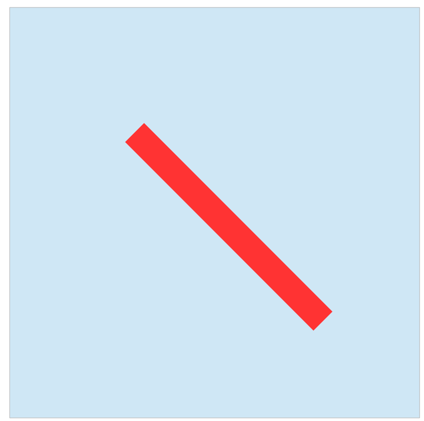
Anti-symmetric Relation Examples

$$\forall x, \forall y \ \left[\left((x, y) \in R \ \wedge \ (y, x) \in R \right) \rightarrow x = y \right]$$

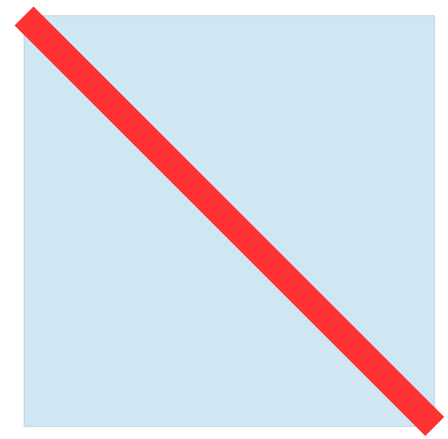
anti-symmetric

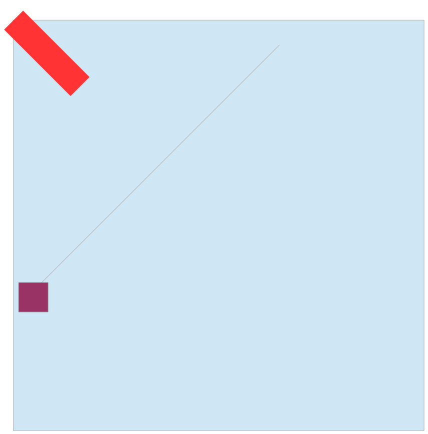


anti-symmetric

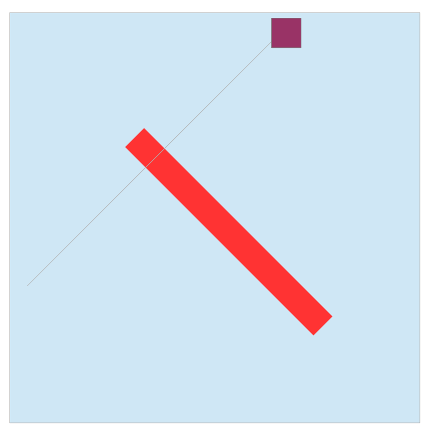


anti-symmetric

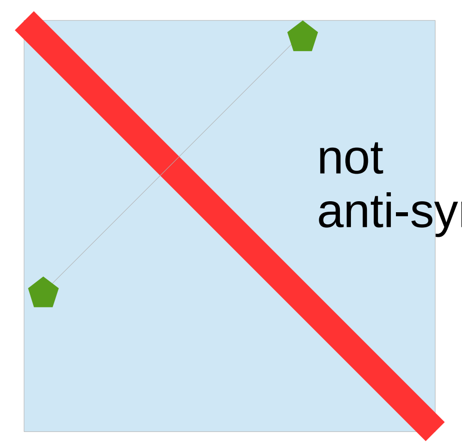




anti-symmetric



anti-symmetric



not
anti-symmetric

Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \rightarrow [\; x = y \;]\}$$

$$\exists x, \exists y \neg\{ \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \rightarrow [\; x = y \;]\}$$

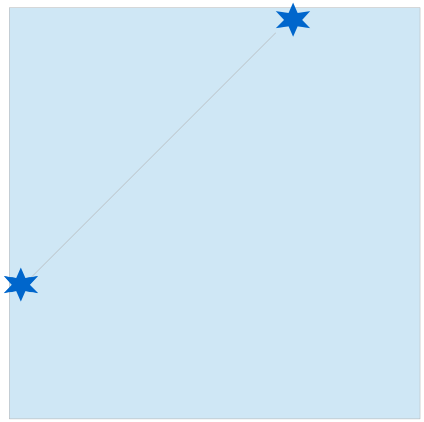
$$\exists x, \exists y \neg\{\neg \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \vee [\; x = y \;]\}$$

$$\exists x, \exists y \{ \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \wedge \neg[\; x = y \;]\}$$

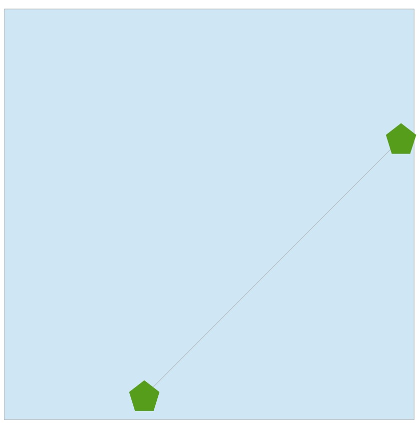
$$\exists x, \exists y \{ \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \wedge [\; x \neq y \;]\}$$

counter example

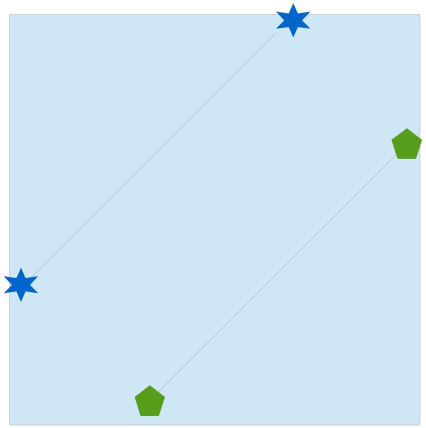
not anti-symmetric



not anti-symmetric



not anti-symmetric



Not Symmetric vs Not Anti-Symmetric Relation

$\neg\{\ \forall x, \forall y\ [\ (x, y) \in R\ ] \rightarrow [\ (y, x) \in R\ ]\ \}$ $\Leftrightarrow$

$\exists x, \exists y\ [\ (x, y) \in R\ ] \wedge [\ (y, x) \notin R\ ]$

not symmetric

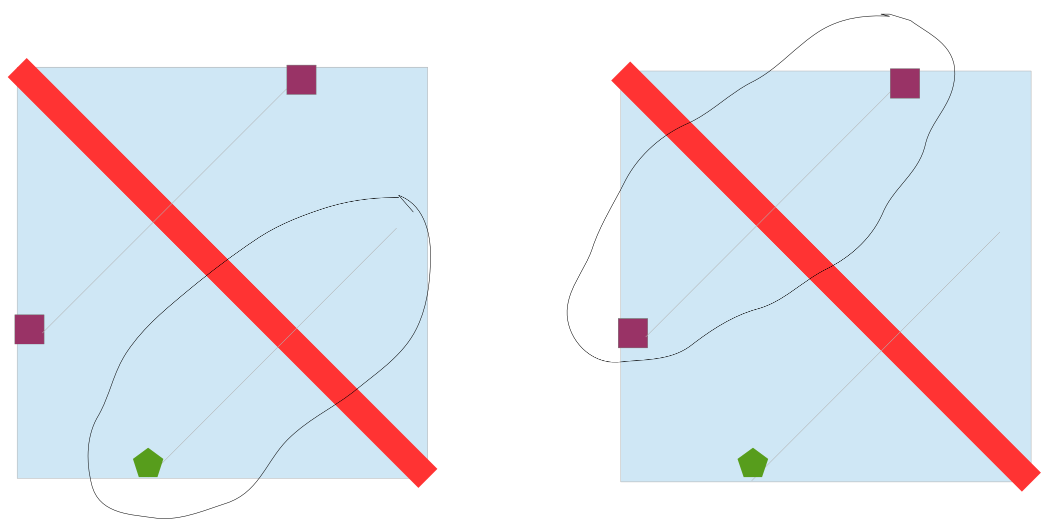
$\neg\{\forall x, \forall y\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \rightarrow [\ x = y\ ]\}$ $\Leftrightarrow$

$\exists x, \exists y\{\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \wedge [\ x \neq y\ ]\}$

not anti-symmetric

neither
symmetric

nor
anti-symmetric

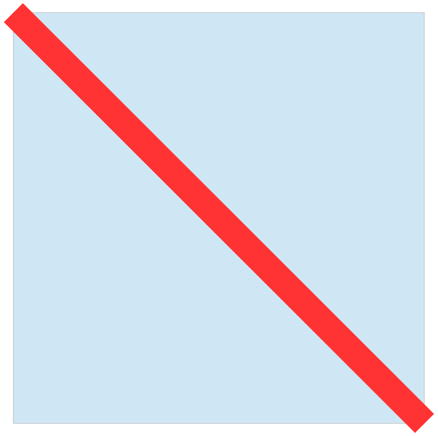


Reflexive, Symmetric, Anti-symmetric

$$\forall x \quad (x, x) \in R$$

$$\forall x, \forall y \; [\; (x, y) \in R \;] \rightarrow [\; (y, x) \in R \;]$$

$$\forall x, \forall y \; [\; (x, y) \in R \;\wedge\; (y, x) \in R \;] \rightarrow [\; x = y \;]$$



Reflexive
Also, symmetric (no relation for (x, y) where x≠y)
Also, anti-symmetric (no relation for (x, y) where x≠y)

Not Anti-symmetric Relation

$$\neg\{\forall x, \forall y\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \rightarrow [\ x = y\ ]\}$$

$$\exists x, \exists y \neg\{\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \rightarrow [\ x = y\ ]\}$$

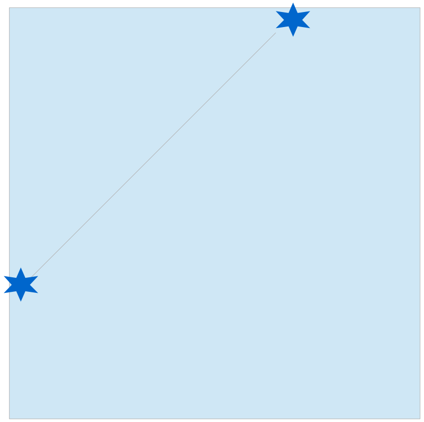
$$\exists x, \exists y \neg\{\neg\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \vee [\ x = y\ ]\}$$

$$\exists x, \exists y \{\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \wedge \neg[\ x = y\ ]\}$$

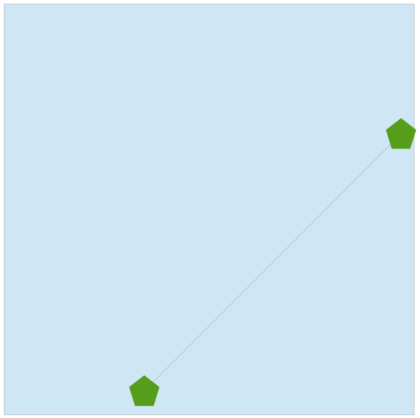
$$\exists x, \exists y \{\ [\ (x, y) \in R\ \wedge\ (y, x) \in R\ ] \wedge [\ x \neq y\ ]\}$$

counter example

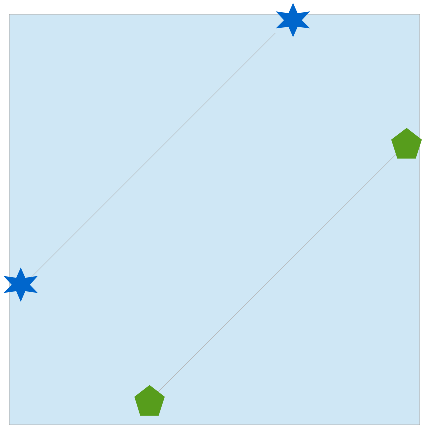
not anti-symmetric



not anti-symmetric

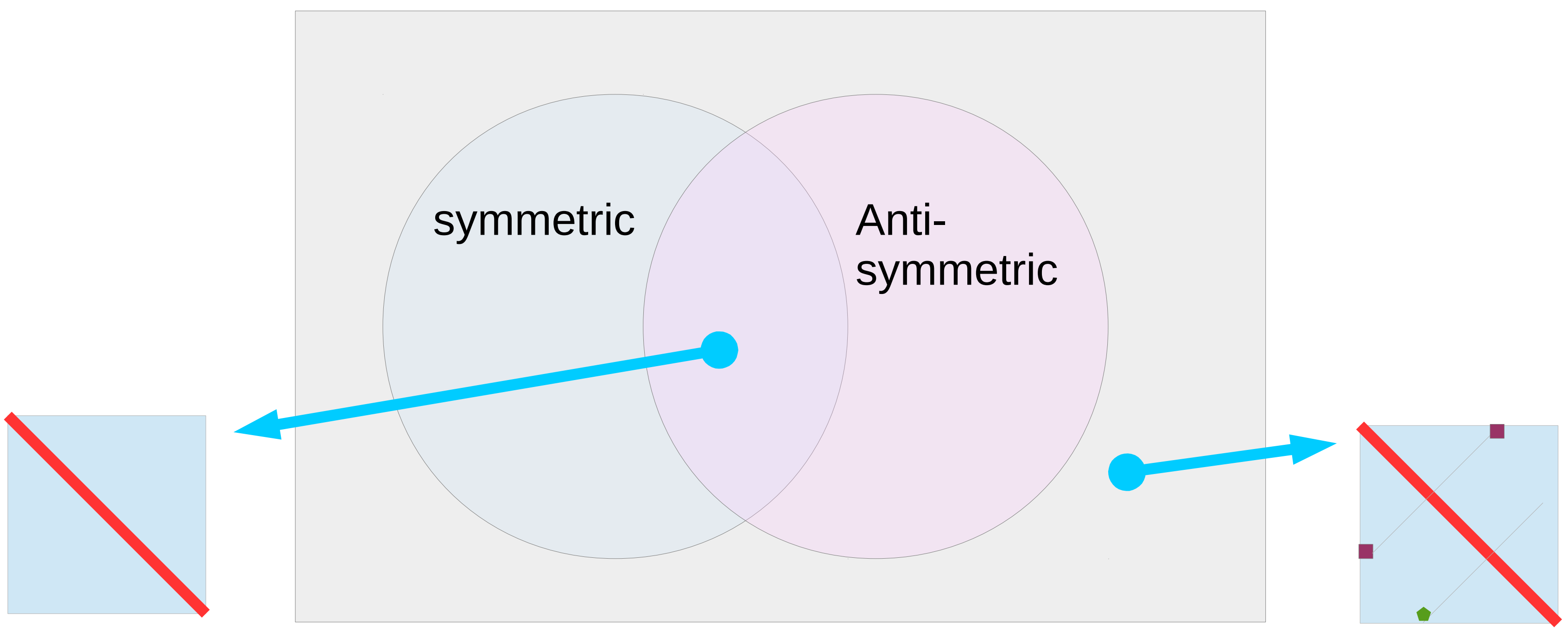


not anti-symmetric



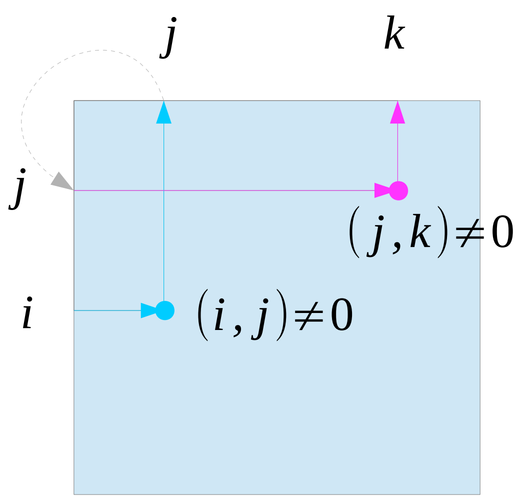
Symmetric vs Anti-Symmetric Relation

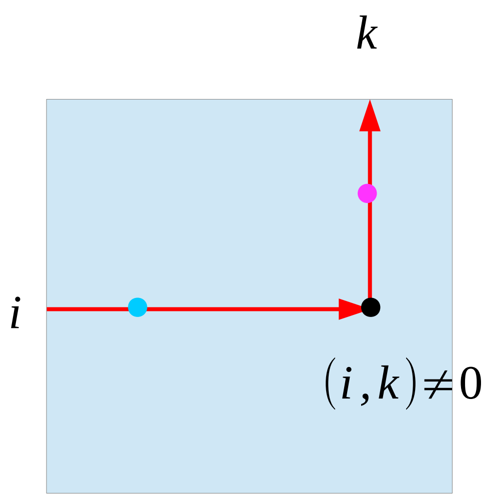
not symmetric	$\neq$	anti-symmetric
not anti-symmetric	$\neq$	symmetric



Transitive Relation

$$\forall x, \forall y, \forall z \ \left[\left((x, y) \in R \ \land \ (y, z) \in R \right) \rightarrow (x, z) \in R \ \right]$$





$(\mathtt{i}\ \mathbf{R}\ \mathtt{j}) \land (\mathtt{j}\ \mathbf{R}\ \mathtt{k})$ $\qquad$ $(\mathtt{i}\ \mathbf{R}\ \mathtt{k})$

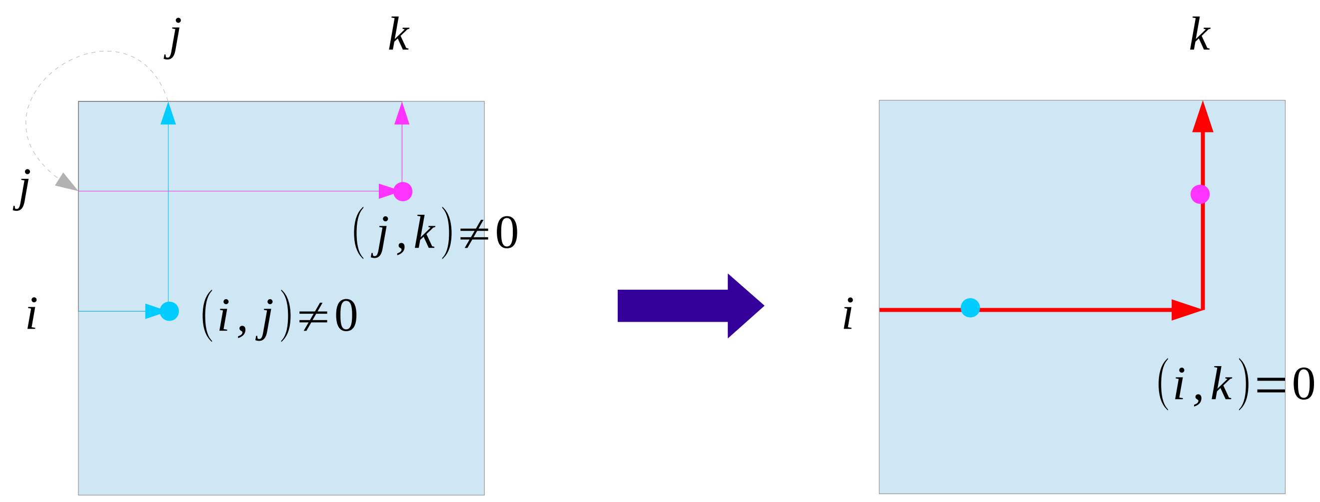
Not Transitive Relation

$$\neg\{\forall x, \forall y, \forall z\ [((x, y) \in R \land (y, z) \in R) \rightarrow (x, z) \in R\ ]\}$$

$$\exists x, \exists y, \exists z\ \neg[((x, y) \in R \land (y, z) \in R) \rightarrow (x, z) \in R\ ]$$

$$\exists x, \exists y, \exists z\ \neg[\neg((x, y) \in R \land (y, z) \in R) \lor ((x, z) \in R)\ ]$$

$$\exists x, \exists y, \exists z\ [(x, y) \in R \land (y, z) \in R \land ((x, z) \notin R)\ ]$$

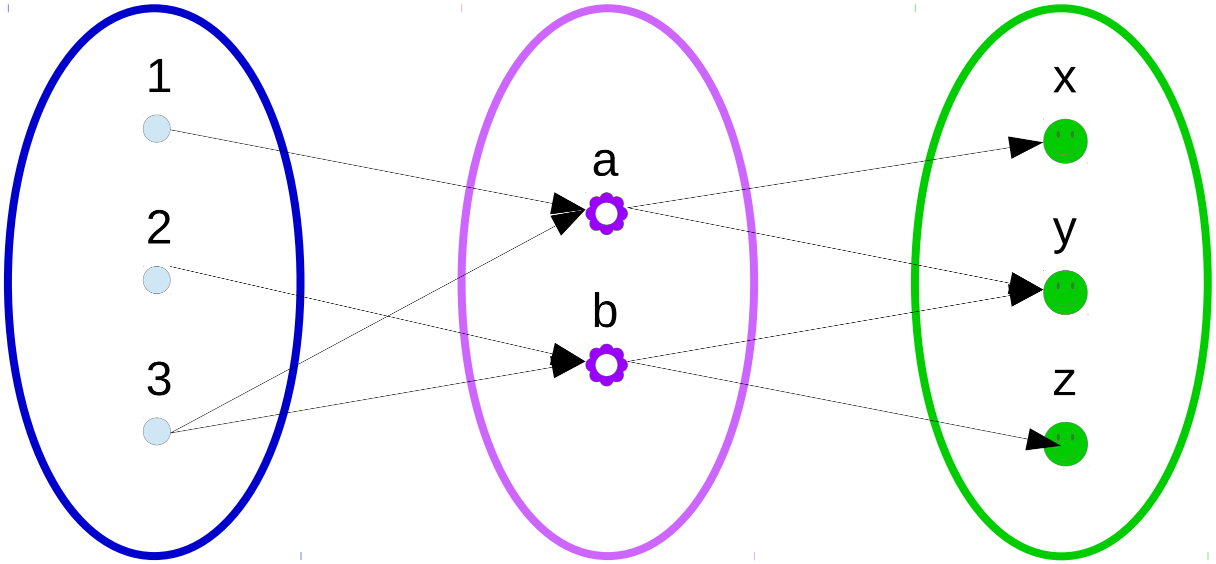


$$(\mathtt{i}\ \mathbf{R}\ \mathtt{j}) \land (\mathtt{j}\ \mathbf{R}\ \mathtt{k}) \qquad (\mathtt{i}\ \cancel{\mathbf{R}}\ \mathtt{k})$$

Relation Examples

$$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$$

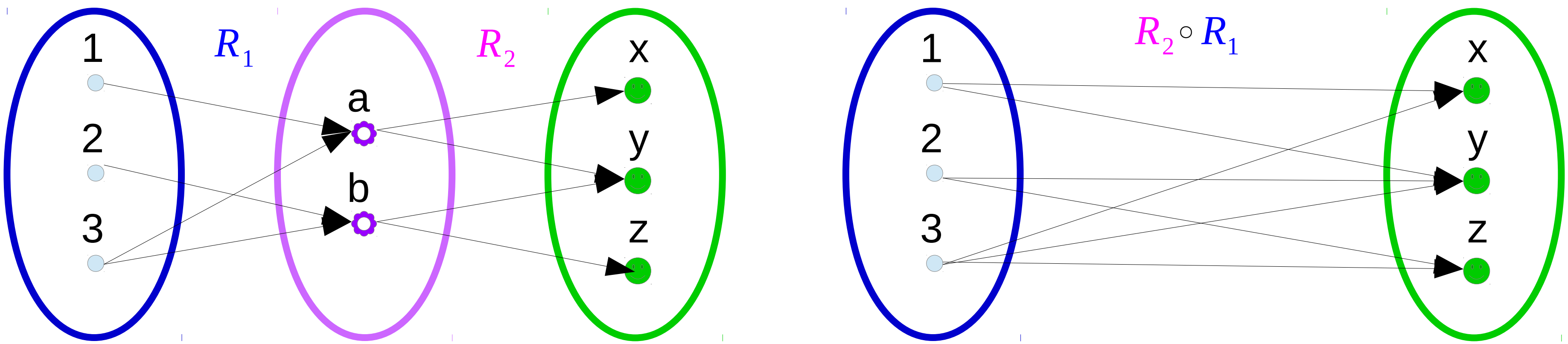
$$R_2 \in \{(a,x),(a,y),(b,y),(b,z)\}$$



Composite Relation Examples

$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$



$R_2 \circ R_1 \in \{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$

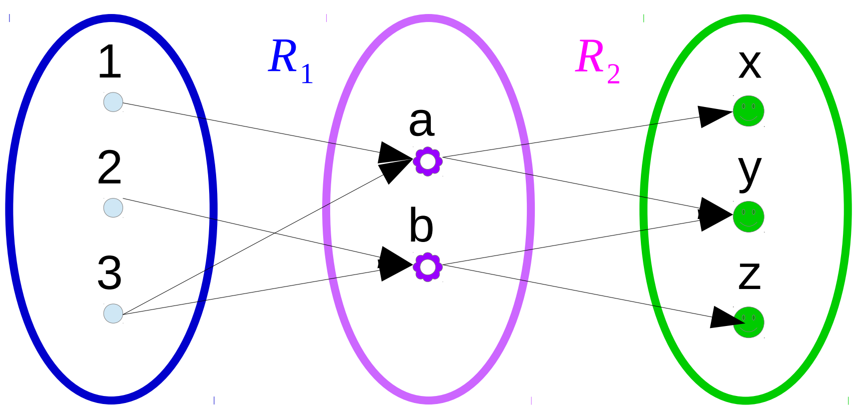
Composite Relation Examples

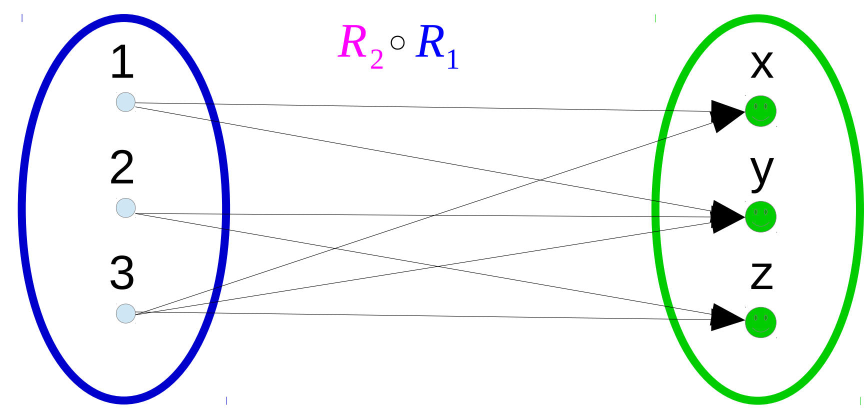
$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$

$R_2 \circ R_1 \in$

$\{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$





$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

Matrix of a Relation

$R_1 \in \{(1,a), (2,b), (3,a), (3,b)\}$

$$A_1 = \begin{array}{c|cc} & a & b \\ \hline 1 & & \\ 2 & \multicolumn{2}{c}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}} \\ 3 & & \end{array}$$

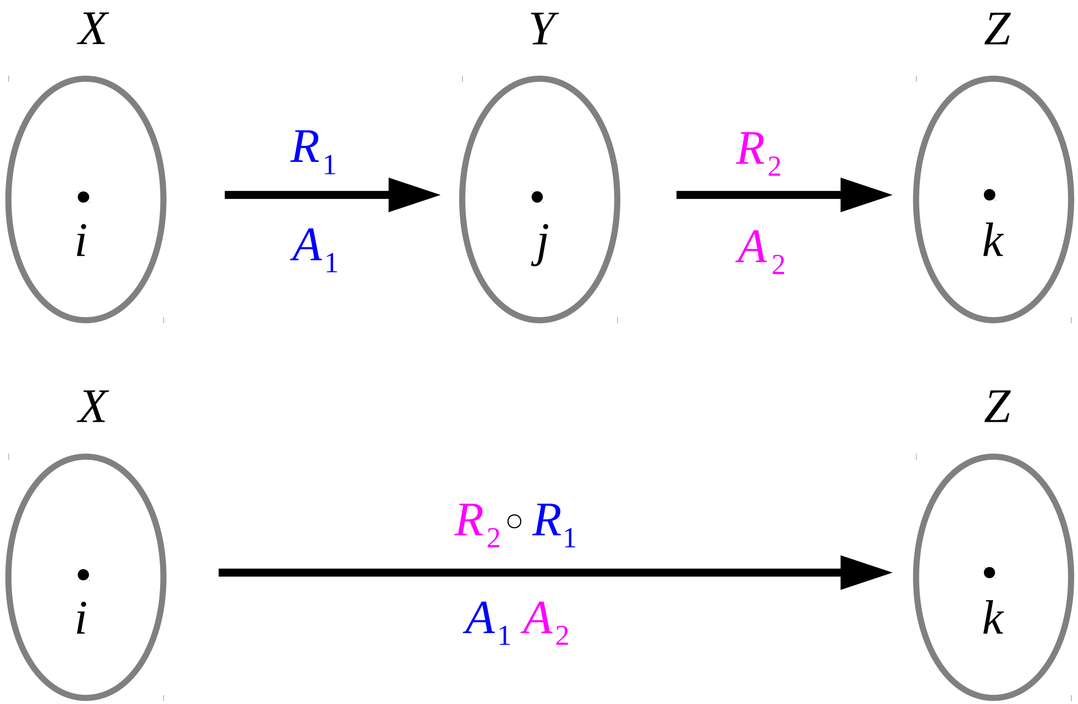
$R_2 \in \{(a,x), (a,y), (b,y), (b,z)\}$

$$A_2 = \begin{array}{cc} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} a \\ b \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$R_2 \circ R_1 \in \{(1,x), (1,y), (2,y), (2,z), (3,x), (3,y), (3,z)\}$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{cc} & \begin{matrix} x & y & z \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

Composite Relation Properties



$$(i, k) \in R_2 \circ R_1 \quad \Longleftrightarrow \quad a_{ik} \neq 0 \;\; of \; A_1 A_2$$

Composite Relation Examples

$R_1 \in \{(1,a),\ (2,b),\ (3,a),\ (3,b)\}$

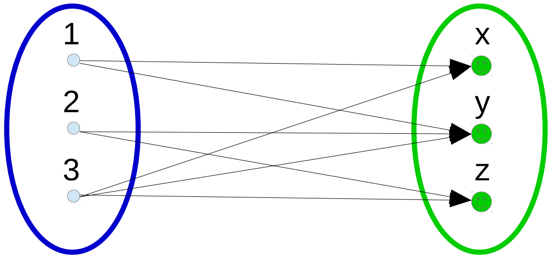
$R_2 \in \{(a,x),\ (a,y),\ (b,y),\ (b,z)\}$

$R_2 \circ R_1 \in \{(1,x),\ (1,y),\ (2,y),\ (2,z),\ (3,x),\ (3,y),\ (3,z)\}$

$$A_1 = \begin{array}{c c} & \begin{array}{cc} a & b \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$A_1 A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

$(i,k) \in R_2 \circ R_1 \iff a_{ik} \neq 0 \quad of\ A_1 A_2$

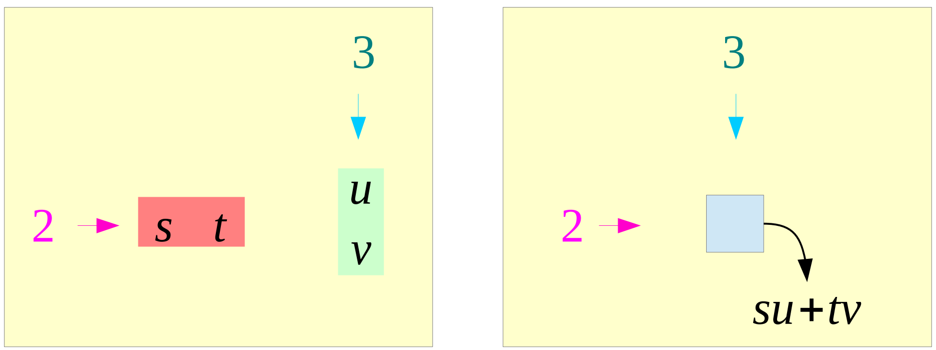


Composite Relation Property Examples

$$A_1 A_2 = \begin{array}{cc} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\end{array} \begin{array}{cc} & \begin{matrix}x&y&z\end{matrix}\\ \begin{matrix}a\\b\end{matrix} & \begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\end{array} = \begin{array}{cc} & \begin{matrix}x&y&z\end{matrix}\\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}1&1&0\\0&1&1\\1&2&1\end{bmatrix}\end{array}$$

$$A_1 = \begin{array}{cc} & \begin{matrix} a & b \end{matrix} \\ \begin{matrix}1\\2\\3\end{matrix} & \begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}\end{array}$$

$$A_2 = \begin{array}{cc} & \begin{matrix}x&y&z\end{matrix}\\ \begin{matrix}a\\b\end{matrix} & \begin{bmatrix}1&1&0\\0&1&1\end{bmatrix}\end{array}$$



$i \in \{1, 2, 3\}$

$k \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$\text{nonzero}\,(i, k)^{th}$ element of $A_1 A_2$ $\Longleftrightarrow$ $(i, k) \in R_2 \circ R_1$

$su+tv \neq 0$

$(su \neq 0) \lor (tv \neq 0)$

$(s=1 \land u=1) \lor (t=1 \land v=1)$

$(2, a) \land (a, y) \lor (2, b) \land (b, y)$ $\quad (2, y)$

Sufficient Part

$$A_1 A_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \qquad A_2 = \begin{array}{c} \\ a \\ b \end{array}\!\!\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

k

$i \rightarrow \begin{bmatrix} s & t \end{bmatrix}$ $\quad \begin{bmatrix} u \\ v \end{bmatrix}$

k

$i \rightarrow \square \rightarrow su+tv$

$i \in \{1, 2, 3\}$
$k \in \{x, y, z\}$

$s \in \{0, 1\}$
$t \in \{0, 1\}$
$u \in \{0, 1\}$
$v \in \{0, 1\}$

$(a_{ik} \neq 0) \quad su+tv \neq 0$

$su = 1$

$(s = 1)$
$(u = 1)$

$tv = 1$

$(t = 1)$
$(v = 1)$

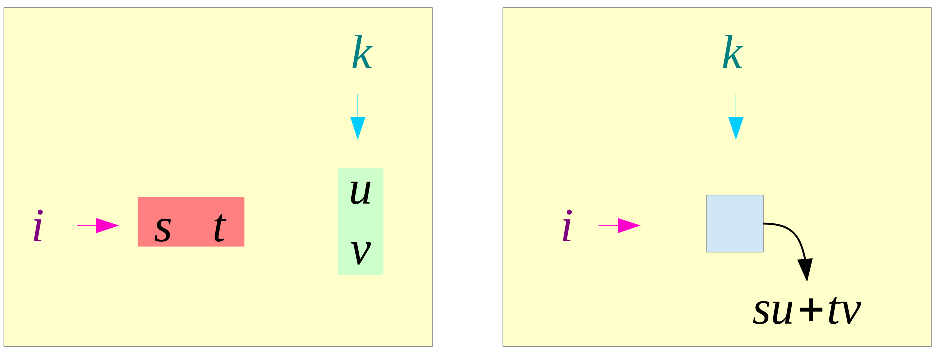
$$\frac{(i, a) \in R_1 \quad (a, k) \in R_2}{(i, k) \in R_2 \circ R_1} \qquad \frac{(i, b) \in R_1 \quad (b, k) \in R_2}{(i, k) \in R_2 \circ R_1}$$

Necessary Part

$$A_1 A_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \end{array}$$

$$A_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

$$A_2 = \begin{array}{c} \\ a \\ b \end{array}\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$



$i \in \{1, 2, 3\}$

$k \in \{x, y, z\}$

$s \in \{0, 1\}$

$t \in \{0, 1\}$

$u \in \{0, 1\}$

$v \in \{0, 1\}$

$(i, k) \in R_2 \circ R_1$

$(i, a) \in R_1$
$(a, k) \in R_2$

$(s = 1)$
$(u = 1)$

$su = 1$

$(i, b) \in R_1$
$(b, k) \in R_2$

$(t = 1)$
$(v = 1)$

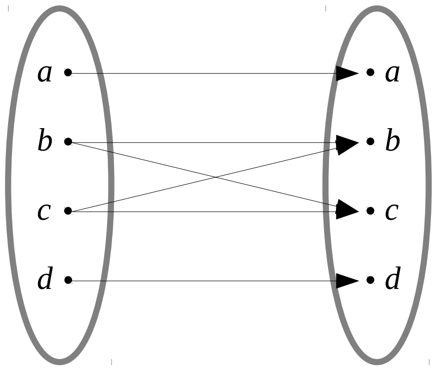
$tv = 1$

$su + tv \neq 0 \quad (a_{ik} \neq 0)$

Young Won Lim
4/14/18

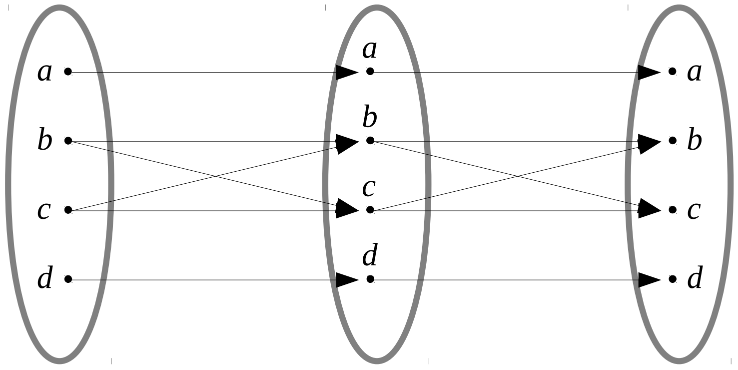
Transitivity Test Examples (1)

$R \in \{(a,a),\ (b,b),\ (c,c),\ (d,d),\ (b,c),\ (c,b)\}$



$$A = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$R \circ R \in \{(a,a),\ (b,b),\ (c,c),\ (d,d),\ (b,c),\ (c,b)\}$



$$A^2 = \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Transitivity Test Examples (2)

$R \in \{(a,a), (b,b), (c,c), (d,d), (b,c), (c,b)\}$

$R \circ R \in \{(a,a), (b,b), (c,c), (d,d), (b,c), (c,b)\}$

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$AA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\Vert ?$

set non-zero element to 1

$$A^2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Transitivity Test Examples (3)

```
(%i8) A1:matrix(
        [1,0,0,0],
        [0,1,1,0],
        [0,1,1,0],
        [0,0,0,1]
      );
```

$$(\%o8) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
(%i9) A2 : A1.A1;
```

$$(\%o9) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
(%i10) A3 : A2.A1;
```

$$(\%o10) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
(%i11) A4 : A3.A1;
```

$$(\%o11) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

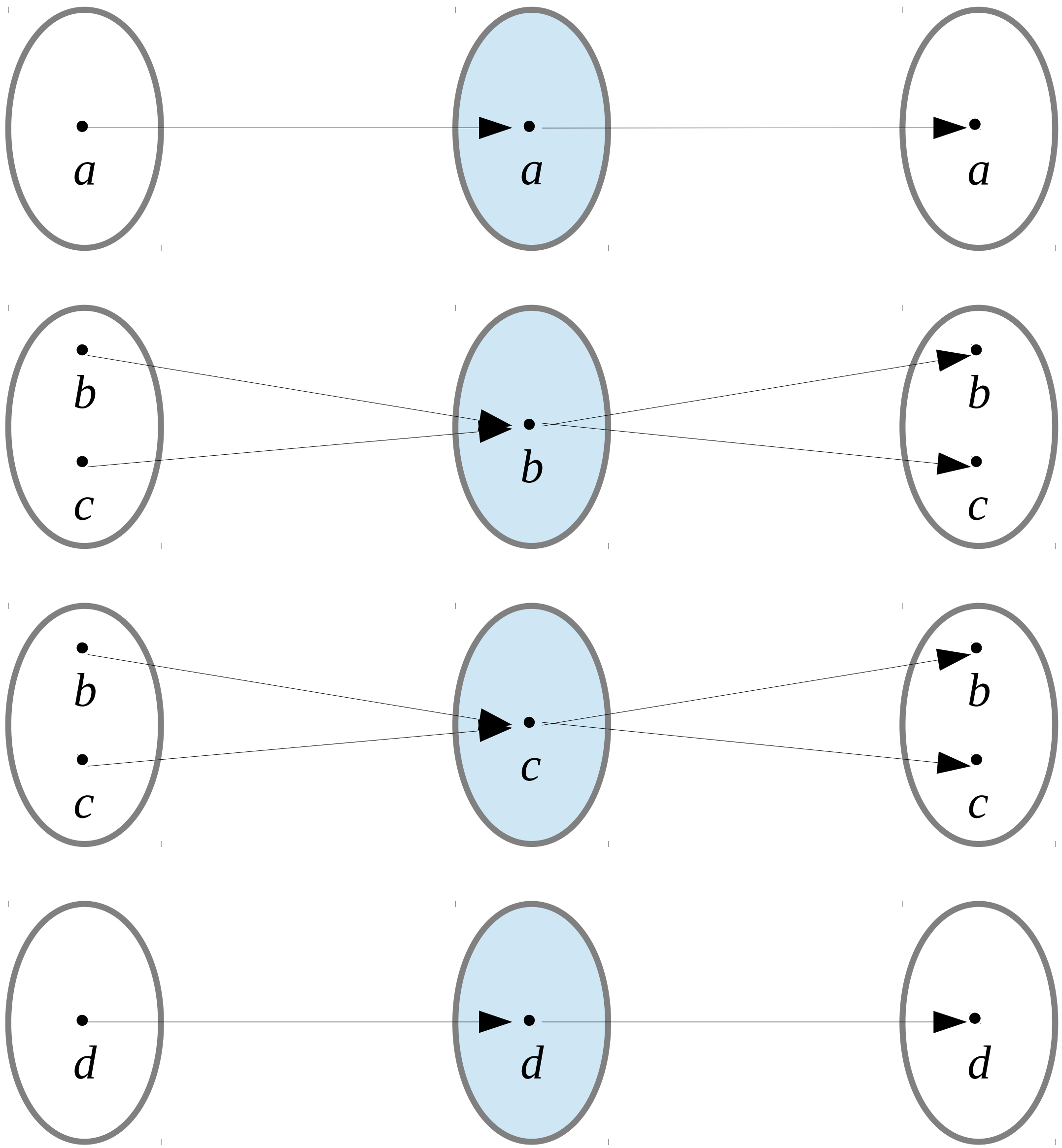
```
(%i12) A5 : A4.A1;
```

$$(\%o12) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 16 & 16 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

```
(%i13) A6 : A5.A1;
```

$$(\%o13) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 32 & 32 & 0 \\ 0 & 32 & 32 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transitivity Test Examples (4)



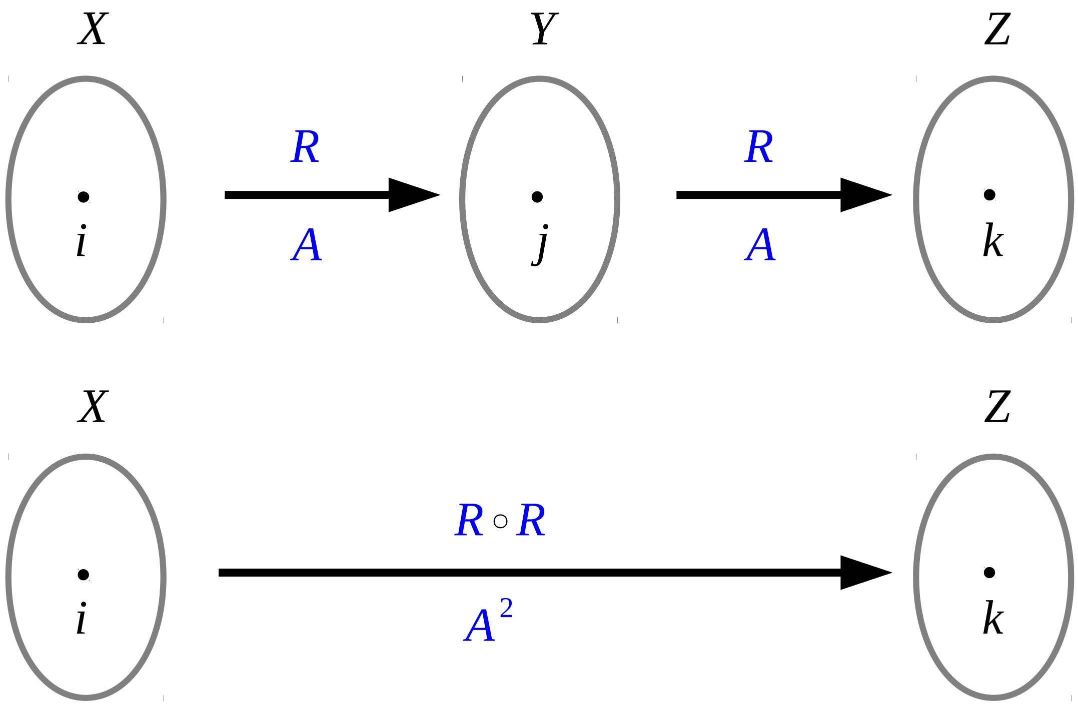
$(a,a) \in R$

$(b,b) \in R$
$(b,c) \in R$
$(c,b) \in R$
$(c,c) \in R$

$(b,b) \in R$
$(b,c) \in R$
$(c,b) \in R$
$(c,c) \in R$

$(d,d) \in R$

Transitivity Test



$$A = A^2 \rightarrow$$
$$A = A^3 \rightarrow$$
$$\cdots$$
$$A = A^n \rightarrow$$
$$\cdots$$

$$A = A^2 \quad \Longleftrightarrow \quad \textit{transitive relation} \;\; R$$

Transitivity Condition

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & a & b & c & d \\ 2 & e & f & \boxed{g} & h \\ 3 & i & j & k & l \\ 4 & m & n & o & p \end{array} = A$$

$g \neq 0$

$e \cdot c + f \cdot g + g \cdot k + h \cdot o \neq 0$

?

$$A^2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & * & * & * & * \\ 2 & e & f & g & h \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{array} \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & * & * & c & * \\ 2 & * & * & g & * \\ 3 & * & * & k & * \\ 4 & * & * & o & * \end{array} = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & * & * & * & * \\ 2 & * & * & \boxed{*} & * \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{array}$$

$nonzero\ (i, j)^{th}\ element\ of\ A^2 \Rightarrow nonzero\ (i, j)^{th}\ element\ of\ A$

A non-zero element of A^2

$$A^2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} * & * & * & * \\ e & f & g & h \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} * & * & c & * \\ * & * & g & * \\ * & * & k & * \\ * & * & o & * \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \end{array} \quad \Rightarrow \quad A$$

$ae = 1$ $\vee$ $bf = 1$ $\vee$ $cg = 1$ $\vee$ $dh = 1$ $\Rightarrow$ $a \cdot e + b \cdot f + c \cdot g + d \cdot h \neq 0$

$(2,1) \in R$, $(1,3) \in R$ | $(2,2) \in R$, $(2,3) \in R$ | $(2,3) \in R$, $(3,3) \in R$ | $(2,4) \in R$, $(4,3) \in R$ $\Rightarrow$ $(2,3) \in R$

$$\forall x, \forall y, \forall z \ \left[\left((x, y) \in R \ \wedge \ (y, z) \in R \right) \rightarrow (x, z) \in R \right]$$

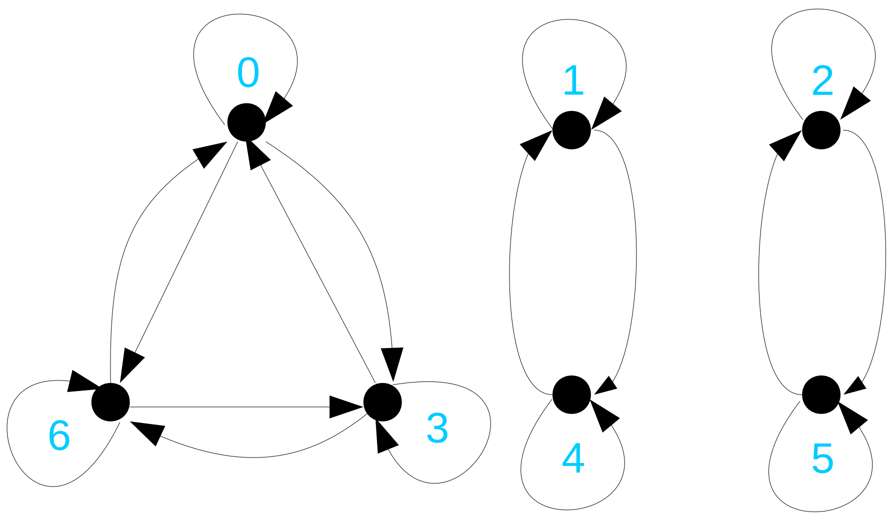
Binary Relations and Digraphs

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
R = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}
\end{array}
$$



http://www.math.fsu.edu/~pkirby/mad2104/SlideShow/s7_1.pdf

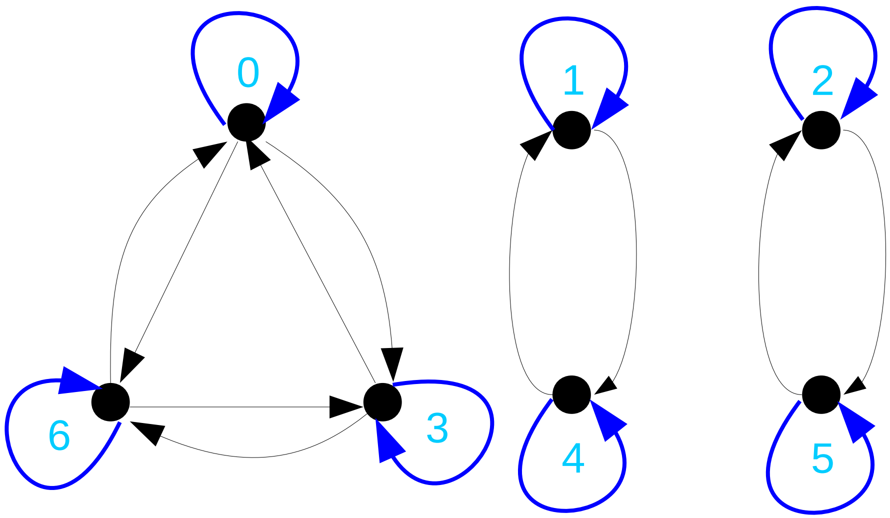
Reflexive Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \; (mod\ 3)\}$$

$$R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



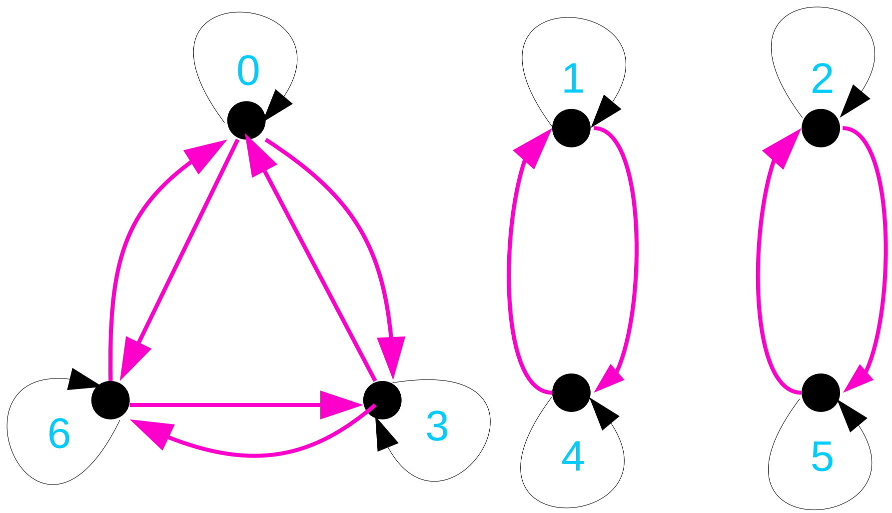
Symmetric Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$$R = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



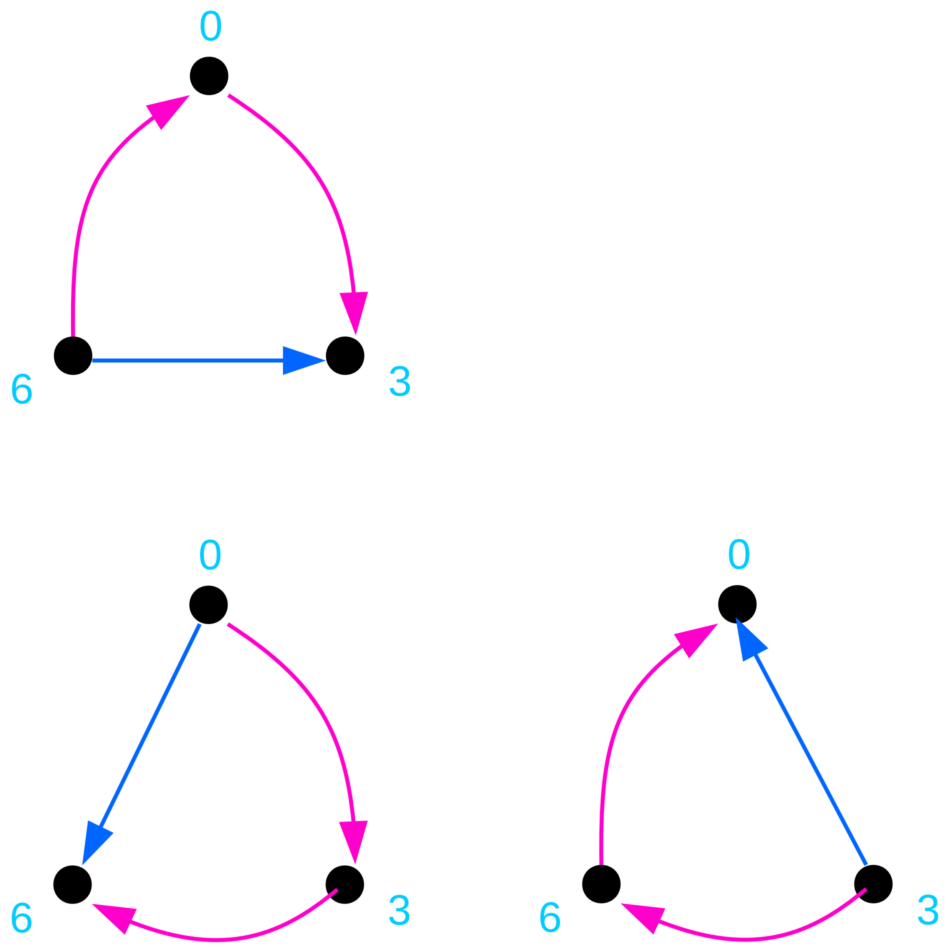
Transitive Relation

$$A = \{0,1,2,3,4,5,6\}$$

$$R \subset A \times A$$

$$R = \{(a,b) \mid a \equiv b \pmod 3\}$$

$$RR = \begin{array}{c|c} & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{bmatrix} \end{array}$$



Transitive Relation

```
(%i2) R:matrix(
       [1,0,0,1,0,0,1],
       [0,1,0,0,1,0,0],
       [0,0,1,0,0,1,0],
       [1,0,0,1,0,0,1],
       [0,1,0,0,1,0,0],
       [0,0,1,0,0,1,0],
       [1,0,0,1,0,0,1]
     );
```

$$(\%o2) \quad \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$R = R^2$ ⇨ transitive

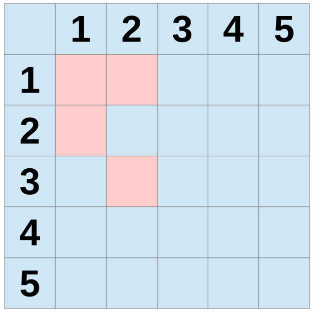
```
(%i4) R2: R.R;
```

$$(\%o4) \quad \begin{bmatrix} 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 3 \end{bmatrix}$$

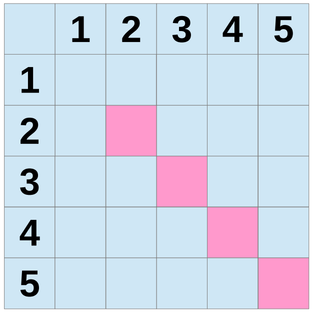
```
(%i7) R3: R.R.R;
```

$$(\%o7) \quad \begin{bmatrix} 9 & 0 & 0 & 9 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 9 & 0 & 0 & 9 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 9 & 0 & 0 & 9 & 0 & 0 & 9 \end{bmatrix}$$

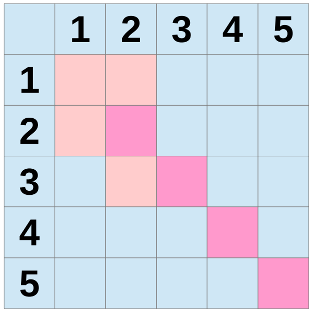
Reflexive and Symmetric Closure



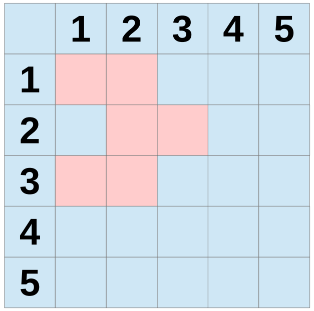
Not Reflexive R



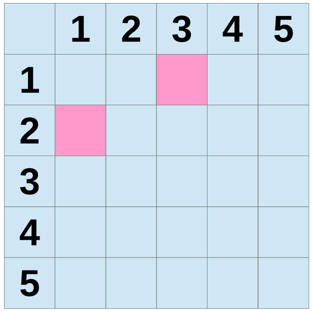
the minimal addition



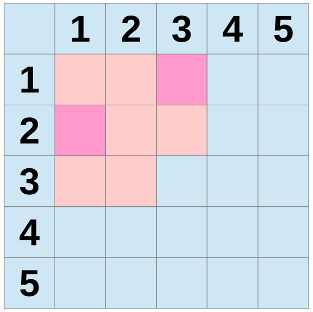
Reflexive Closure of R



Not Symmetric R

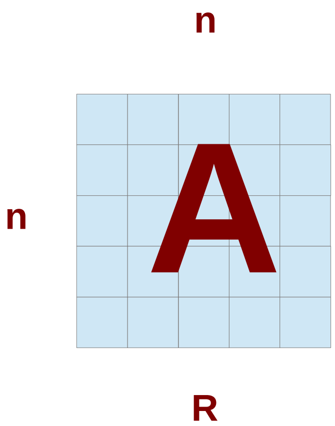


the minimal addition



Symmetric Closure of R

Transitive Closure



$$R^* = \cup_{n=1}^{\infty} R^n$$

$$= R \cup R^2 \cup \cdots \cup R^n$$

$$A \vee A^2 \vee \cdots \vee A^n$$

set non-zero element to 1

Transitive Closure Example 1

```
(%i9) A: matrix(
       [1,0,1],
       [0,1,0],
       [1,1,0]
      );
```

(%o9) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

```
(%i11) A2: A.A;
```

(%o11) $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

```
(%i12) A3: A.A.A;
```

(%o12) $\begin{bmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$

```
(%i13) A+A2+A3;
```

(%o13) $\begin{bmatrix} 6 & 3 & 4 \\ 0 & 3 & 0 \\ 4 & 4 & 2 \end{bmatrix}$

```
(%i18) A4: A.A.A.A;
```

(%o18) $\begin{bmatrix} 5 & 4 & 3 \\ 0 & 1 & 0 \\ 3 & 3 & 2 \end{bmatrix}$

```
(%i20) A5: A.A.A.A.A;
```

(%o20) $\begin{bmatrix} 8 & 7 & 5 \\ 0 & 1 & 0 \\ 5 & 5 & 3 \end{bmatrix}$

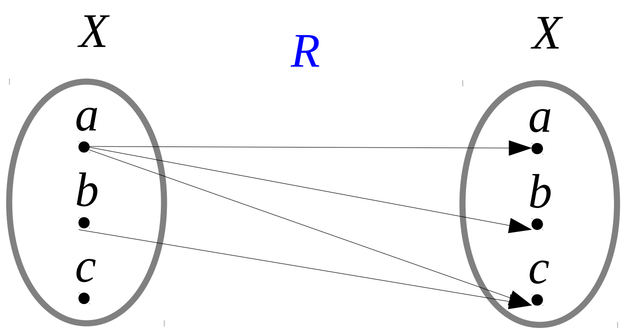
```
(%i19) matrix(
        [1,1,1],
        [0,1,0],
        [1,1,1]
       );
```

(%o19) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ transitive closure of A

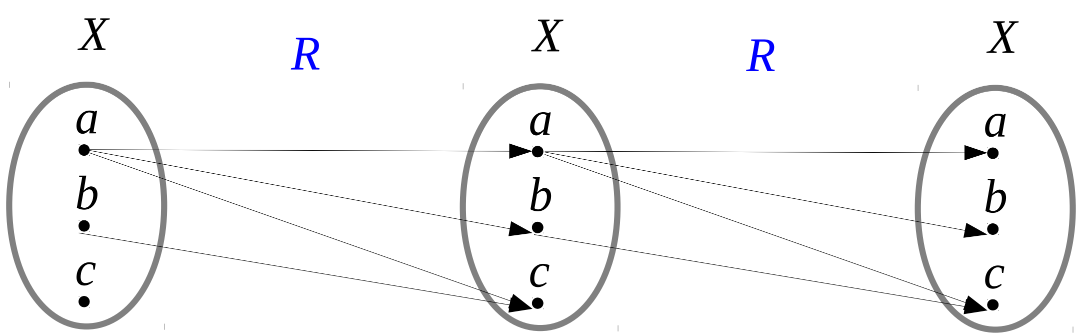
$$A \neq A^2$$

$$A \neq tc(A)$$ ⇨ not transitive

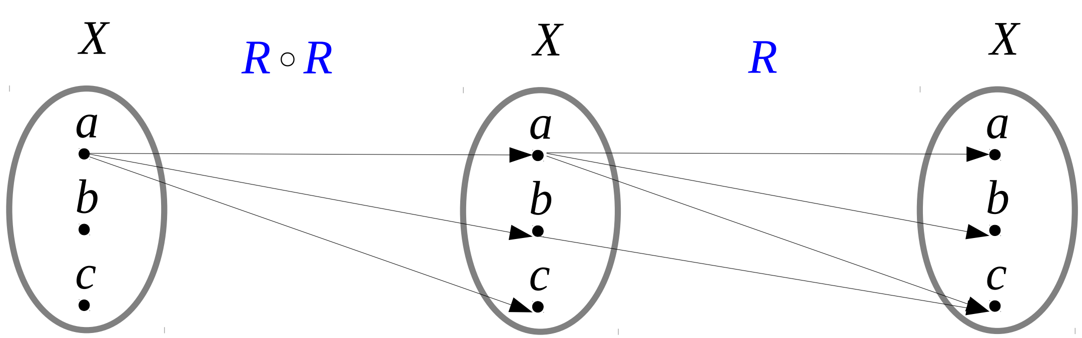
Transitive Closure Example 2-a



$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

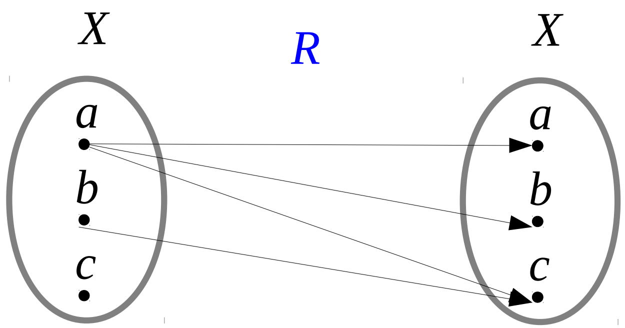


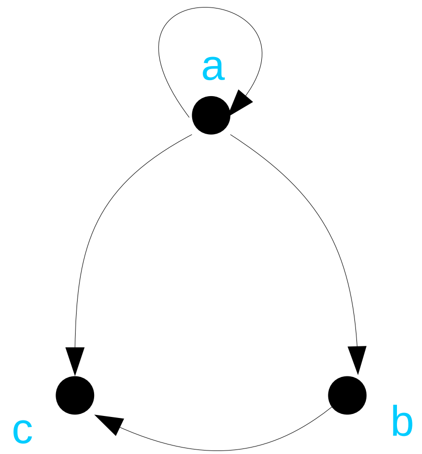
$$A^2 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$



$$A^3 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Transitive Closure Example 2-b





$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^3 = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^* = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$A \neq A^2$

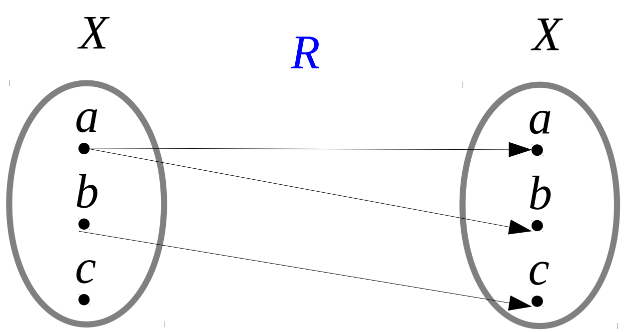
But the <u>transitive</u> <u>closure</u> of A is equal to A

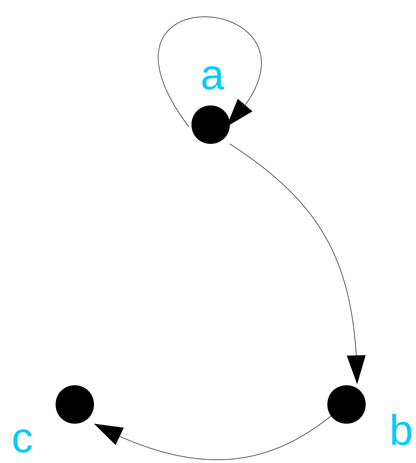
⇨ transitive

$A \neq A^2$

$A = tc(A)$ ⇨ transitive

Transitive Closure Example 3





$$A = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^2 = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^3 = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$A^* = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

$A \neq A^2$

And the transitive closure of A is not equal to A

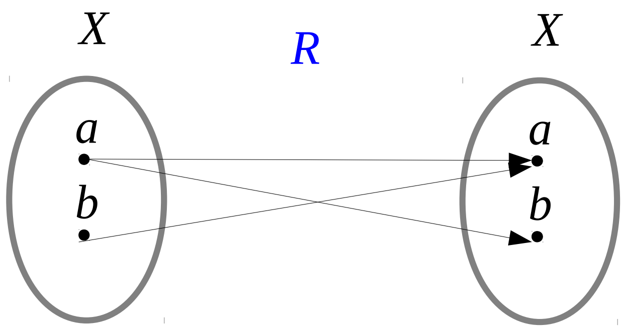
⇨ not transitive

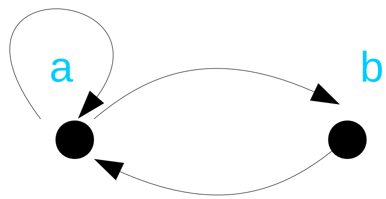
$A \neq A^2$

$A \neq tc(A)$

⇨ not transitive

Transitive Closure Example 4





$$A = \begin{array}{c|cc} & a & b \\ \hline a & 1 & 1 \\ b & 1 & 0 \end{array}$$

$$A^2 = \begin{array}{c|cc} & a & b \\ \hline a & 2 & 1 \\ b & 1 & 1 \end{array}$$

$$A^* = \begin{array}{c|cc} & a & b \\ \hline a & 1 & 1 \\ b & 1 & 1 \end{array}$$

$$A \neq A^2$$

$$A \neq tc(A)$$

not transitive

Transitivity Test

$$\begin{cases} A = A^2 & \Rightarrow \text{Transitive} \\ A \neq A^2 & \end{cases}$$

$$\begin{cases} (A \vee A^2 \vee \cdots \vee A^n) = A & \Rightarrow \text{Transitive} \\ (A \vee A^2 \vee \cdots \vee A^n) \neq A & \Rightarrow \text{Not Transitive} \end{cases}$$

References

[1] http://en.wikipedia.org/
[2]