

# DLTI Difference Equation

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# Causal LTI Systems (1)

$$a_N y[n-N] + \cdots + a_1 y[n-1] + a_0 y[n] = b_M x[n-M] + \cdots + b_1 x[n-1] + b_0 x[n]$$

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] \quad M = N \\ = & b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

$n \rightarrow n - N$

$$\boxed{\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}}$$

$$\boxed{\begin{aligned} & y[n] + a_1 y[n-1] + \cdots + a_{N-1} y[n-N+1] + a_N y[n-N] \\ = & b_0 x[n] + b_1 x[n-1] + \cdots + b_{N-1} x[n-N+1] + b_N x[n-N] \end{aligned}}$$

$$(E^N + a_1 E^{N-1} + \cdots + a_{N-1} E + a_N) y[n] = (b_0 E^M + b_{N-M+1} E^{M-1} + \cdots + b_{N-1} E + b_N) x[n]$$

$$Q(E) y(t) = P(E) x(t)$$

# Causal LTI Systems (2)

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_{N-M} x[n+M] + b_{N-M+1} x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

Causal System: output cannot depend on future input values

Causality  $\iff M \leq N$

If  $M > N$        $y[n+N]$  would depend on  $x[n+M]$

  
later instance

If  $M = N$

$$\begin{aligned} & y[n+N] + a_1 y[n+N-1] + \cdots + a_{N-1} y[n+1] + a_N y[n] \\ = & b_0 x[n+M] + b_1 x[n+M-1] + \cdots + b_{N-1} x[n+1] + b_N x[n] \end{aligned}$$

# Causal LTI Systems (3)

$$\begin{aligned} & y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ = & b_{N-M}x[n+M] + b_{N-M+1}x[n+M-1] + \cdots + b_{N-1}x[n+1] + b_Nx[n] \end{aligned}$$

If  $M = N$

$$\begin{aligned} & y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ = & b_0x[n+M] + b_1x[n+M-1] + \cdots + b_{N-1}x[n+1] + b_Nx[n] \end{aligned}$$

advance operator from  
advance operator from

$n \rightarrow n - N$

$$\begin{aligned} & y[n] + \color{red}{a_1}y[n-1] + \cdots + \color{red}{a_{N-1}}y[n-N+1] + \color{red}{a_N}y[n-N] \\ = & b_0x[n] + \color{green}{b_1}x[n-1] + \cdots + \color{green}{b_{N-1}}x[n-N+1] + \color{green}{b_N}x[n-N] \end{aligned}$$

The advance operator  $E x[n] = x[n+1]$        $E^k x[n] = x[n+k]$

$$\begin{aligned} (\color{blue}{E}^N + \color{red}{a_1}\color{blue}{E}^{N-1} + \cdots + \color{red}{a_{N-1}}\color{blue}{E} + \color{red}{a_N})y[n] &= (\color{green}{b_0}\color{blue}{E}^M + \color{green}{b_{N-M+1}}\color{blue}{E}^{M-1} + \cdots + \color{green}{b_{N-1}}\color{blue}{E} + \color{green}{b_N})x[n] \\ Q(\color{blue}{E})y[n] &= P(\color{blue}{E})x[n] \end{aligned}$$

# Zero Input Response $y_0(t)$ (1)

$$\begin{aligned} & y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ &= \color{blue}{b_0}x[n+M] + \color{blue}{b_1}x[n+M-1] + \cdots + \color{blue}{b_{N-1}}x[n+1] + \color{blue}{b_N}x[n] \end{aligned}$$

$$(\color{blue}{E^N} + \color{red}{a_1}\color{blue}{E^{N-1}} + \cdots + \color{red}{a_{N-1}}\color{blue}{E} + \color{red}{a_N})y[n] = (\color{blue}{b_0}\color{blue}{E^M} + \color{blue}{b_{N-M+1}}\color{blue}{E^{M-1}} + \cdots + \color{blue}{b_{N-1}}\color{blue}{E} + \color{blue}{b_N})x[n]$$

$$Q(\color{blue}{E})y[n] = P(\color{blue}{E})x[n]$$

$$y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] = 0$$

$$(\color{blue}{E^N} + \color{red}{a_1}\color{blue}{E^{N-1}} + \cdots + \color{red}{a_{N-1}}\color{blue}{E} + \color{red}{a_N}) \cdot y_0[n] = 0$$

linear combination of  $y_0[n]$  and advanced  $y_0[n]$  is zero for all  $n$

iff  $y_0[n]$  and advanced  $y_0[n]$  have the same form

only exponential function  $\gamma^n$

$$E^k\{\gamma^n\} = \gamma^{n+k} = \gamma^k \gamma^n$$

$$y_0[n] = c \gamma^n$$

$$E^k\{y_0[n]\} = y_0[n+k] = c \gamma^{n+k}$$

$$c(\color{blue}{\gamma^N} + \color{red}{a_1}\color{blue}{\gamma^{N-1}} + \cdots + \color{red}{a_{N-1}}\color{blue}{\gamma} + \color{red}{a_N}) \cdot y_0[n] = 0$$

$$(\color{blue}{\gamma^N} + \color{red}{a_1}\color{blue}{\gamma^{N-1}} + \cdots + \color{red}{a_{N-1}}\color{blue}{\gamma} + \color{red}{a_N}) = 0 \quad \leftrightarrow \quad Q(\gamma) = 0$$

# Zero Input Response $y_0(t)$ (2)

$$\begin{aligned} & y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ &= \color{green}{b_0}x[n+M] + \color{green}{b_1}x[n+M-1] + \cdots + \color{green}{b_{N-1}}x[n+1] + \color{green}{b_N}x[n] \end{aligned}$$

$$(\color{blue}{E}^N + \color{red}{a_1}\color{blue}{E}^{N-1} + \cdots + \color{red}{a_{N-1}}\color{blue}{E} + \color{red}{a_N})y[n] = (\color{green}{b_0}\color{blue}{E}^M + \color{green}{b_{N-M+1}}\color{blue}{E}^{M-1} + \cdots + \color{green}{b_{N-1}}\color{blue}{E} + \color{green}{b_N})x[n]$$

$$Q(\color{blue}{E})y[n] = P(\color{blue}{E})x[n]$$

$$y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] = 0$$

$$(\color{blue}{E}^N + \color{red}{a_1}\color{blue}{E}^{N-1} + \cdots + \color{red}{a_{N-1}}\color{blue}{E} + \color{red}{a_N}) \cdot y_0[n] = 0$$

$$c(\color{blue}{\gamma}^N + \color{red}{a_1}\color{blue}{\gamma}^{N-1} + \cdots + \color{red}{a_{N-1}}\color{blue}{\gamma} + \color{red}{a_N}) \cdot y_0[n] = 0$$

$$(\color{blue}{\gamma}^N + \color{red}{a_1}\color{blue}{\gamma}^{N-1} + \cdots + \color{red}{a_{N-1}}\color{blue}{\gamma} + \color{red}{a_N}) = 0 \quad \leftrightarrow \quad Q(\color{blue}{\gamma}) = 0$$

$$(\color{blue}{\gamma} - \color{blue}{\gamma}_1)(\color{blue}{\gamma} - \color{blue}{\gamma}_2) \cdots (\color{blue}{\gamma} - \color{blue}{\gamma}_N) = 0$$

$$y_0[n] = c_1\color{blue}{\gamma}_1^n + c_2\color{blue}{\gamma}_2^n + \cdots + c_N\color{blue}{\gamma}_N^n \quad \color{blue}{\gamma}_i \quad \text{characteristic roots}$$

$$\color{blue}{\gamma}_i^n \quad \text{characteristic modes}$$

ZIR: a linear combination of the characteristic modes of the system

# Closed Form $h[n]$ (1)

$$\begin{aligned} & y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ &= \color{blue}{b_0}x[n+M] + \color{blue}{b_1}x[n+M-1] + \cdots + \color{blue}{b_{N-1}}x[n+1] + \color{blue}{b_N}x[n] \end{aligned}$$

$$(E^N + \color{red}{a_1}E^{N-1} + \cdots + \color{red}{a_{N-1}}E + \color{red}{a_N})y[n] = (\color{blue}{b_0}E^M + \color{blue}{b_{N-M+1}}E^{M-1} + \cdots + \color{blue}{b_{N-1}}E + \color{blue}{b_N})x[n]$$

$$Q(E)y[n] = P(E)x[n]$$

**$h[n]$  : system response to input  $\delta[n]$**

$$Q(E)h[n] = P(E)\delta[n] \quad \text{with initial condition}$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0$$

When  $n < 0$ ,  $h[n] = 0$

When  $n > 0$ ,  $h[n]$  must be made up of **characteristic modes**

When the input is zero, only the characteristic modes can be sustained

When  $n = 0$ , it may have non-zero value  $A_0$

$$h[n] = A_0\delta[n] + \color{yellow}{y_c[n]}u[n]$$

 linear combination of the characteristic modes

# Closed Form $h[n]$ (2)

$$y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ = \color{green}{b_0}x[n+M] + \color{green}{b_1}x[n+M-1] + \cdots + \color{green}{b_{N-1}}x[n+1] + \color{green}{b_N}x[n]$$

$$(E^N + \color{red}{a_1}E^{N-1} + \cdots + \color{red}{a_{N-1}}E + \color{red}{a_N})y[n] = (\color{green}{b_0}E^M + \color{green}{b_{N-M+1}}E^{M-1} + \cdots + \color{green}{b_{N-1}}E + \color{green}{b_N})x[n]$$

$$Q(E)y[n] = P(E)x[n]$$

$$Q(E)y(t) = P(E)x(t) \quad \rightarrow$$

$$h[n] = \underline{A_0\delta[n]} + y_c[n]u[n]$$



$$Q(E)(\underline{A_0\delta[n]} + y_c[n]u[n]) = P(E)\delta(t)$$

*y<sub>c</sub> is made up of characteristic modes*

$$Q(E)h(t) = P(E)\delta(t) \quad \text{causal } h[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0 \quad \text{initial condition}$$

$$\begin{cases} Q(E)(y_c[n]u[n]) = 0 \\ Q(E)(A_0\delta[n]) = P(E)\delta(t) \end{cases}$$

$$A_0(\delta[n+N] + \color{red}{a_1}\delta[n+N-1] + \cdots + \color{red}{a_{N-1}}\delta[n+1] + \color{yellow}{a_N}\delta[n]) \\ = \color{green}{b_0}\delta[n+M] + \color{green}{b_1}\delta[n+M-1] + \cdots + \color{green}{b_{N-1}}\delta[n+1] + \color{yellow}{b_N}\delta[n]$$

$$\mathbf{n=0} \quad A_0 \quad a_N = b_N \quad A_0 = \frac{a_N}{b_N}$$

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

# Closed Form $h[n]$ (3)

$$y[n+N] + \color{red}{a_1}y[n+N-1] + \cdots + \color{red}{a_{N-1}}y[n+1] + \color{red}{a_N}y[n] \\ = \color{green}{b_0}x[n+M] + \color{green}{b_1}x[n+M-1] + \cdots + \color{green}{b_{N-1}}x[n+1] + \color{green}{b_N}x[n]$$

$$(E^N + \color{red}{a_1}E^{N-1} + \cdots + \color{red}{a_{N-1}}E + \color{red}{a_N})y[n] = (\color{green}{b_0}E^M + \color{green}{b_{N-M+1}}E^{M-1} + \cdots + \color{green}{b_{N-1}}E + \color{green}{b_N})x[n]$$

$$Q(E)y[n] = P(E)x[n]$$

$$Q(E)y(t) = P(E)x(t) \quad \rightarrow$$

$$h[n] = \underline{A_0\delta[n]} + y_c[n]u[n]$$



$$Q(E)h(t) = P(E)\delta(t) \quad \text{causal } h[n]$$

$$h[-1] = h[-2] = \cdots = h[-N] = 0 \quad \text{initial condition}$$

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n]$$

*N unknown coefficients in  $y_c[n]$   
– determined from N values of  $h[n]$*   
 $h[0], h[1], \dots, h[N-1]$

# Example (1) - ZIR

$$y[n+2] - 0.6y[n+1] - 0.16y[n] = 5x[n+2]$$

$$(E^2 - 0.6E - 0.16) y(t) = 5E^2 x[n]$$

initial condition  $y[-1] = 0, y[-2] = \frac{25}{4}$   
input  $x(t) = 4^{-n} u[n]$

Characteristic polynomial

$$\gamma - 0.6\gamma - 0.16 = (\gamma + 0.2)(\gamma - 0.8)$$

Characteristic Equation  $(\gamma + 0.2)(\gamma - 0.8) = 0$

Characteristic Roots  $\gamma = -0.2, \gamma = 0.8$

Zero Input Response  $y_0[n]$

$$y_0[n] = c_1(-0.2)^n + c_2(0.8)^n$$



$$y_0[n] = \frac{1}{5}(-0.2)^n + \frac{4}{5}(0.8)^n$$

$$y_0[-1] = -5c_1 + \frac{5}{4}c_2 = 0 \quad c_1 = \frac{1}{5}$$

$$y_0[-2] = 25c_1 + \frac{25}{16}c_2 = \frac{25}{4} \quad c_2 = \frac{4}{5}$$

## Example (2) - ZSR

$$y[n] - 0.6y[n-1] - 0.16y[n-2] = 5x[n]$$

$$h[n] - 0.6h[n-1] - 0.16h[n-2] = 5\delta[n]$$

$$h[-1] = 0, \quad h[-2] = 0$$

Calculate iteratively

$$h[0] = 5, \quad h[1] = 3, \quad \dots$$

$$(E^2 - 0.6E - 0.16) y(t) = 5E^2 x[n]$$

$$y_c[n] = c_1(-0.2)^n + c_2(0.8)^n$$

$$h[n] = 0 \cdot \delta[n] + [c_1(-0.2)^n + c_2(0.8)^n] u[n]$$

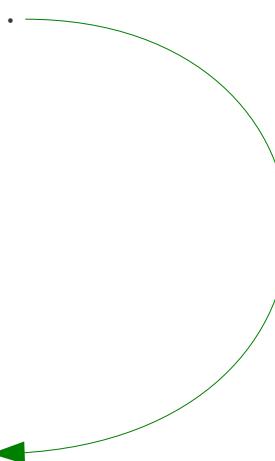
$$h[0] = 5 \quad h[0] = c_1 + c_2 \quad c_1 = 1$$

$$h[1] = 1 \quad h[1] = -0.2c_1 + 0.8c_2 \quad c_2 = 4$$

$$h[n] = [(-0.2)^n + 4(0.8)^n] u[n] \quad \text{Closed form } h[n]$$

$$y[n] = \sum_{m=0}^n x[m] h[n-m]$$

Zero State Response



## References

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- [4] [ocw.usu.edu, Electrical and Computer Engineering, Signals and Systems](http://ocw.usu.edu/Electrical%20and%20Computer%20Engineering/Signals%20and%20Systems)