DLTI Convolution (1A)

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using LibreOffice.

Based on

Introduction to Signal Processing

S. J. Ofranidis

The necessities in DSP C Programming

Filtering C codes (A.pdf) 20190307

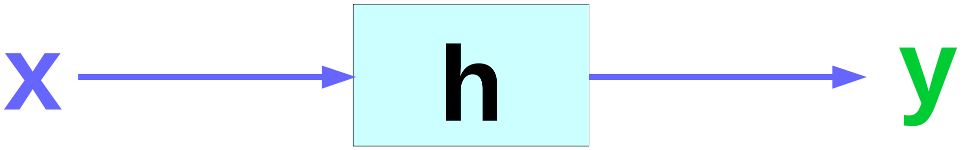
conv

```
#include <stdlib.h>       // to use max, min
/* conv.c - convolution of x[n] with h[n], resulting in y[n] */
/* h : filter array, M : filter order */
/* x : input array, L : input length */
/* y : output array  with length of L+M */

void conv(int M, double *h, int L, double *x, double *y)
{
        int n, m;

        for (n = 0; n< L+M-1; n++)
                for (y[n] = 0, m = max(0, n-L+1); m <= min(n, M-1); m++)
                        y[n] += h[m] * x[n-m];
}
```

Index Variable Constraints



x[0..L-1]
input array
L input length

h[0..M-1]
filter array,
M (filter length)
M-1 (filter order)

y[0..L+M-2]
output array
(M+L-1) output length

Assume
M < L

Case A

$$y\overset{M+L-1}{[n]} \mathrel{+}= h\overset{M}{[m]} * x\overset{L}{[n-m]};$$

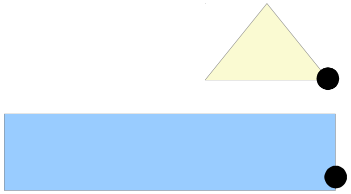
n $\in$ **[0, L+M-2]**
m $\in$ **[0, M-1]**
n-m $\in$ **[0, L-1]**

Case B

$$y\overset{M+L-1}{[n]} \mathrel{+}= x\overset{L}{[m]} * h\overset{M}{[n-m]};$$

n $\in$ **[0, L+M-2]**
m $\in$ **[0, L-1]**
n-m $\in$ **[0, M-1]**

- Flipped and shifted functions
  - Case **A**: **h[n-m]**
  - Case **B**: **x[n-m]**
- Range partitions for **n**
- Effective index ranges for **n**, **m**, **n-m**



Flipped and shifted waveforms

Case A

$h[m]$ the original waveform

$h[-m]$ The flipped against y-axis

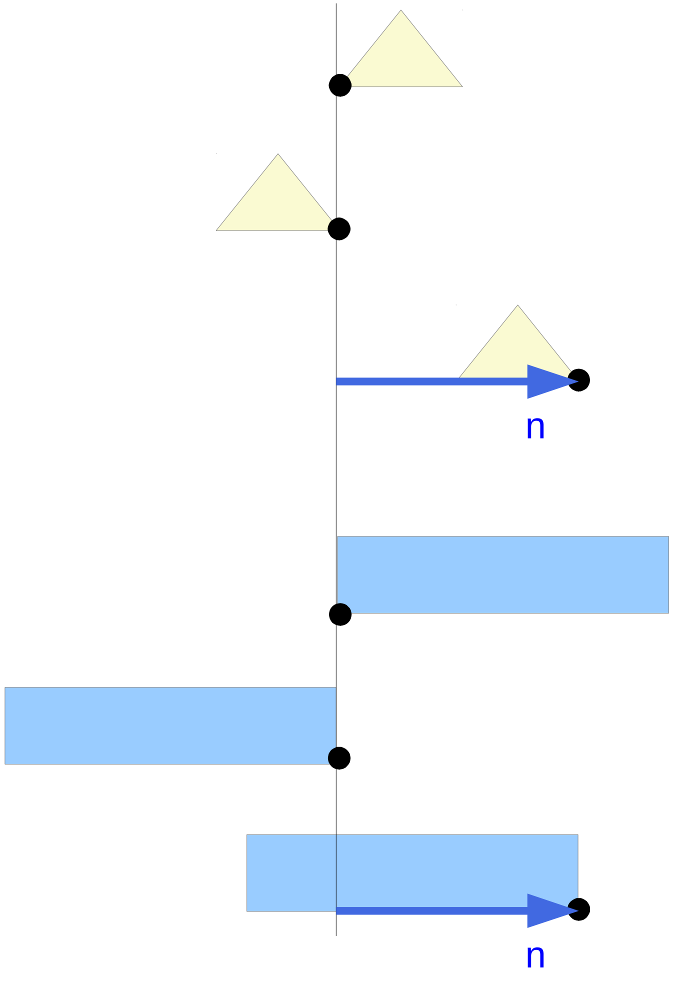
$h[-(m-n)] = h[n-m]$ shift to the right by n

Case B

$x[m]$ the original waveform

$x[-m]$ The flipped against y-axis

$x[-(m-n)] = x[n-m]$ shift to the right by n



Flipped and shifted function of **h[n-m]**

Case A

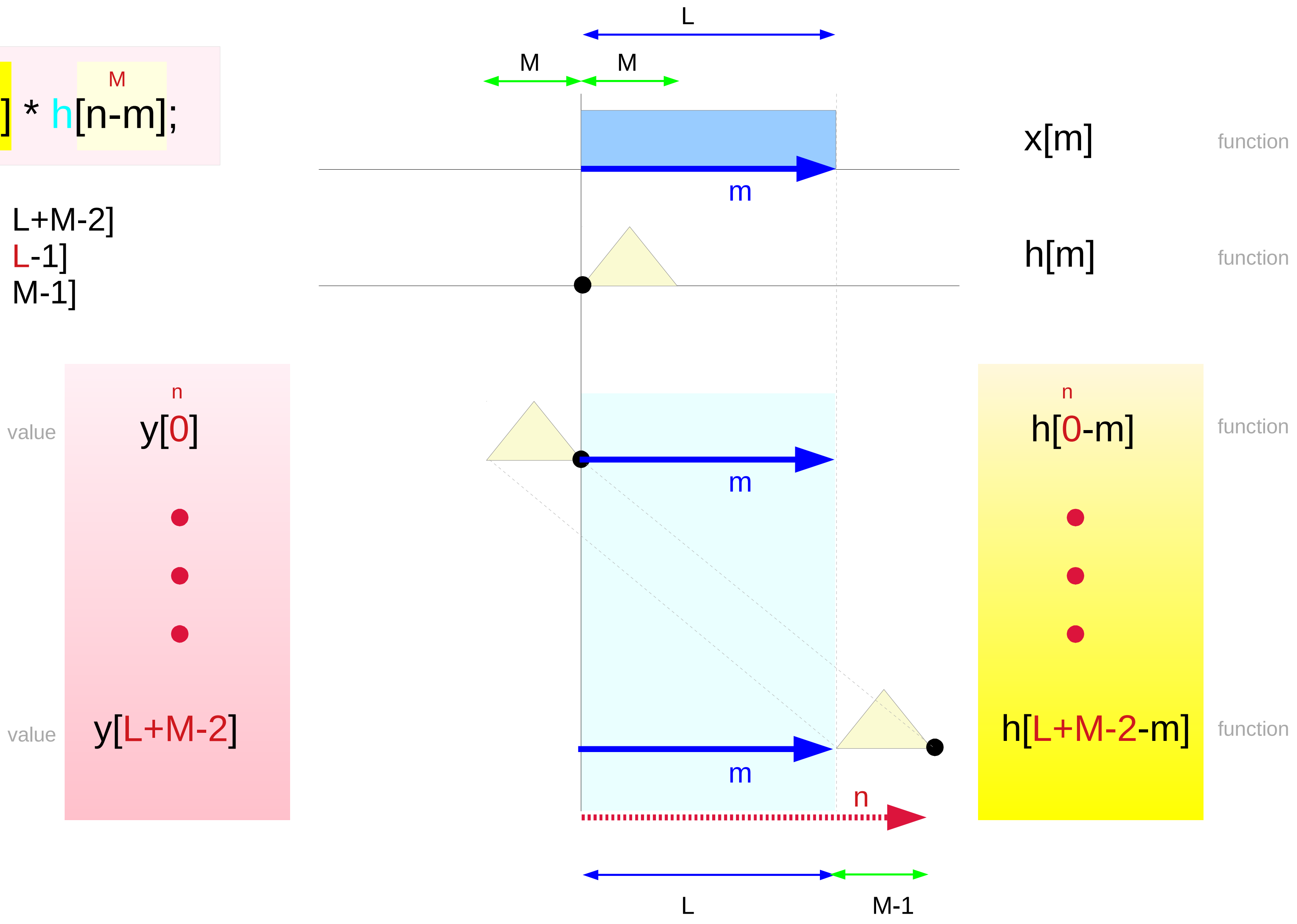
M+L-1 L M

`y[n] += x[m] * h[n-m];`

$n \in [0, L+M-2]$

$m \in [0, L-1]$

$n-m \in [0, M-1]$



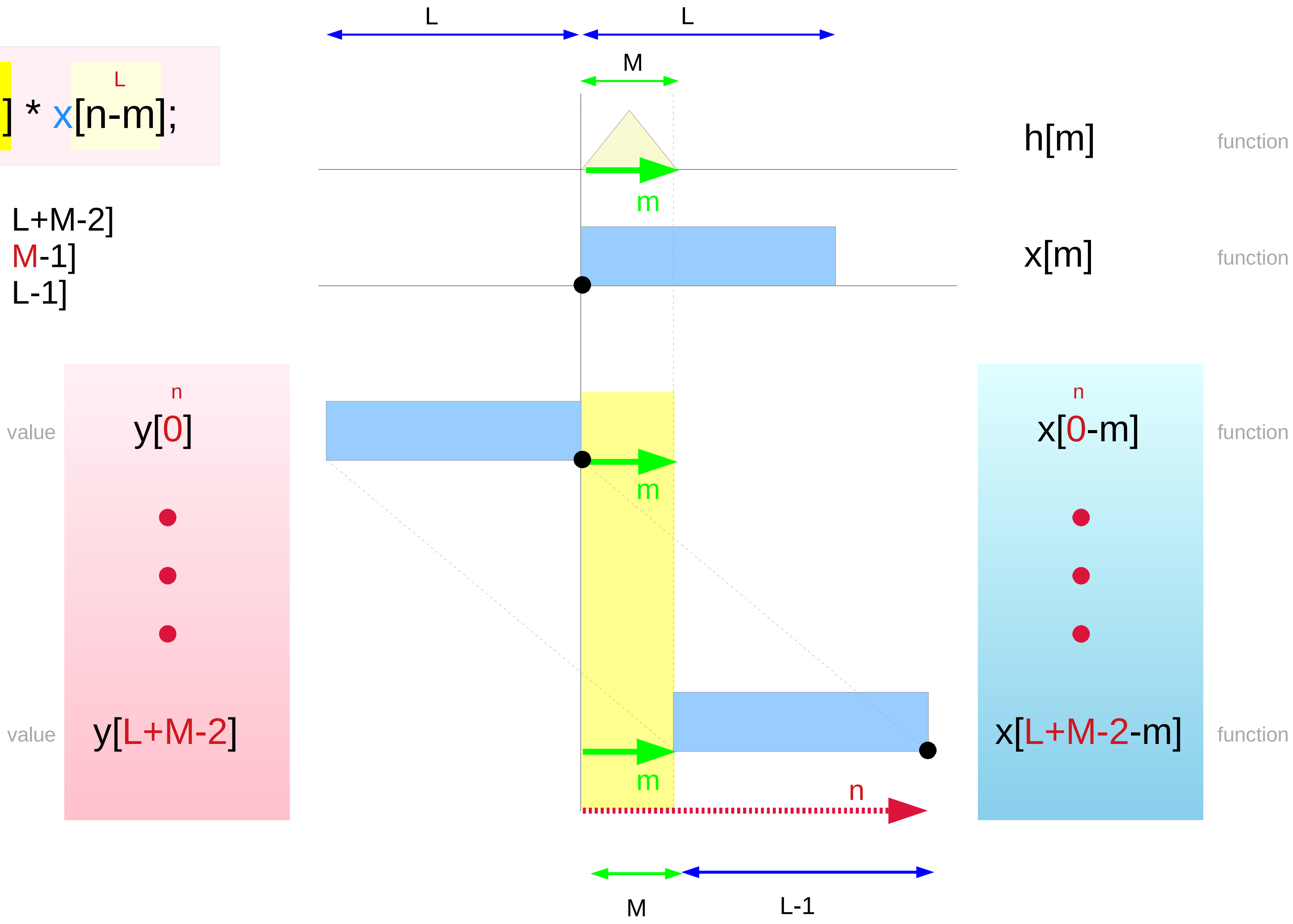
Flipped and shifted function of **x[n-m]**

Case B

M+L-1 M L

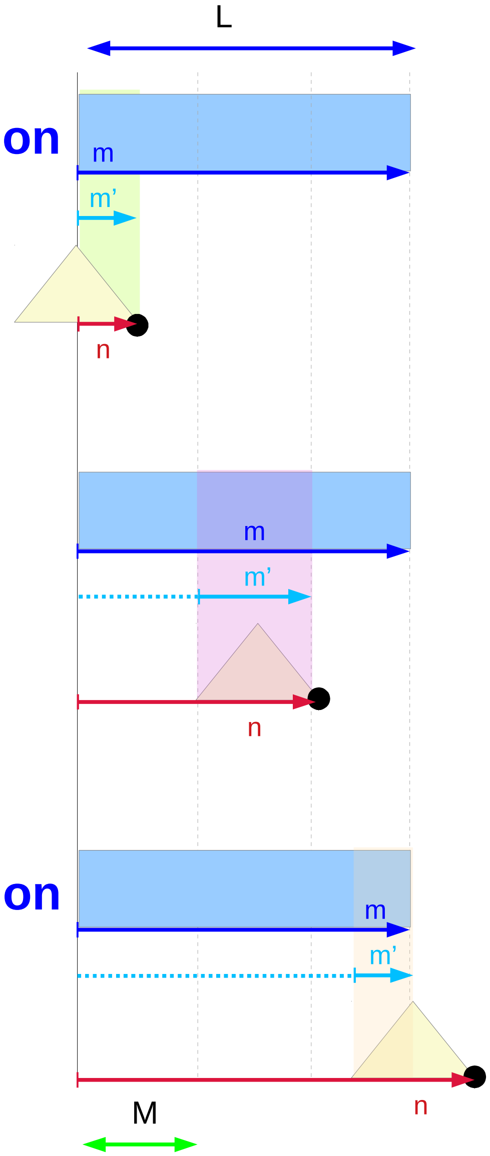
y[n] += h[m] * x[n-m];

n ∈ [0, L+M-2]
m ∈ [0, M-1]
n-m ∈ [0, L-1]



Range partitions for **n** (1)

Case A



partial inclusion

x[m]

h[n-m]

n ∈ [0, M-2]

full inclusion

x[m]

h[n-m]

n ∈ [M-1, L-1]

partial inclusion

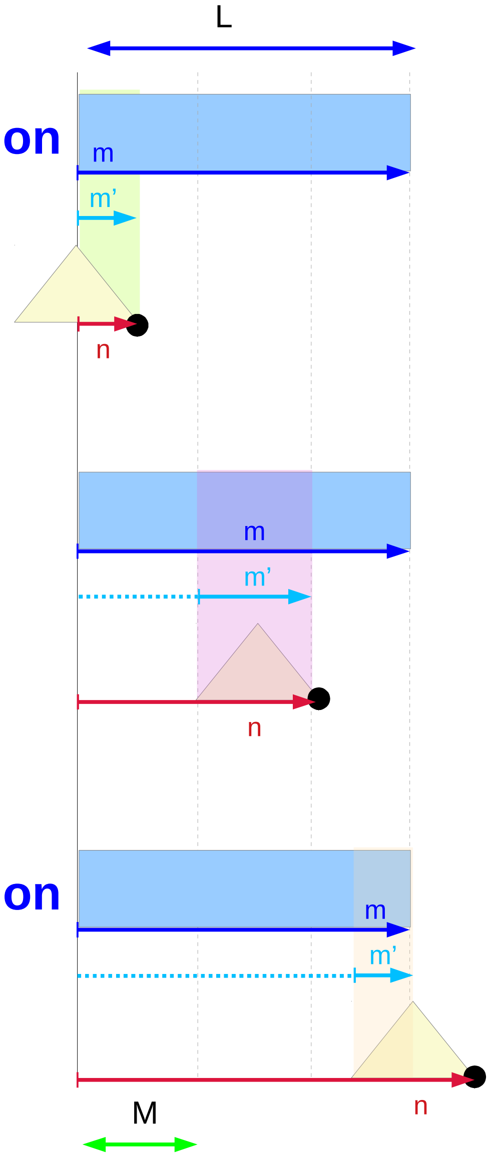
x[m]

h[n-m]

n ∈ [L, L+M-2]

Effective index for **x[m]** (2)

Case A



partial inclusion

x[m]

h[n-m]

m' ∈ [0, n]

for a given **n**

n ∈ [0, M-2]

full inclusion

x[m]

h[n-m]

m' ∈ [n-M+1, n]

for a given **n**

n ∈ [M-1, L-1]

partial inclusion

x[m]

h[n-m]

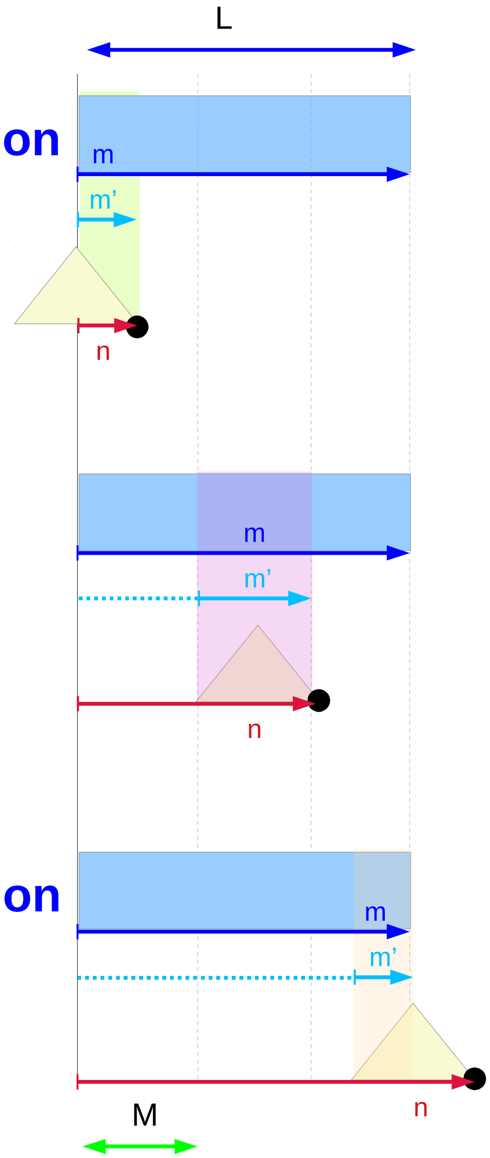
m' ∈ [n-M+1, L-1]

for a given **n**

n ∈ [L, L+M-2]

Effective index for **h[n-m]** (3)

Case A



partial inclusion

x[m] $m' \in [0, n]$

h[n-m] $n\text{-}m' \in \overleftarrow{[n, 0]}$

for a given n

$n \in [0, M\text{-}2]$

full inclusion

x[m] $m' \in [n\text{-}M\text{+}1, n]$

h[n-m] $n\text{-}m' \in \overleftarrow{[M\text{-}1, 0]}$

for a given n

$n \in [M\text{-}1, L\text{-}1]$

partial inclusion

x[m] $m' \in [n\text{-}M\text{+}1, L\text{-}1]$

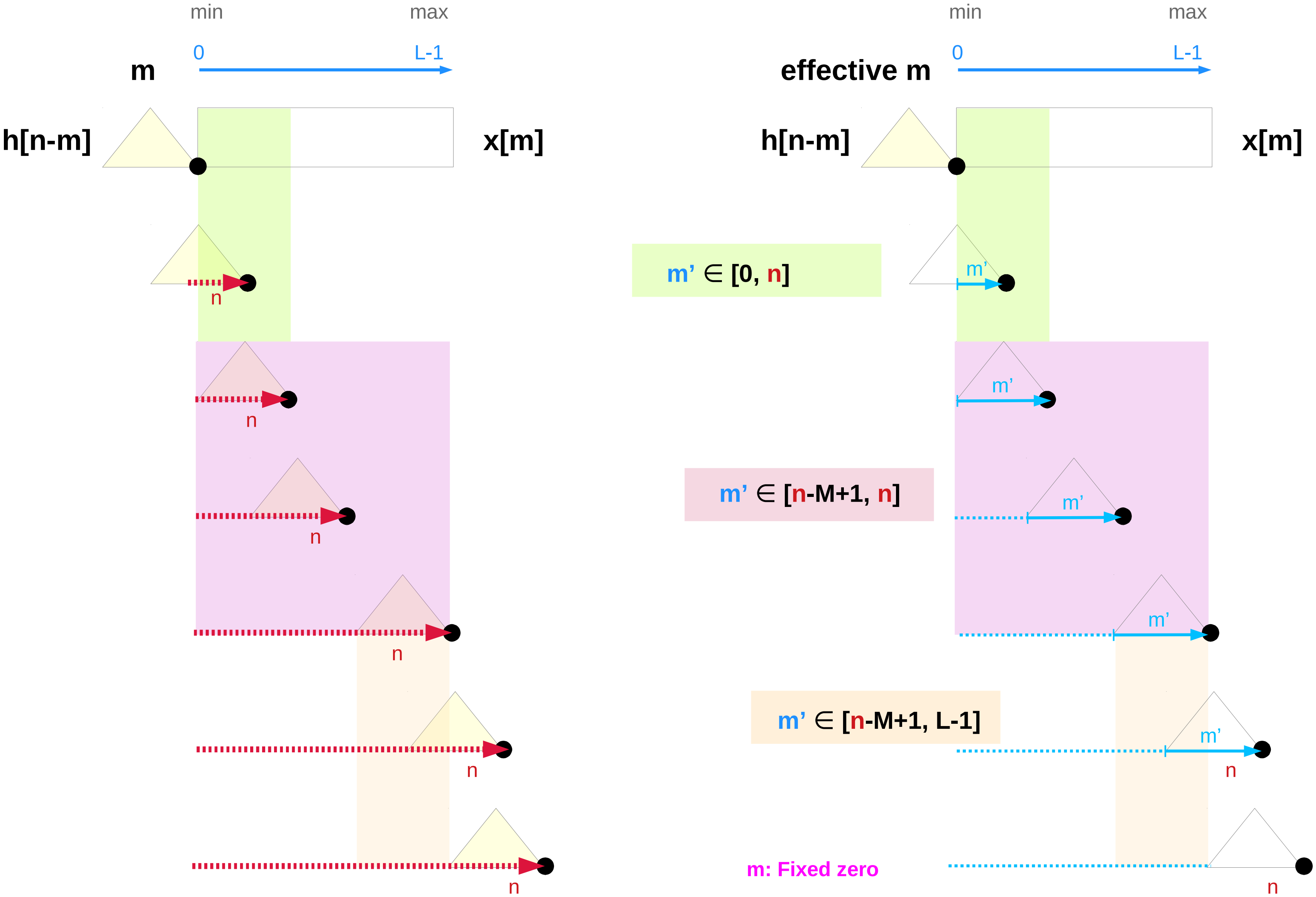
h[n-m] $n\text{-}m' \in \overleftarrow{[M\text{-}1, n\text{-}L\text{+}1\ ]}$

for a given n

$n \in [L, L\text{+}M\text{-}2]$

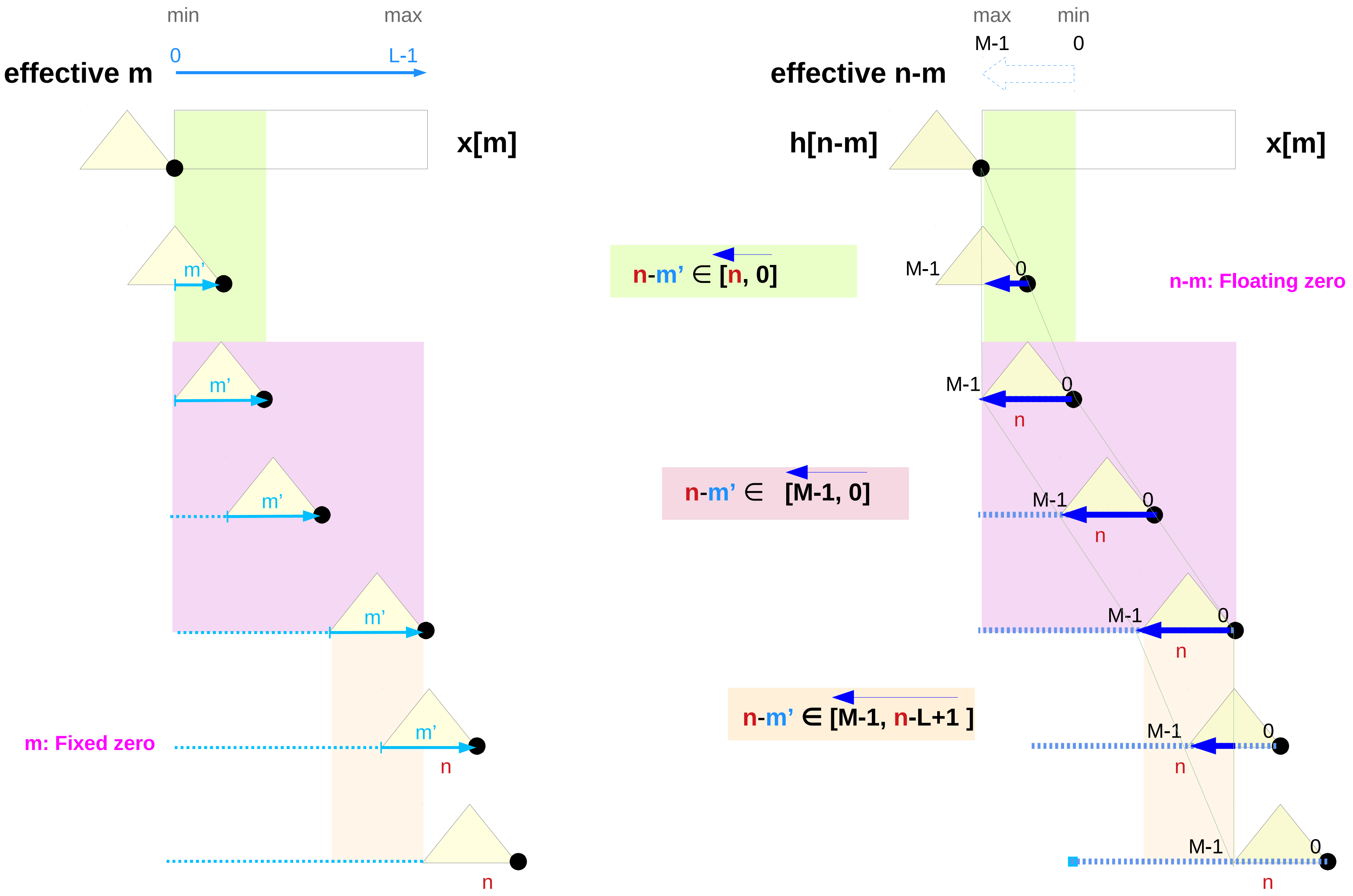
Index **n** and **m**

Case A



Index **m** and **n-m**

Case A



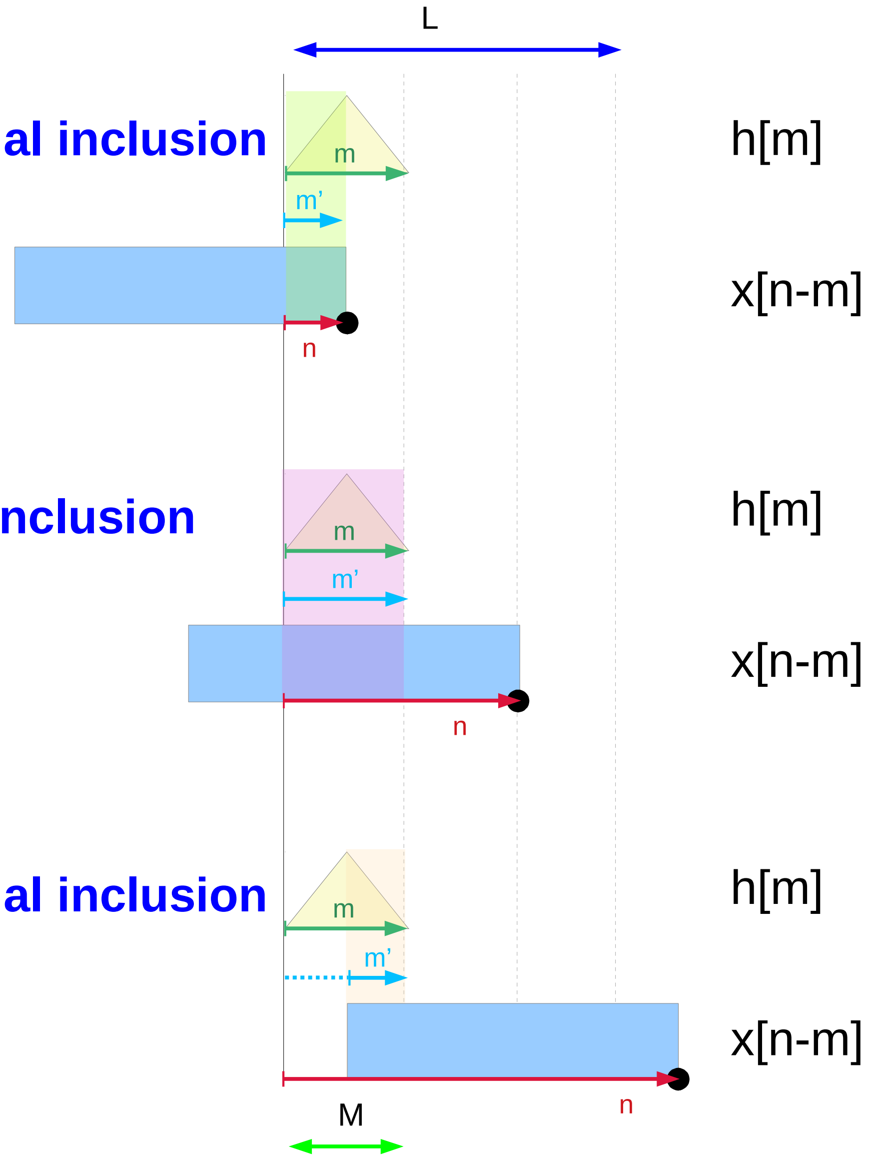
$n-m' \in [n, 0]$

$n-m' \in [M-1, 0]$

$n-m' \in [M-1, n-L+1]$

Range partitions for **n** (1)

Case B



partial inclusion

n ∈ **[0, M-2]**

full inclusion

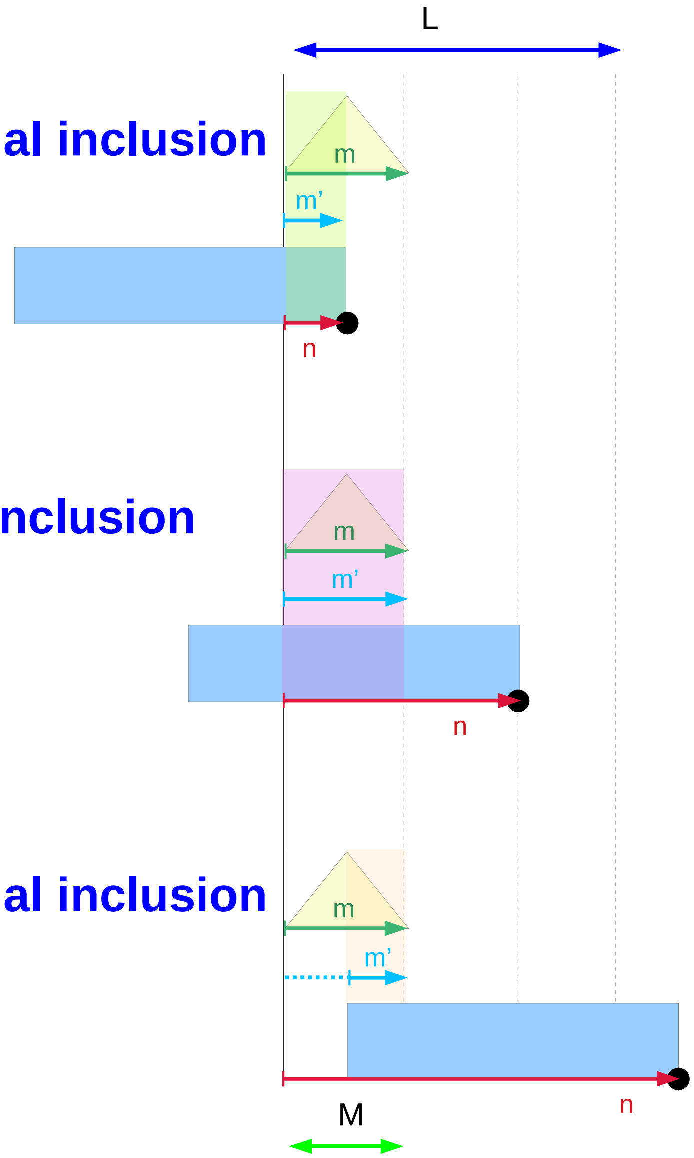
n ∈ **[M-1, L-1]**

partial inclusion

n ∈ **[L, L+M-2]**

Effective index for **h[m]** (2)

Case B



partial inclusion

h[m]

x[n-m]

$m' \in [\;0,\; n]$

for a given n

$n \in [0, M-2]$

full inclusion

h[m]

x[n-m]

$m' \in [\;0,\; M-1]$

for a given n

$n \in [M-1, L-1]$

partial inclusion

h[m]

x[n-m]

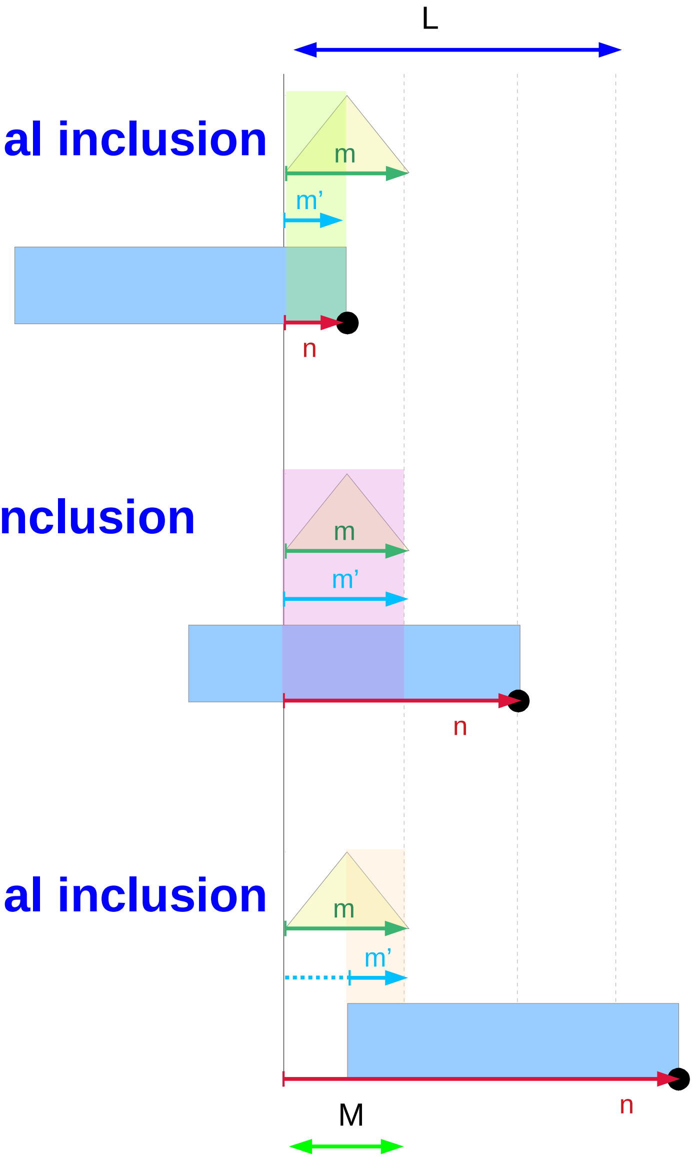
$m' \in [n-L+1, M-1]$

for a given n

$n \in [L, L+M-2]$

Effective index for **x[n-m]** (3)

Case B



partial inclusion

h[m] $m' \in [0, n]$

x[n-m] $n-m' \in \overleftarrow{[n, 0]}$

for a given n

$n \in [0, M-2]$

full inclusion

h[m] $m' \in [0, M-1]$

x[n-m] $n-m' \in \overleftarrow{[n, n-M+1]}$

for a given n

$n \in [M-1, L-1]$

partial inclusion

h[m] $m' \in [n-L+1, M-1]$

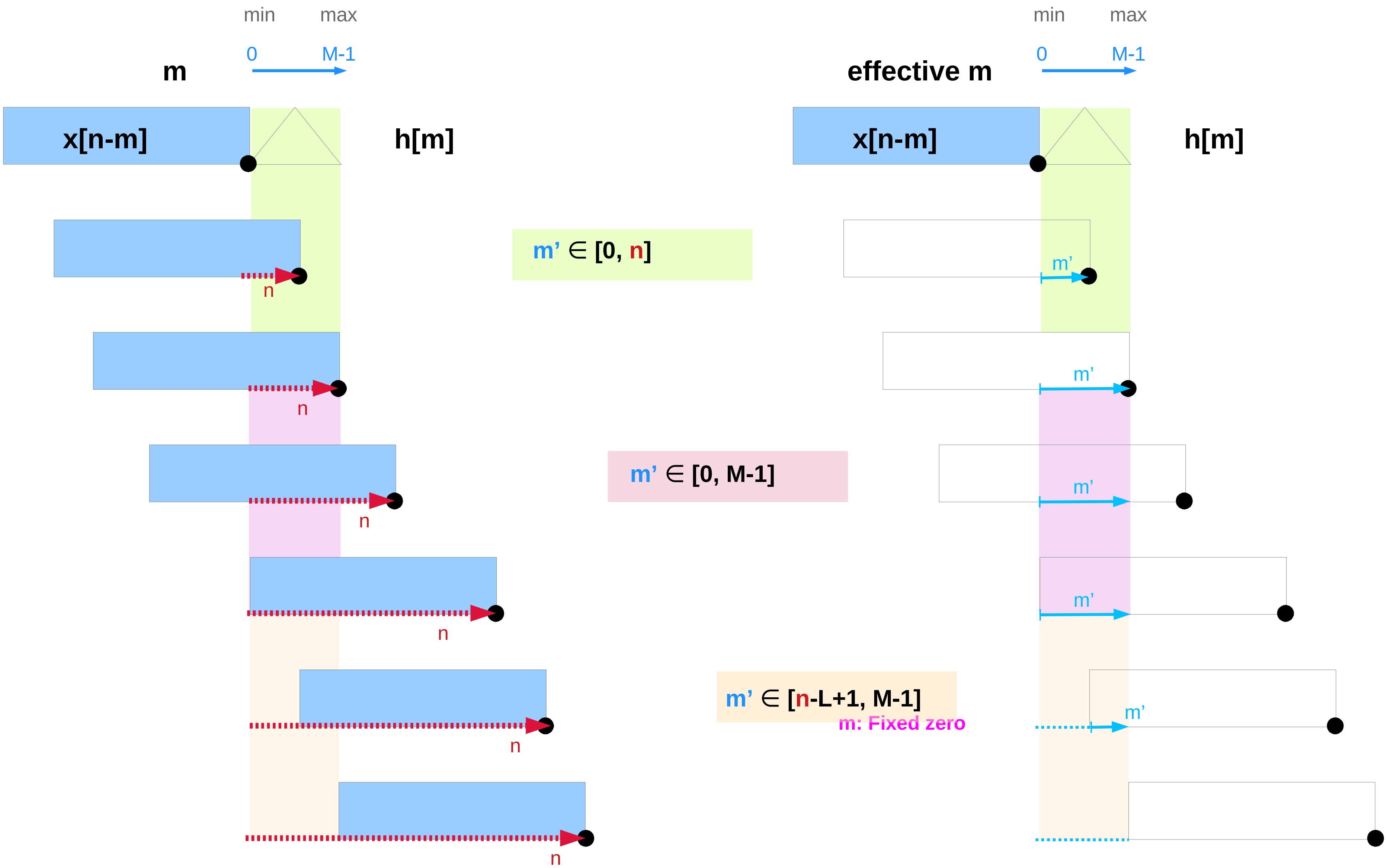
x[n-m] $n-m' \in \overleftarrow{[L-1, n-M+1]}$

for a given n

$n \in [L, L+M-2]$

Index **n** and **m**

Case B



m’ ∈ [0, n]

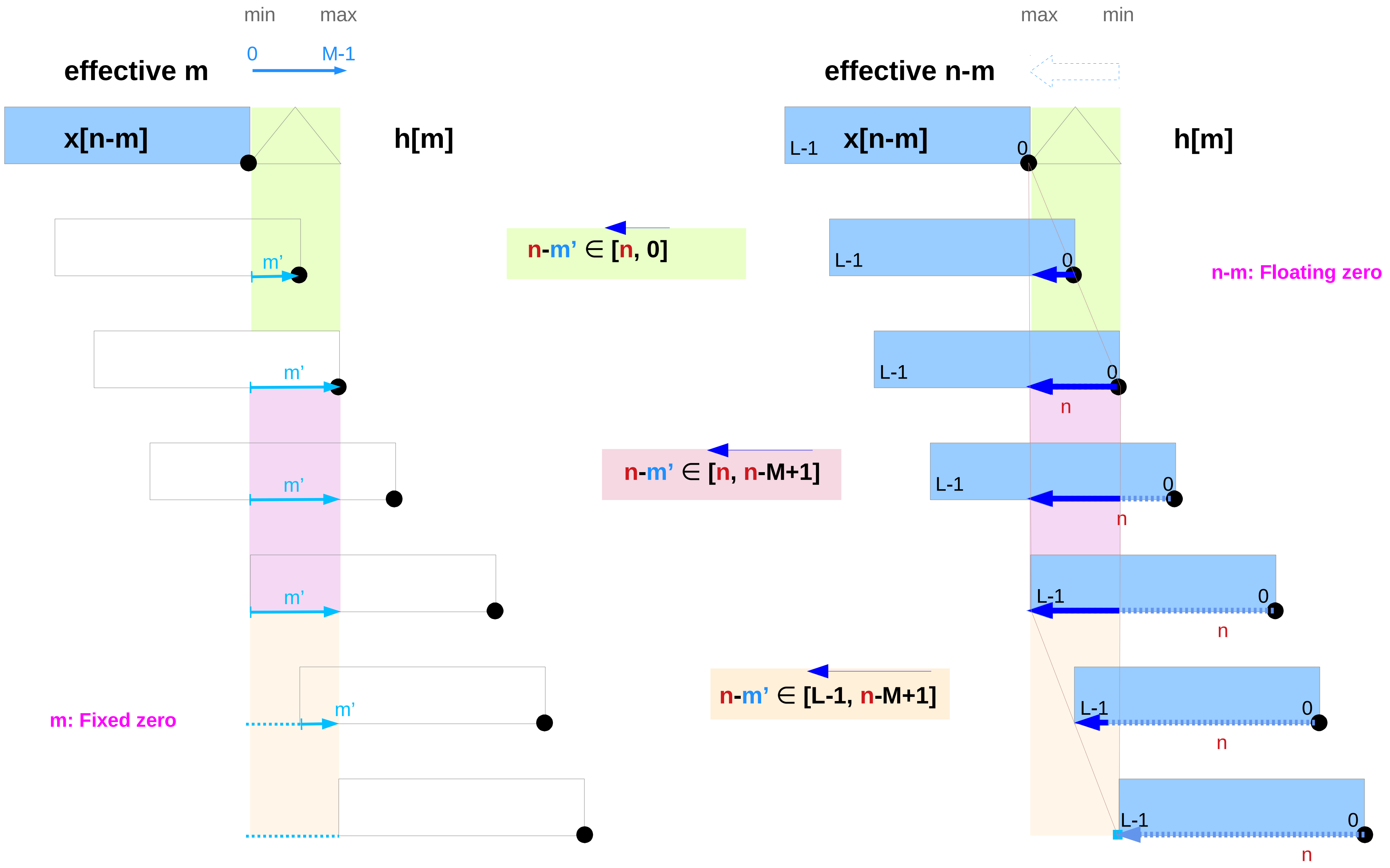
m’ ∈ [0, M-1]

m’ ∈ [n-L+1, M-1]

m: Fixed zero

Index **m** and **n-m**

Case B



n-m' ∈ [n, 0]

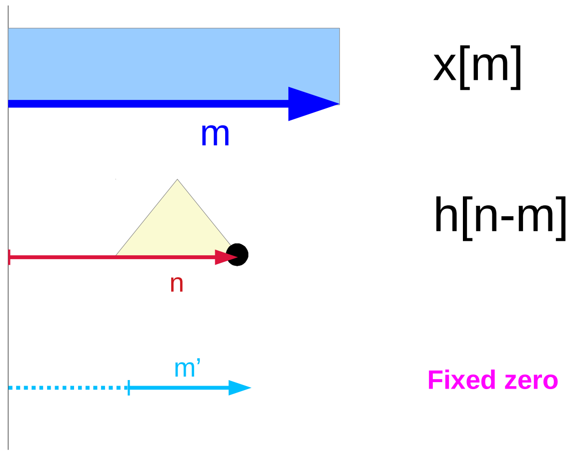
n-m' ∈ [n, n-M+1]

n-m' ∈ [L-1, n-M+1]

- Summary
  - Effective ranges for **m**
  - Effective ranges for **n-m**
  - Memorizing effective ranges for **m**
  - Lower and upper bounds for **m**

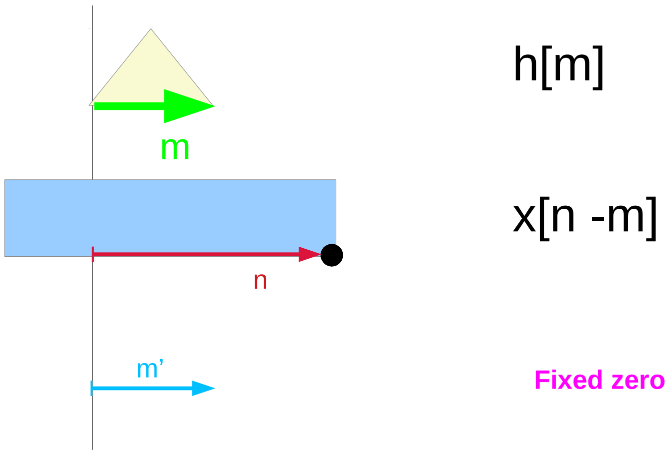
Summary (1) : effective ranges for **m**

Case A



Part. 1	**m' ∈ [0, n]**
Part. 2	**m' ∈ [n-M+1, n]**
Part. 3	**m' ∈ [n-M+1, L-1]**

Case B

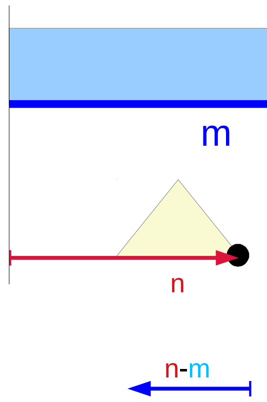


Part. 1	**m' ∈ [0, n]**
Part. 2	**m' ∈ [0, M-1]**
Part. 3	**m' ∈ [n-L+1, M-1]**

Summary (2) : effective ranges for **n-m**

Case A, B

Case A



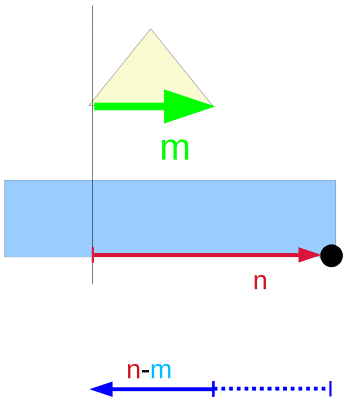
x[m]

h[n-m]

Floating zero

Part. 1	**n-m’** ∈ **[n, 0]**
Part. 2	**n-m’** ∈ **[M-1, 0]**
Part. 3	**n-m’** ∈ **[M-1, n-L+1]**

Case B



h[m]

x[n -m]

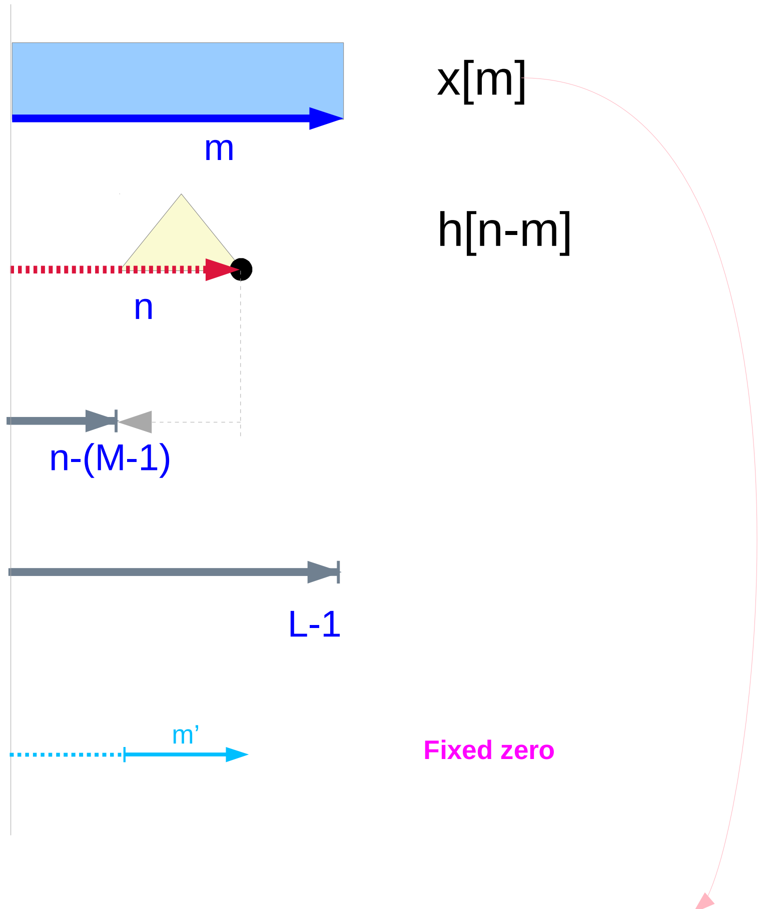
Floating zero

Part. 1	**n-m’** ∈ **[n, 0]**
Part. 2	**n-m’** ∈ **[n, n-M+1]**
Part. 3	**n-m’** ∈ **[L-1, n-M+1]**

Summary (3) : memorizing effective ranges for **m**

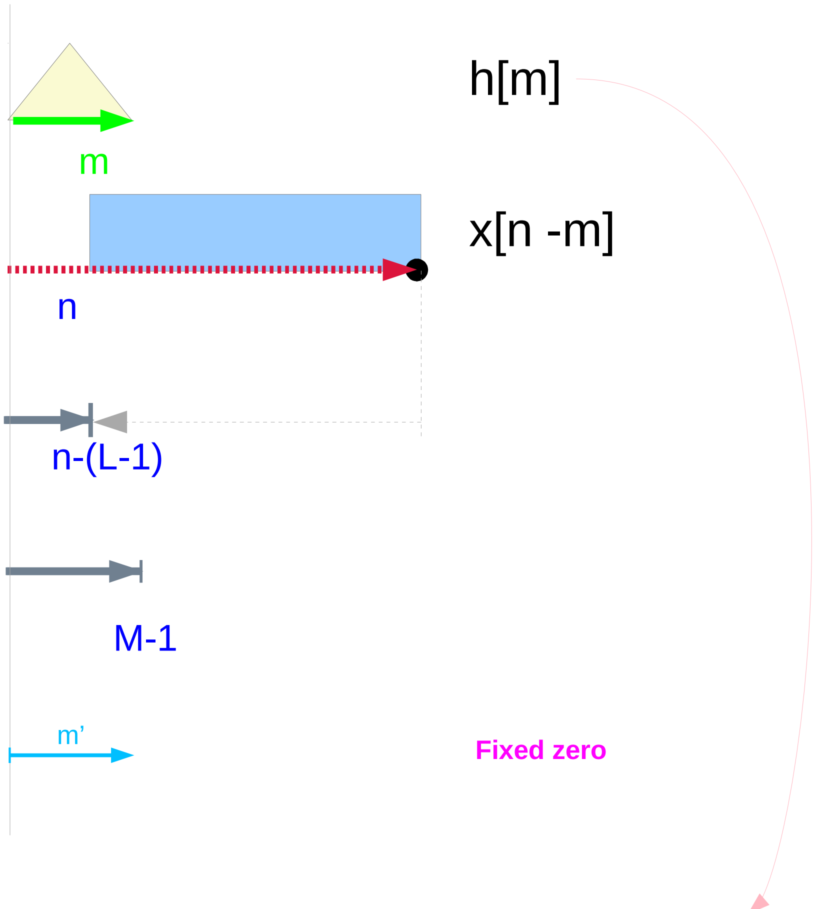
Case A, B

Case A



[max(0, n-(M-1)), min(n, L-1)]

Case B

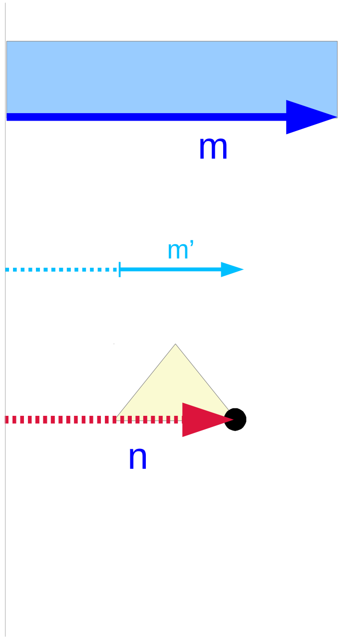


[max(0, n-(L-1)), min(n, M-1)]

Summary (4) : lower and upper bounds for **m**

Case A, B

Case A



upper bound

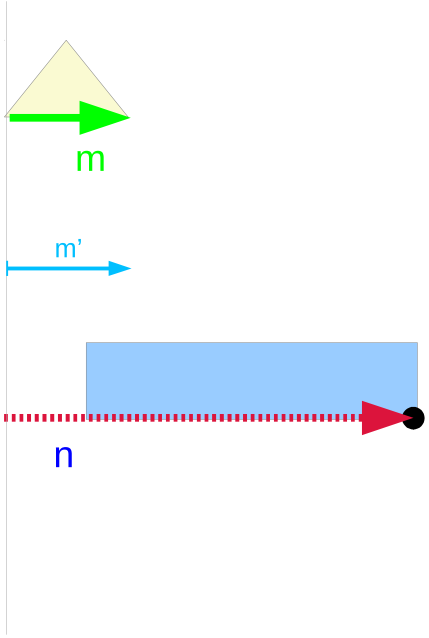
min(n, L-1)

Fixed zero

lower bound

max(0, n-(M-1))

Case B



upper bound

min(n, M-1)

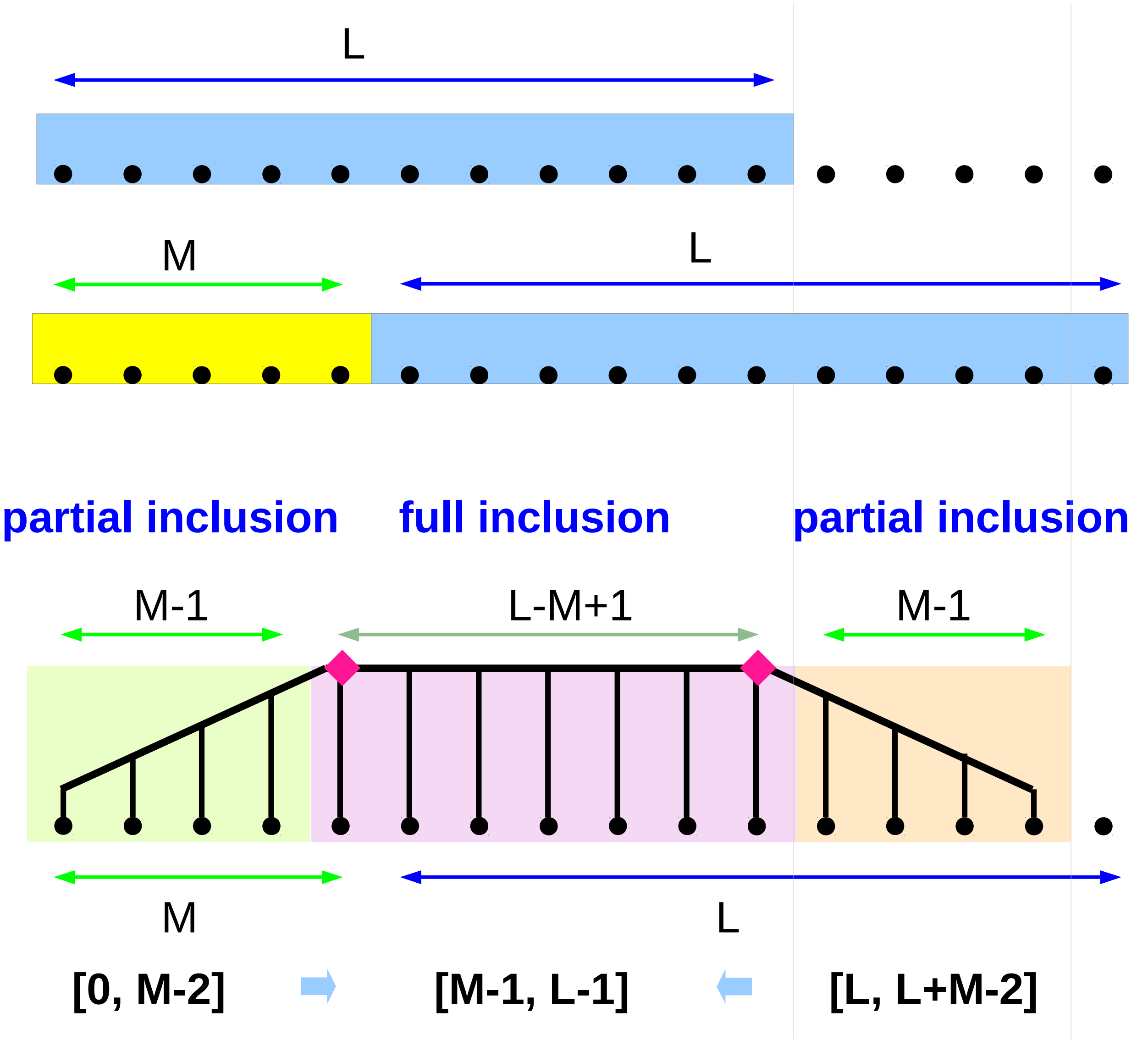
Fixed zero

lower bound

max(0, n-(L-1))

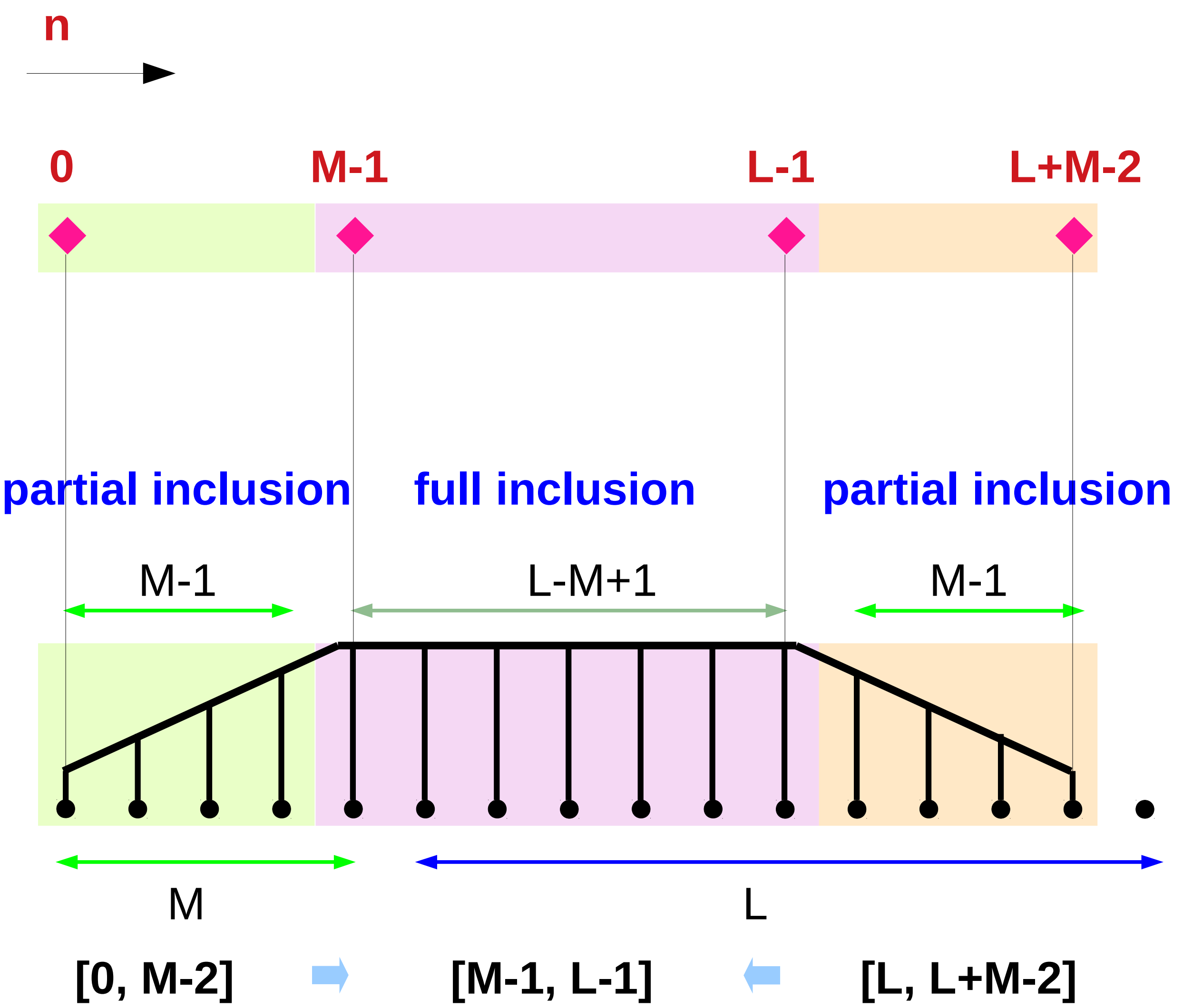
- Range Partitions for **n**

Index sizes of including regions



amount of the
including indices

Four boundary points



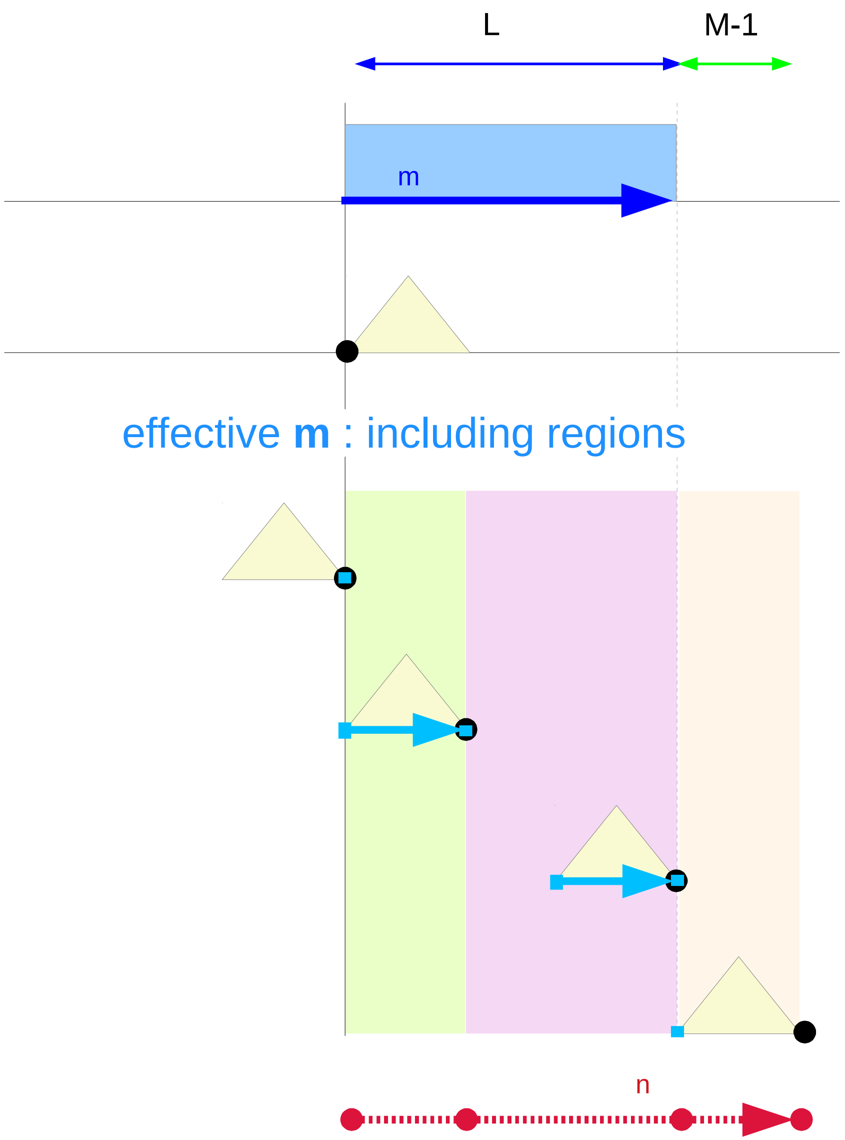
h[n-m] at the boundary points (1)

Case A

M+L-1 L M

```
y[n] += x[m] * h[n-m];
```

n ∈ [0, L+M-2]
m ∈ [0, L-1]
n-m ∈ [0, M-1]



x[m]

h[m]

effective **m** : including regions

Pt 1 value y[0] h[0-m] function

partial overlap

Pt 2 value y[M-1] h[M-1-m] function

full overlap

Pt 3 value y[L-1] h[L-1-m] function

partial overlap

Pt 4 value y[L+M-2] h[L+M-2-m] function

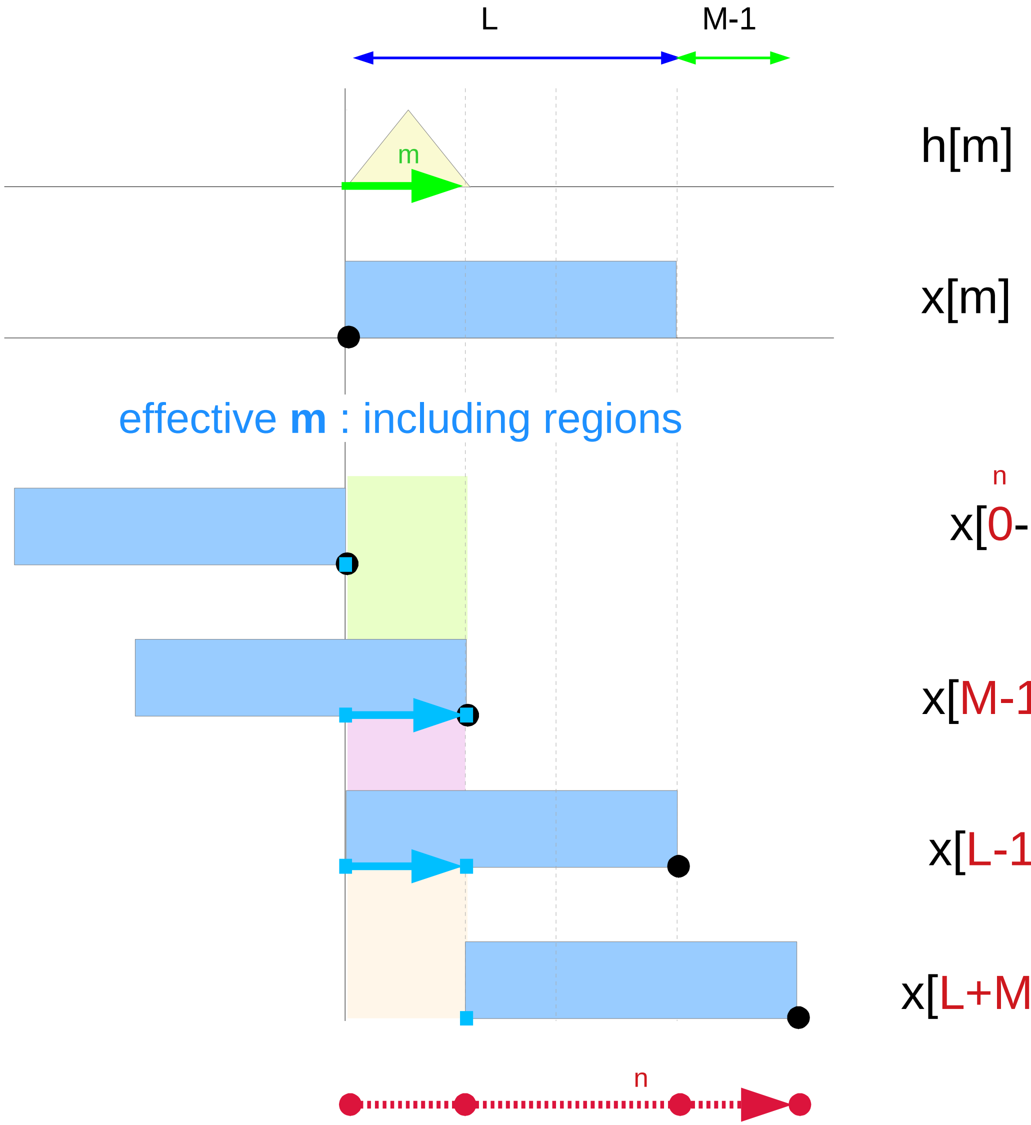
x[n-m] at the boundary points (2)

Case B

M+L-1 M L

y[n] += h[m] * x[n-m];

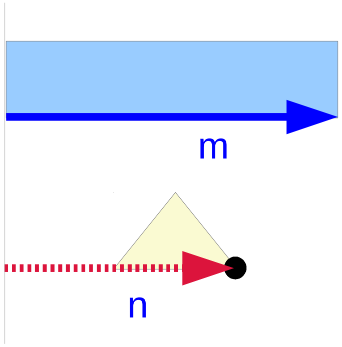
n ∈ [0, L+M-2]
m ∈ [0, M-1]
n-m ∈ [0, L-1]



Pt 1	value	y[0]	x[0-m]	function
partial overlap				
Pt 2	value	y[M-1]	x[M-1-m]	function
full overlap				
Pt 3	value	y[L-1]	x[L-1-m]	function
partial overlap				
Pt 4	value	y[L+M-2]	x[L+M-2-m]	function

Effective ranges of **m** and **n-m**

Case A



upper bound

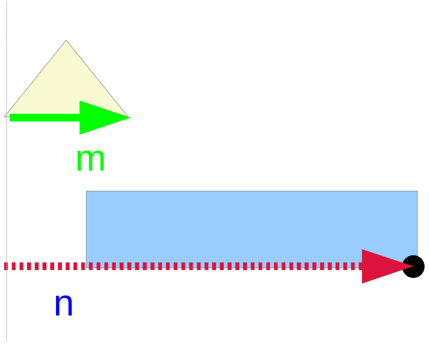
min(n, L-1)

lower bound

max(0, n-(M-1))

	m	**n-m**
0	[0, 0]	[0, 0]
	[0, 1]	[1, 0]
⋮	‖ ‖	‖
M-1	[0, M-1]	[M-1, 0]
⋮		‖ ‖
L-1	[L-M, L-1]	[M-1, 0]
	[L-M+1, L-1]	[M-1, 1]
⋮	‖	‖
L+M-2	[L-1, L-1]	[M-1, M-1]

Case B



upper bound

min(n, M-1)

lower bound

max(0, n-(L-1))

	m	**n-m**
0	[0, 0]	[0, 0]
	[0, 1]	[1, 0]
⋮	‖ ‖	‖ ‖
M-1	[0, M-1]	[M-1, 0]
⋮	‖ ‖	
L-1	[0, M-1]	[L-1, L-M]
	[1, M-1]	[L-1, L-M+1]
⋮	‖	‖
L+M-2	[M-1, M-1]	[L-1, L-1]

Effective ranges of **m** in **x[m]** (1)

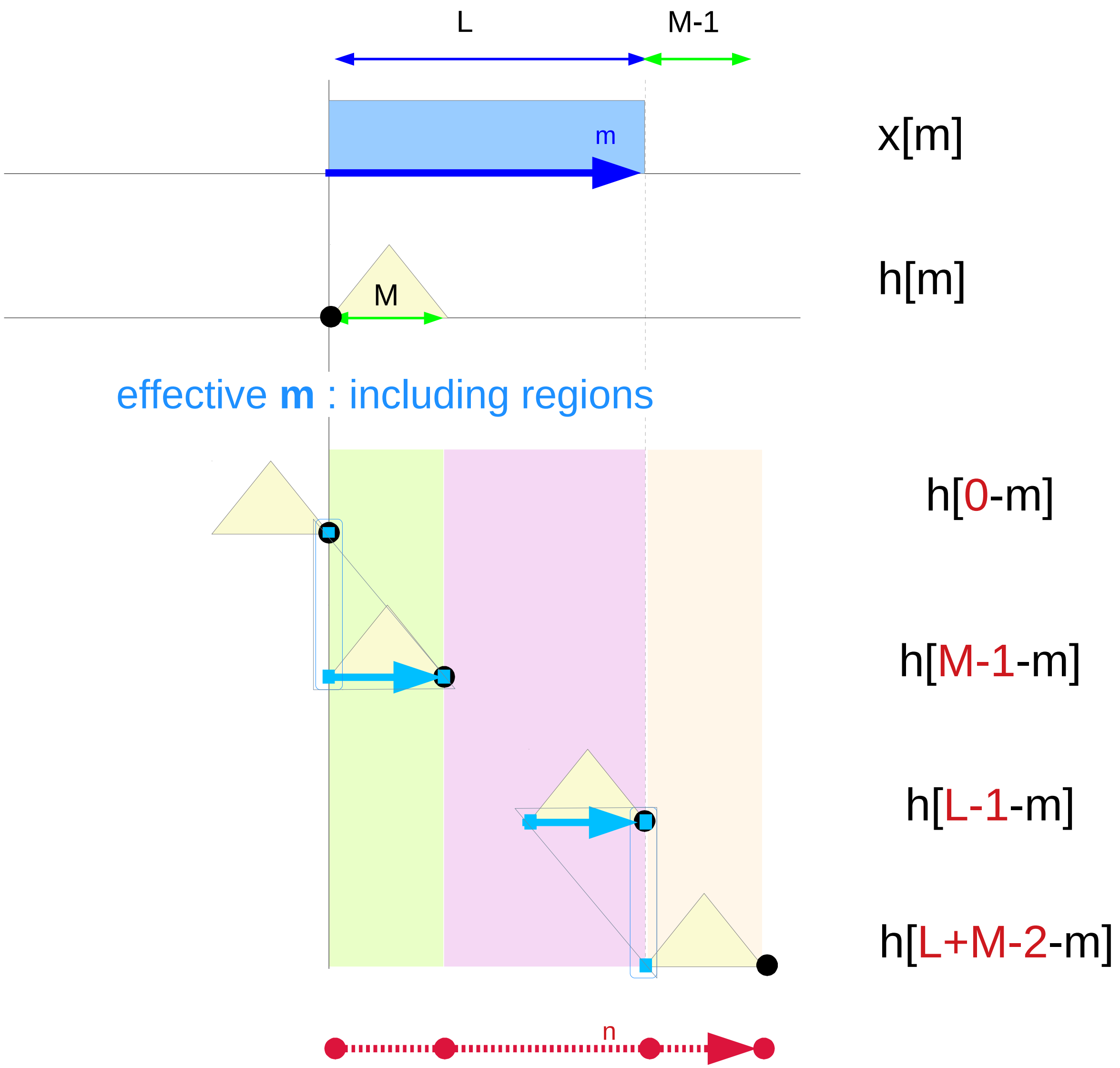
Case A

M+L-1 L M

y[n] += x[m] * h[n-m];

n ∈ [0, L+M-2]
m ∈ [0, L-1]
n-m ∈ [0, M-1]

	max(0, n-(M-1)) lower bound	min(n, L-1) upper bound
y[0]	[0,	0]
	[0,	1]
⋮	‖	↓
y[M-1]	[0,	M-1]
⋮	↓	↓
y[L-1]	[L-M,	L-1]
	[L-M+1,	L-1]
⋮	↓	‖
y[L+M-2]	[L-1,	L-1]
n	n-m	



Effective ranges of **m** in **h[m]** (2)

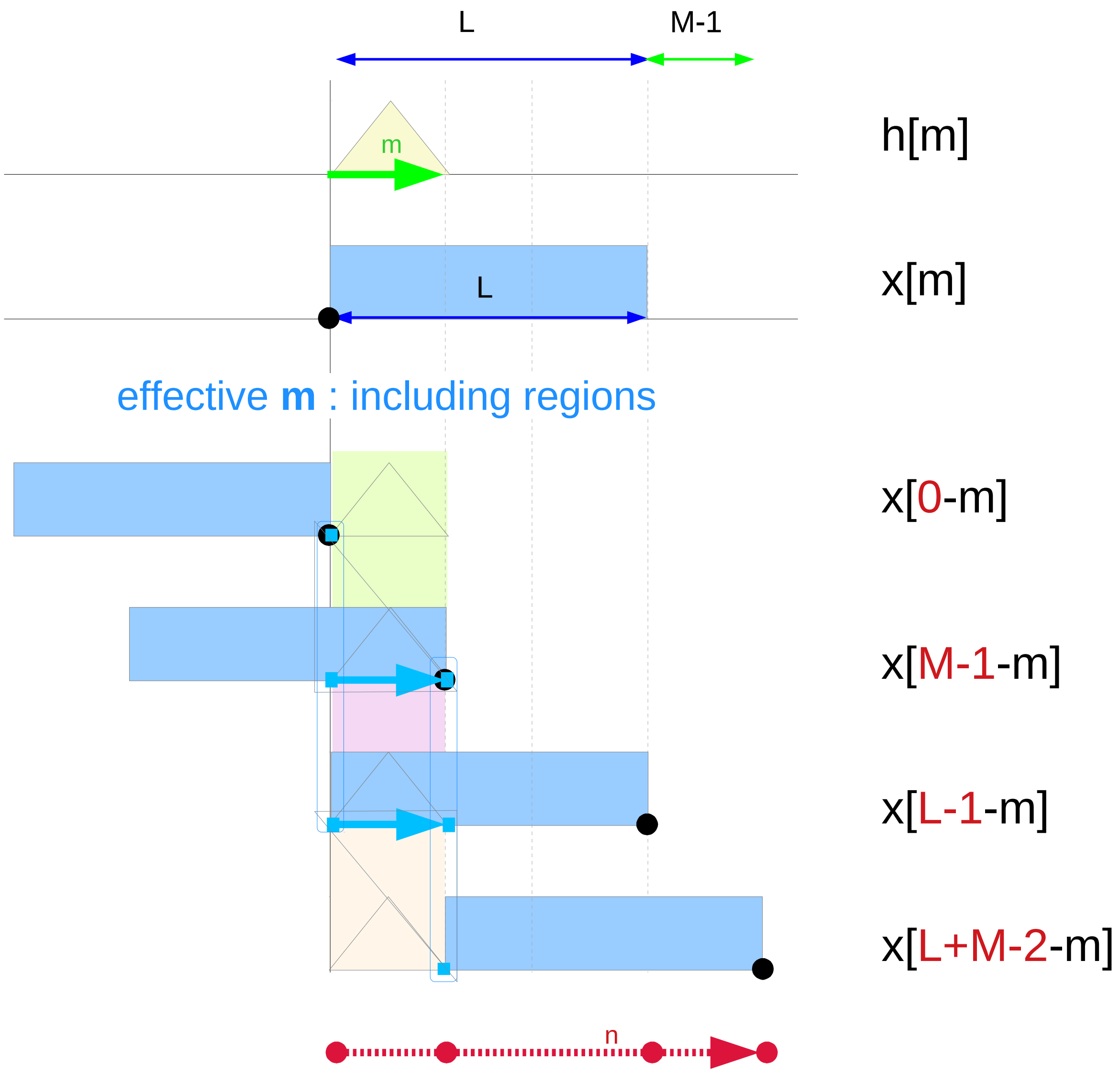
Case B

M+L-1 M L

y[n] += h[m] * x[n-m];

n ∈ [0, L+M-2]
m ∈ [0, M-1]
n-m ∈ [0, L-1]

	max(0, n-(L-1)) lower bound	min(n, M-1) upper bound
y[0]	[0,	0]
	[0,	1]
⋮	‖	↓
y[M-1]	[0,	M-1]
⋮	‖	‖
y[L-1]	[0,	M-1]
	[1,	M-1]
⋮	↓	‖
y[L+M-2]	[M-1,	M-1]
n	n-m	


effective **m** : including regions

Effective ranges of **n-m** in **h[n-m]** (1)

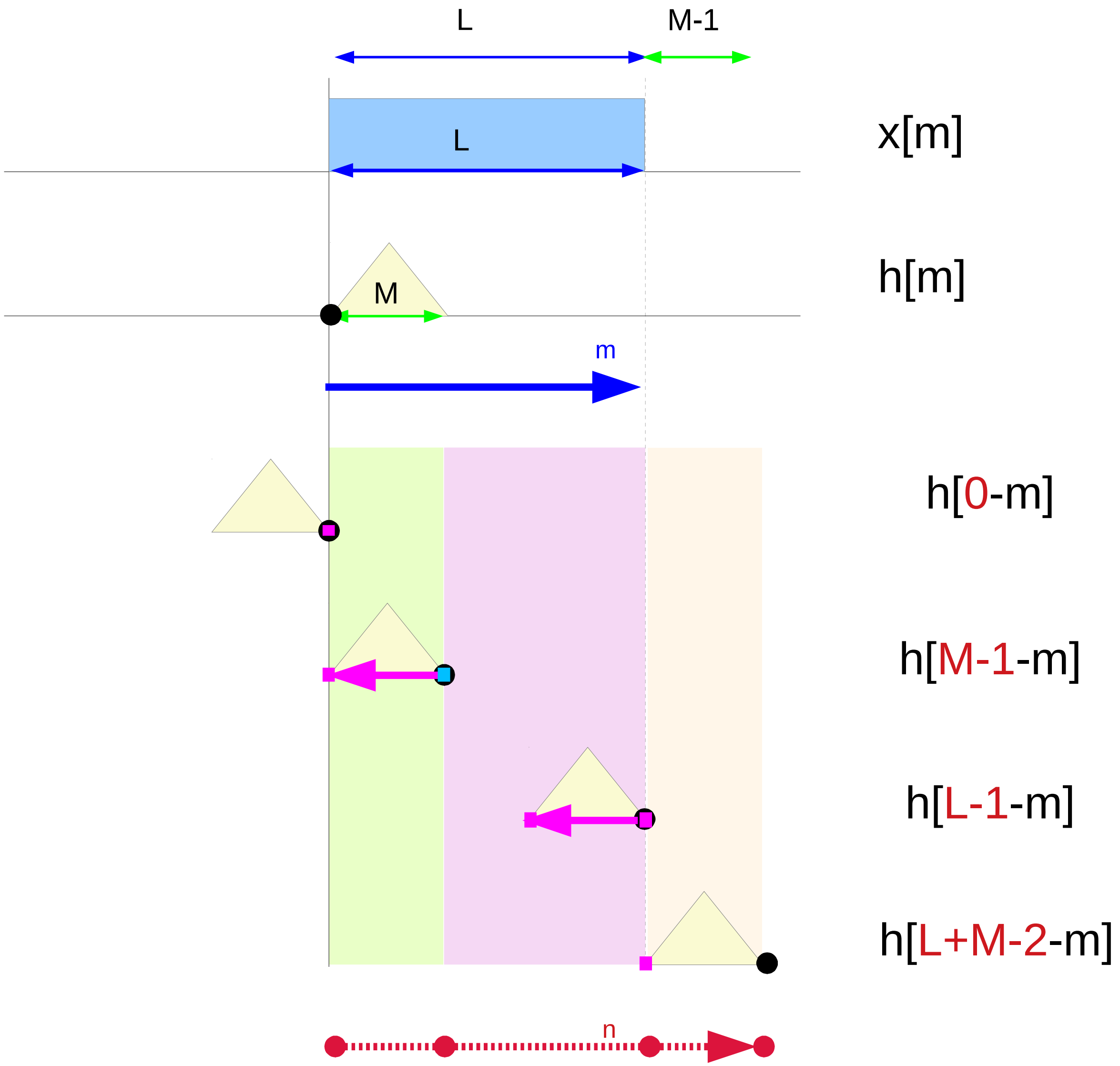
Case A

M+L-1 L M

y[n] += x[m] * h[n-m];

n ∈ [0, L+M-2]
m ∈ [0, L-1]
n-m ∈ [0, M-1]

	max(0, n-(M-1)) lower bound	min(n, L-1) upper bound
y[0]	[0,	0]
	[1,	0]
⋮	↓	‖
y[M-1]	[M-1,	0]
⋮	‖	‖
y[L-1]	[M-1,	0]
	[M-1,	1]
⋮	‖	↓
y[L+M-2]	[M-1,	M-1]
n	n-m	



Effective ranges of **n-m** in **x[n-m]** (2)

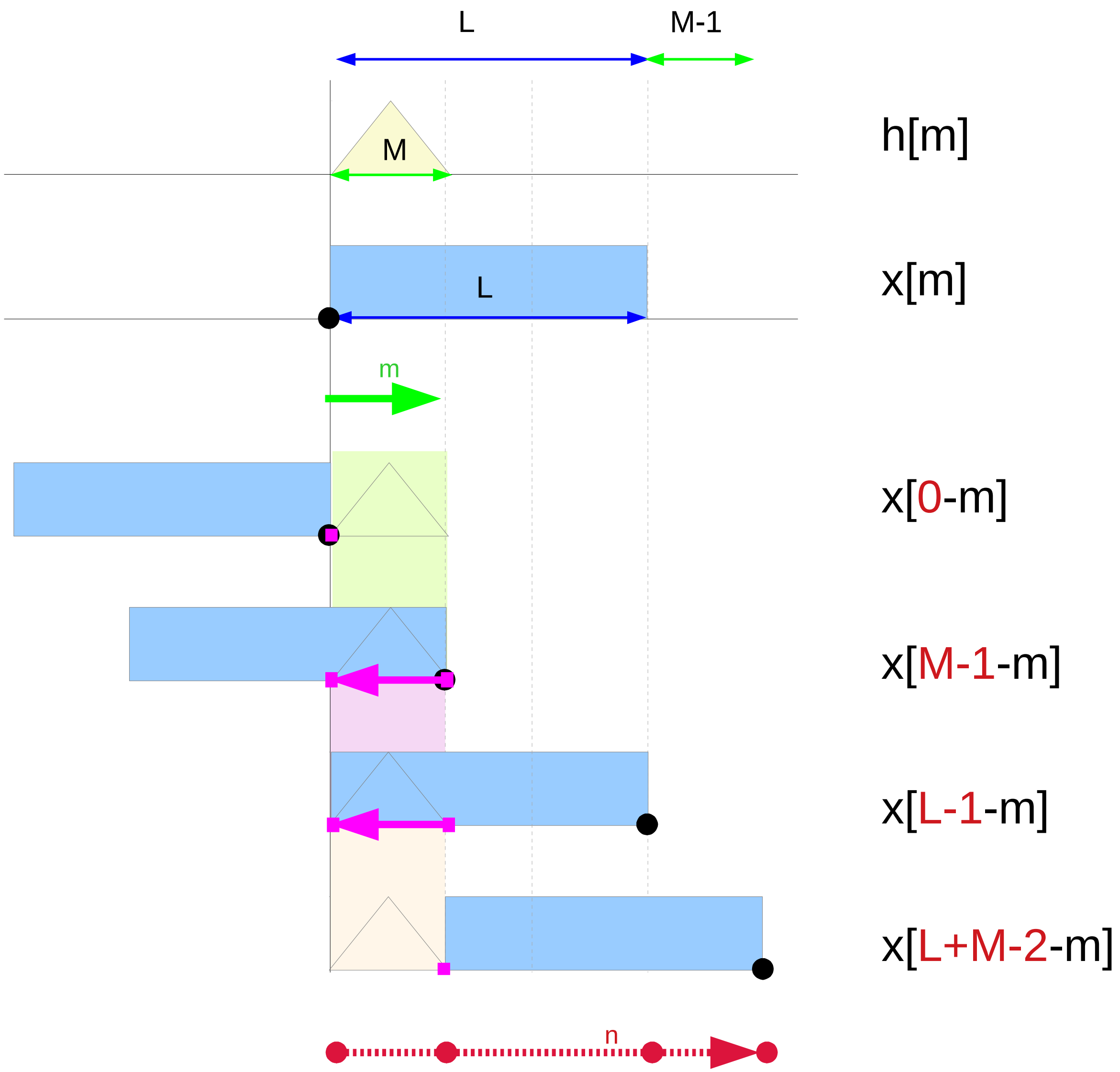
Case B

M+L-1 M L

y[n] += h[m] * x[n-m];

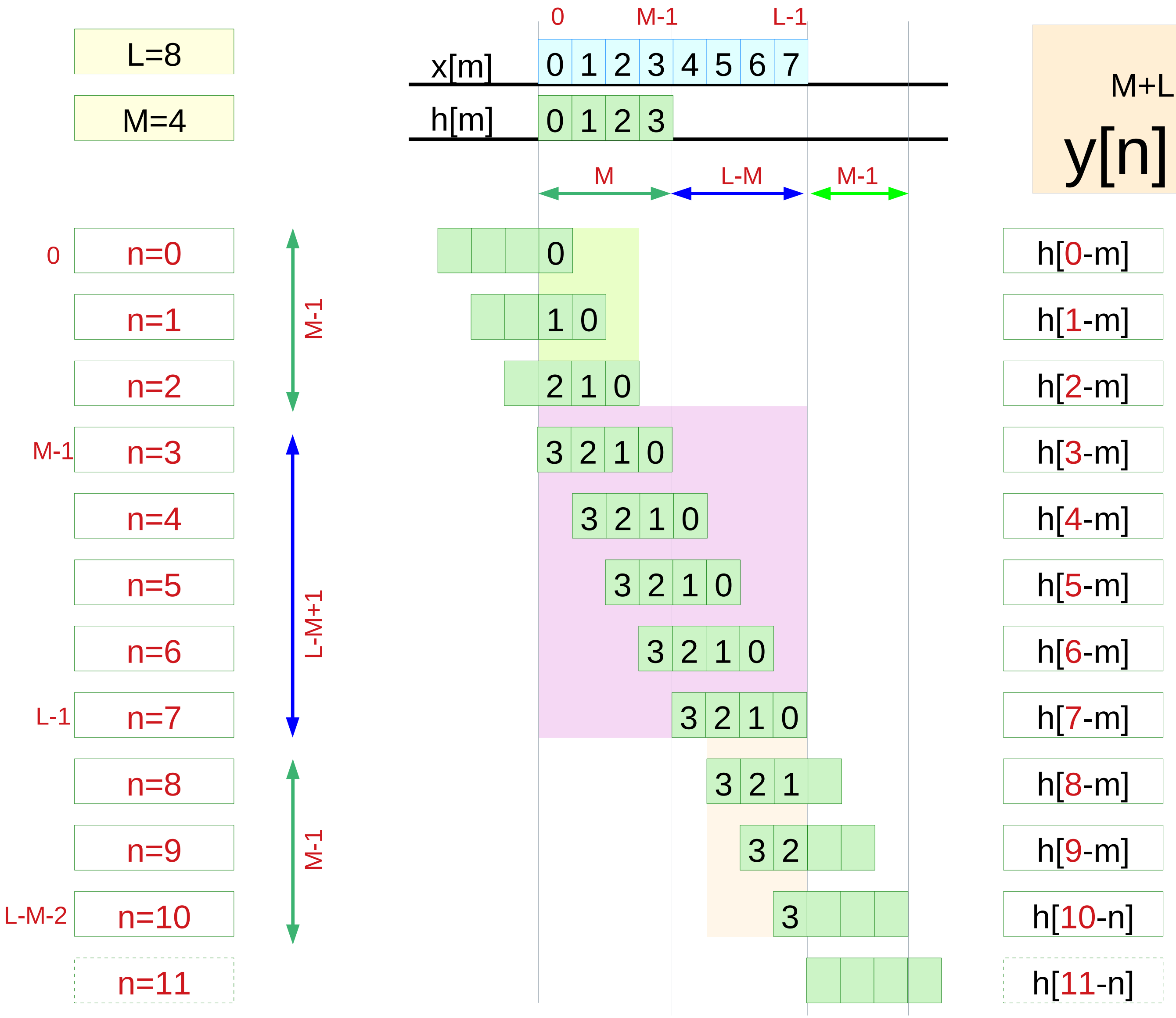
n ∈ [0, L+M-2]
m ∈ [0, M-1]
n-m ∈ [0, L-1]

	max(0, n-(L-1)) lower bound	min(n, M-1) upper bound
y[0]	[0,	0]
	[1,	0]
⋮		
y[M-1]	[M-1,	0]
⋮		
y[L-1]	[L-1,	L-M]
	[L-1,	L-M+1]
⋮		
y[L+M-2]	[L-1,	L-1]
n	n-m	



Index **n-m** value example

Case A



M+L-1 L M

$$y[n] \mathrel{+}= x[m] * h[n-m];$$

Index **n-m** value example

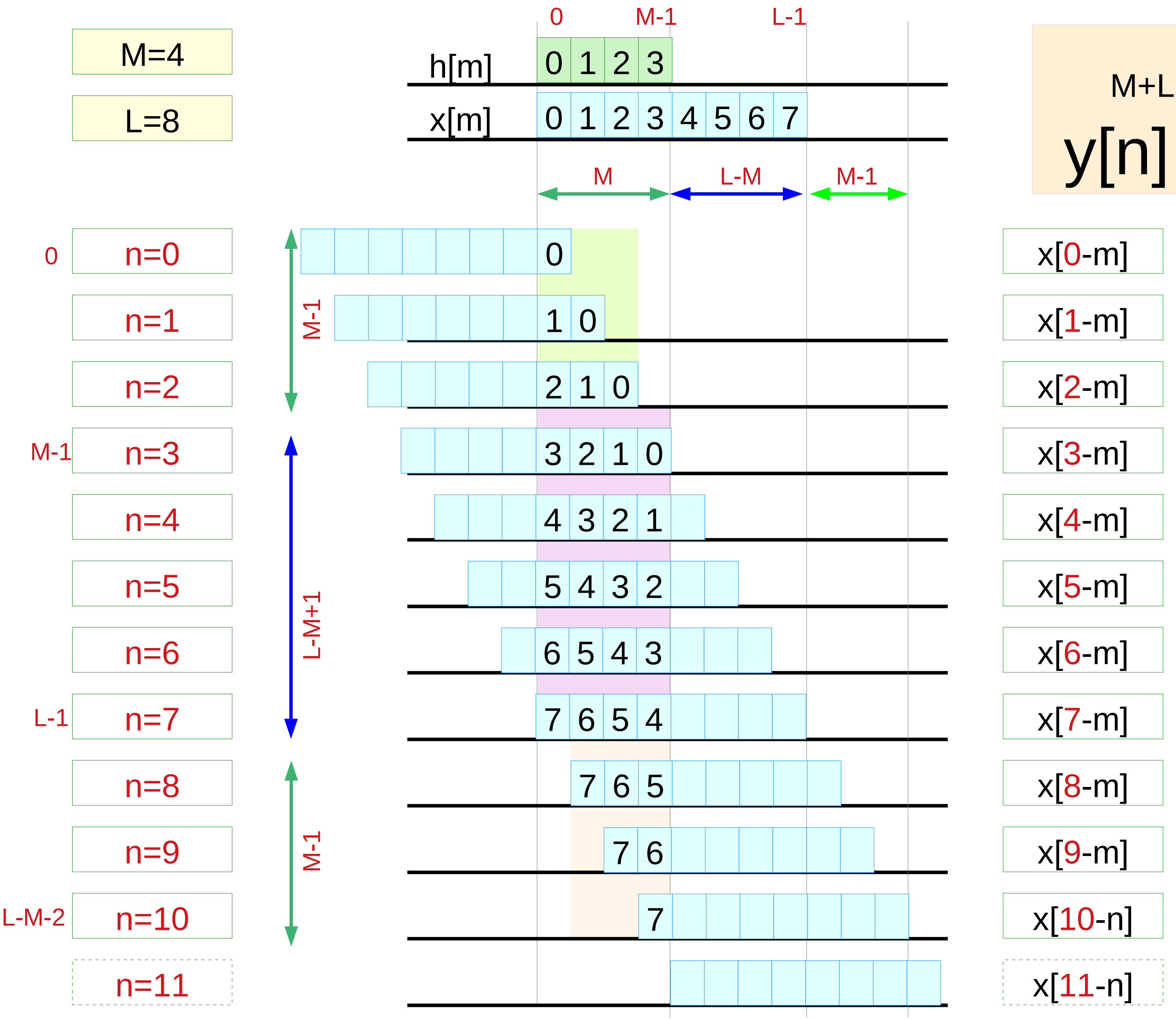
Case B

M=4

L=8

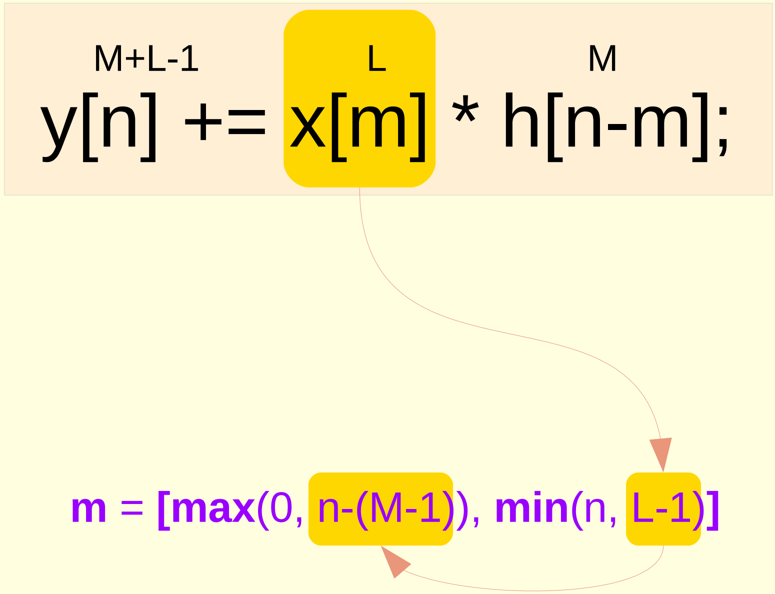
$$y[n] \mathrel{+}= h[m] * x[n-m];$$

M+L-1 M L



- Reasoning about lower and upper bounds of **m** **Case A**

Case A



$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

m = **[max**(0, n-(M-1)), **min**(n, L-1)**]**

Case B

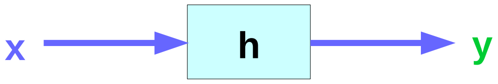
$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

m = **[max**(0, n-(L-1)), **min**(n, M-1)**]**

Index Variable Constraints

Case A

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$



Constraint 1 : **n ∈ [0, L+M-2]**

Constraint 2 : **n-m ∈ [0, M-1]**

Constraint 3 : **m ∈ [0, L-1]**

y[ ] : array with size of **L+M-1**

h[ ] : array with size of **M**

x[ ] : array with size of **L**

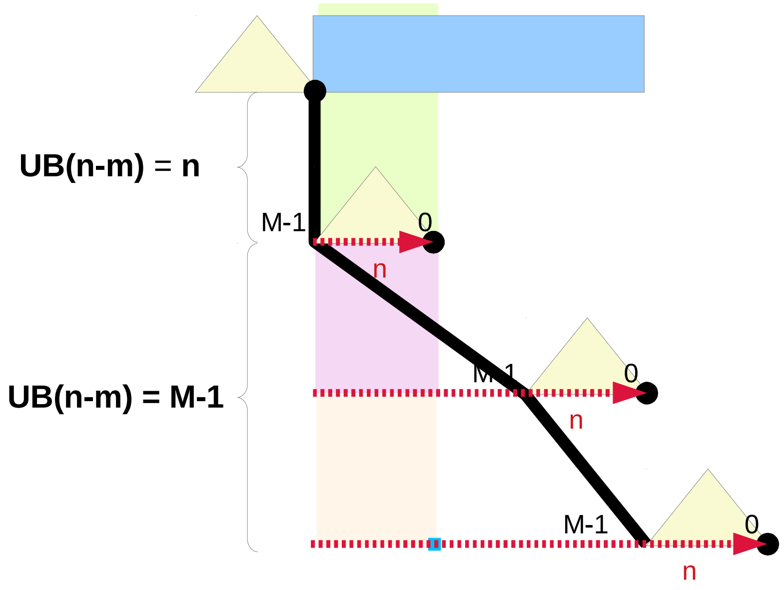
Lower Bound of **m** = max(**0**, n-(M-1))

Case A

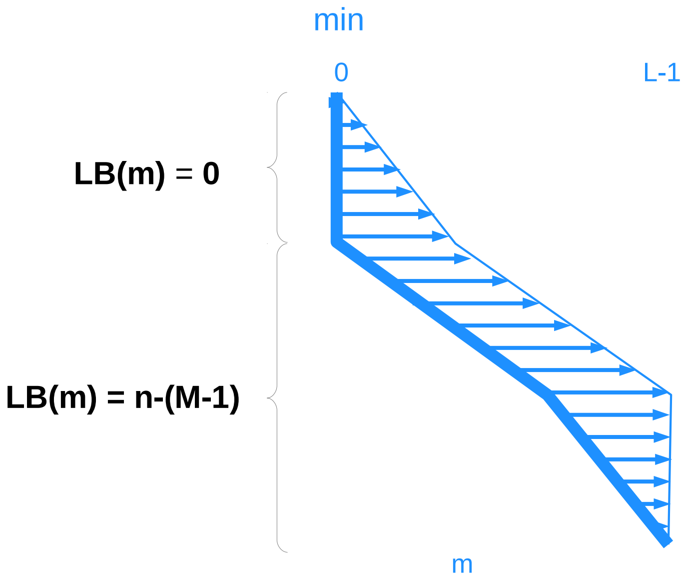
UB(n-m)

LB(m) = max(0, n-(M-1))

LB(m)



Floating zero



Fixed zero

M+L-1 L M

y[n] += x[m] * h[n-m];

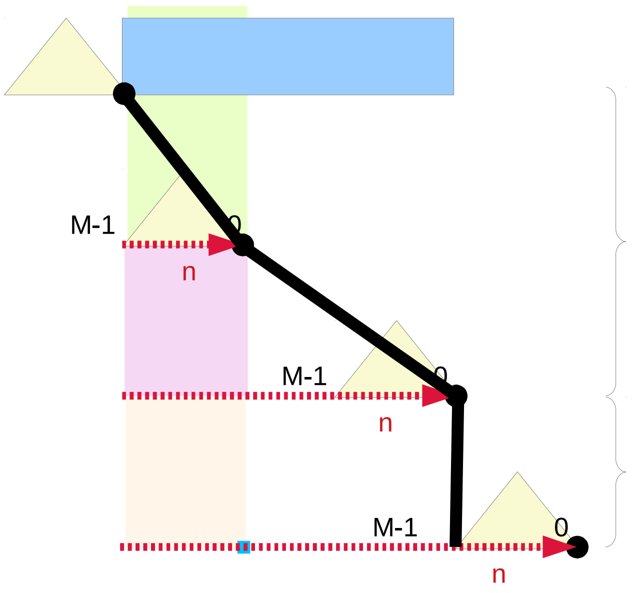
Upper Bound of **m** = min(n, **L-1**)

Case A

LB(n-m)

UB(m) = min(n, L-1)

UB(m)

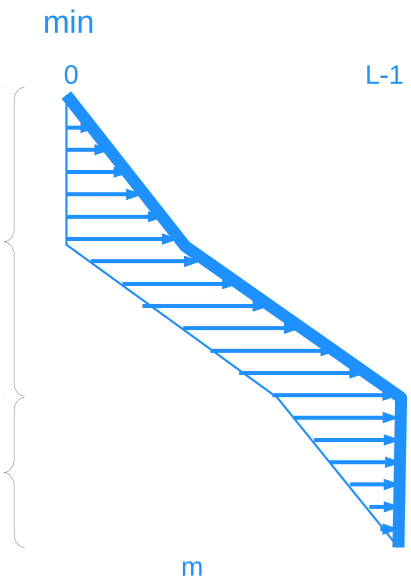


LB(n-m) = **0**

UB(m) = **n**

LB(n-m) = n-(L-1)

UB(m) = (L-1)



Floating zero

Fixed Zero

M+L-1 L M

y[n] += x[m] * h[n-m];

Constraint 1 & 2 – UB(**n-m**) ➡ LB(**m**)

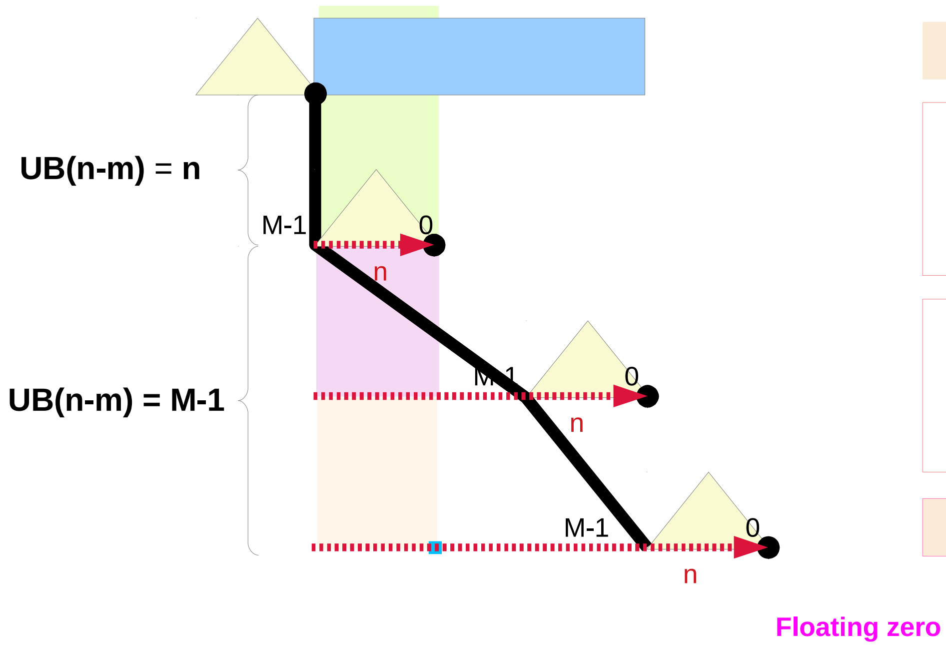
Case A

Constraint 2 : **n-m** ∈ **[0, M-1]** ➡ n - m : **M-1** ... 0

max min

(0, n+1-M)

for **UB(n-m)** values
m should be least possible

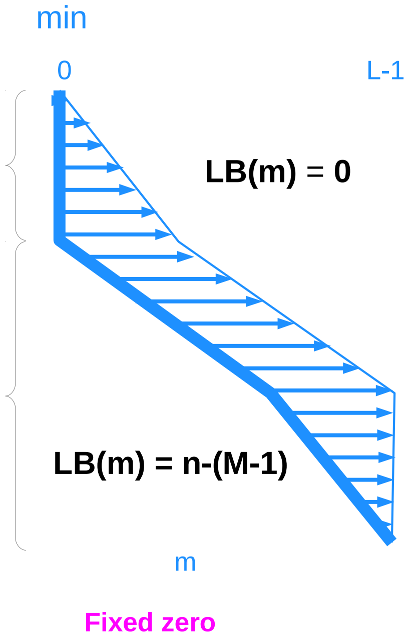


0 ≤ (n-m) ≤ M-1

Case A) n ≤ M-1
➡UB(**n-m**) = **n**
➡LB(**m**) = **0**

Case B) n ≥ M
➡UB(**n-m**) = **M-1**
➡LB(**m**) = **n-(M-1)**

LB(m) = max(0, n-(M-1))



M+L-1 L M

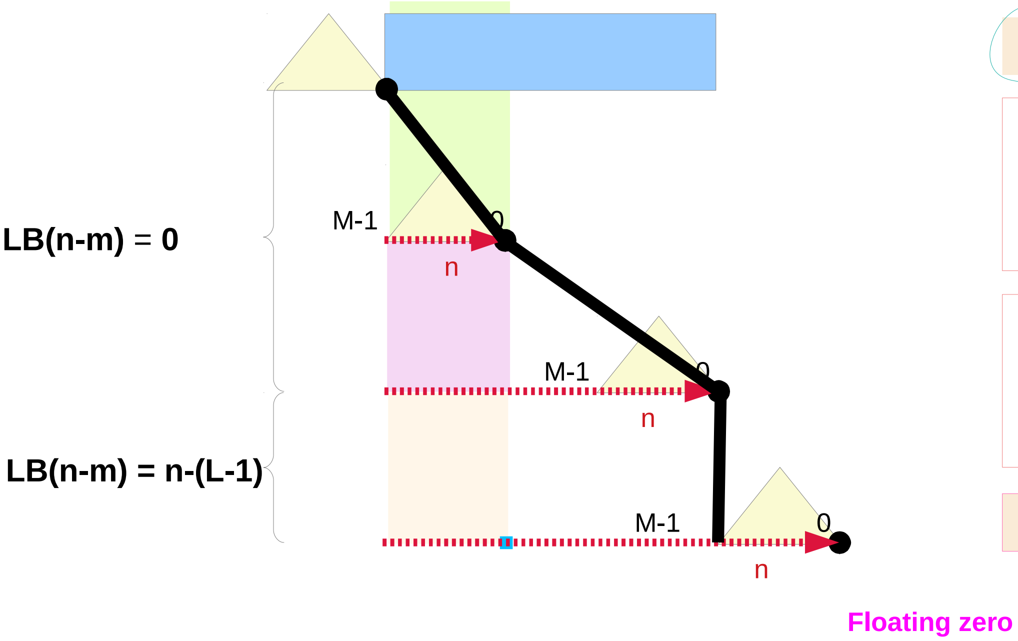
y[n] += x[m] * h[n-m];

Constraint 1 & 3 – UB(**m**) ➡ LB(**n-m**)

Case A

Constrain 3 : **m ∈ [0, L-1]** ➡ n - m : max M-1 ... **0** min (n, M-1)

for **LB(n-m)** values
m should be greatest possible

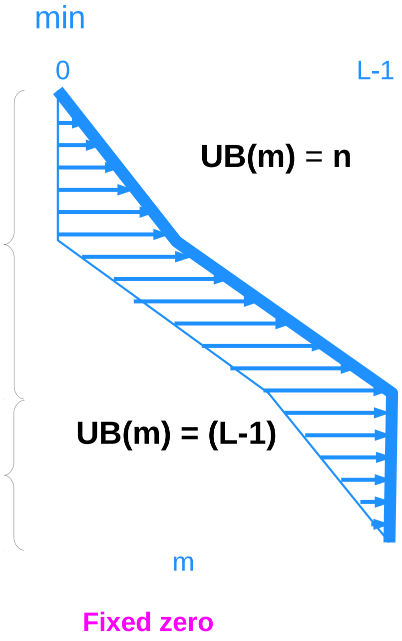


0 ≤ (n-m) ≤ M-1

Case A) **n ≤ M-1**
➡ UB(**m**) = **n**
➡ LB(**n-m**) = **0**

Case B) **n ≥ M**
➡ UB(m) = **L-1**
➡ LB(**n-m**) = **n-(L-1)**

UB(m) = min(n, L-1)



M+L-1 L M

y[n] += x[m] * h[n-m];

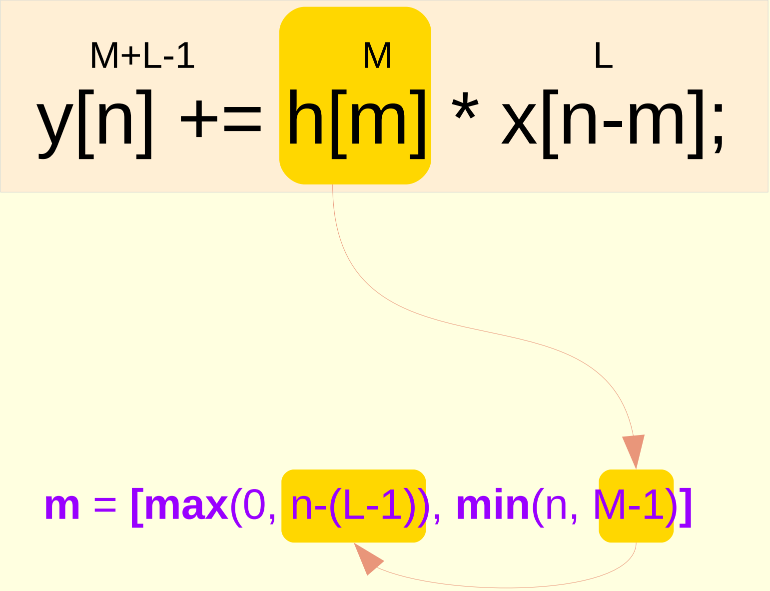
- Reasoning about lower and upper bounds of **m** **Case B**

Case A

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

m = **[max**(0, n-(M-1)), **min**(n, L-1)**]**

Case B



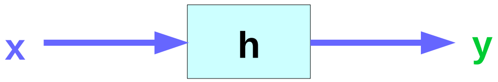
$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

m = **[max**(0, n-(L-1)), **min**(n, M-1)**]**

Index Variable Constraints

M+L-1 M L

`y[n] += h[m] * x[n-m];`



Constraint 1 : **n** ∈ **[0, L+M-2]**

Constraint 2 : **n-m** ∈ **[0, L-1]**

Constraint 3 : **m** ∈ **[0, M-1]**

y[ ] : array with size of **L+M-1**

x[ ] : array with size of **L**

h[ ] : array with size of **M**

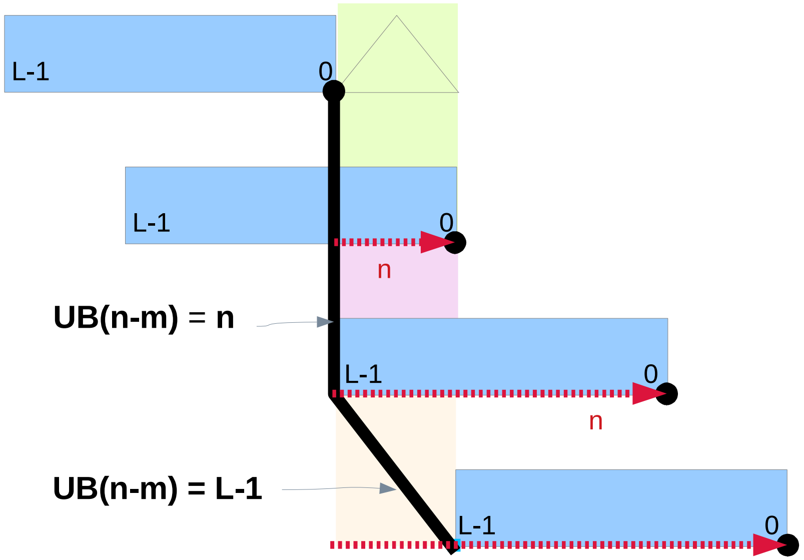
Lower Bound of **m** = max(**0**, n-(L-1))

Case B

UB(n-m)

LB(m) = max(0, n-(L-1))

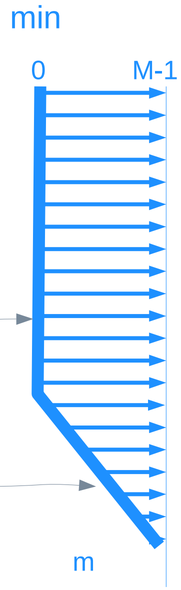
LB(m)



UB(n-m) = n

UB(n-m) = L-1

Floating zero



LB(m) = 0

LB(m) = n-(L-1)

Fixed zero

M+L-1 M L

y[n] += h[m] * x[n-m];

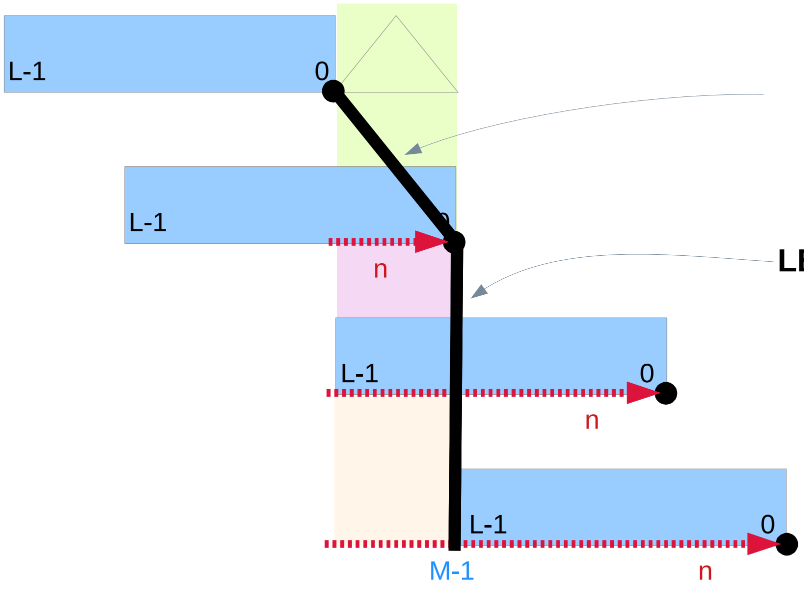
Upper Bound of **m** = min(n, **M-1**)

Case B

LB(n-m)

UB(m) = min(n, M-1)

UB(m)



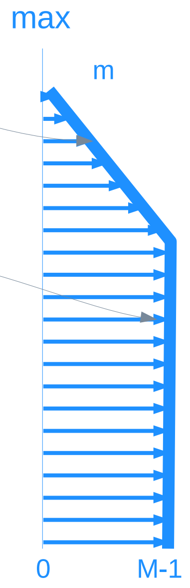
LB(n-m) = 0

LB(n-m) = n-(M-1)

Floating zero

UB(m) = n

UB(m) = M-1



Fixed zero

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

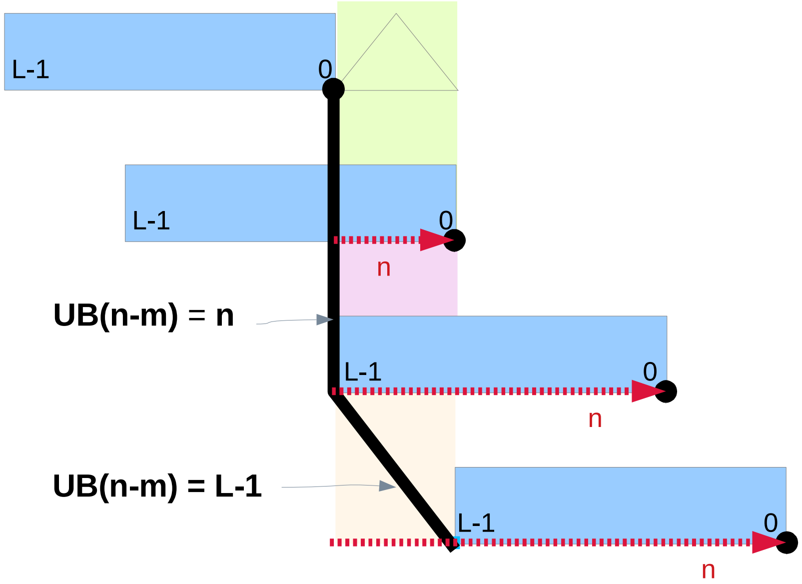
Constraint 1 & 2 – UB(**n-m**) ➡ LB(**m**)

Case B

Constraint 2 : **n-m ∈ [0, L-1]** ➡ n - m : **L-1** ... 0 (0, n+1-L)

max min

for **UB(n-m)** values
m should be least possible

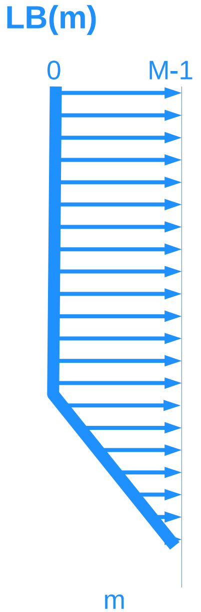


0 ≤ (n-m) ≤ L-1

Case A) n ≤ L-1
➡UB(**n-m**) = **n**
➡LB(**m**) = **0**

Case B) n ≥ L
➡UB(**n-m**) = **L-1**
➡LB(**m**) = **n-(L-1)**

LB(m) = max(0, n-(L-1))



Floating zero

Fixed zero

M+L-1 M L

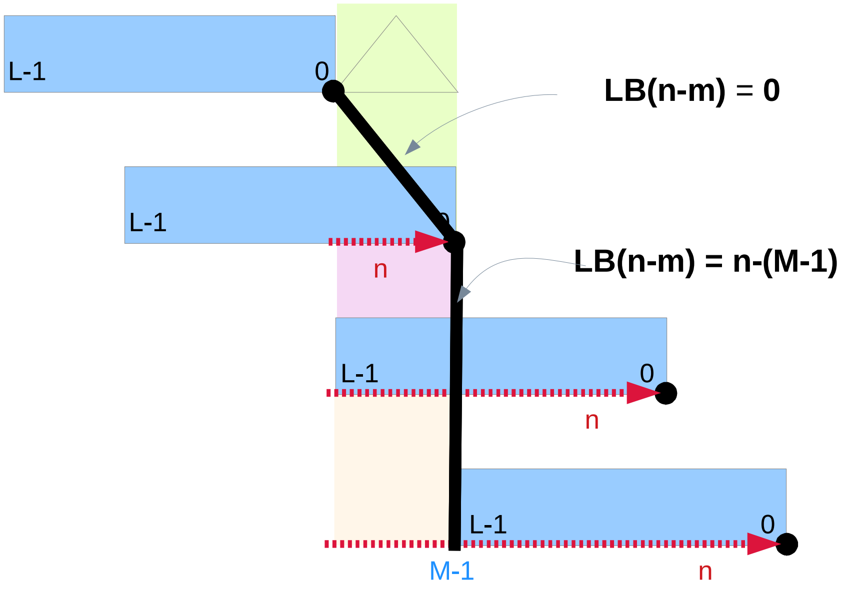
y[n] += h[m] * x[n-m];

Constraint 1 & 3 – UB(**m**) ➡ LB(**n-m**)

Case B

Constrain 3 : **m ∈ [0, M-1]** ➡ n - m : max L-1 ... **0** min (n, M-1)

for **LB(n-m)** values
m should be greatest possible



LB(n-m) = **0**

LB(n-m) = **n-(M-1)**

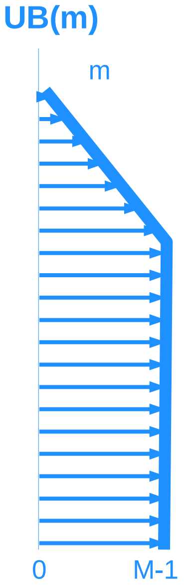
Floating zero

0 ≤ (n-m) ≤ L-1

Case A) n ≤ M-1
➡ UB(**m**) = **n**
➡ LB(**n-m**) = **0**

Case B) n ≥ M
➡ UB(m) = **M-1**
➡ LB(**n-m**) = **n-(M-1)**

UB(m) = min(n, M-1)

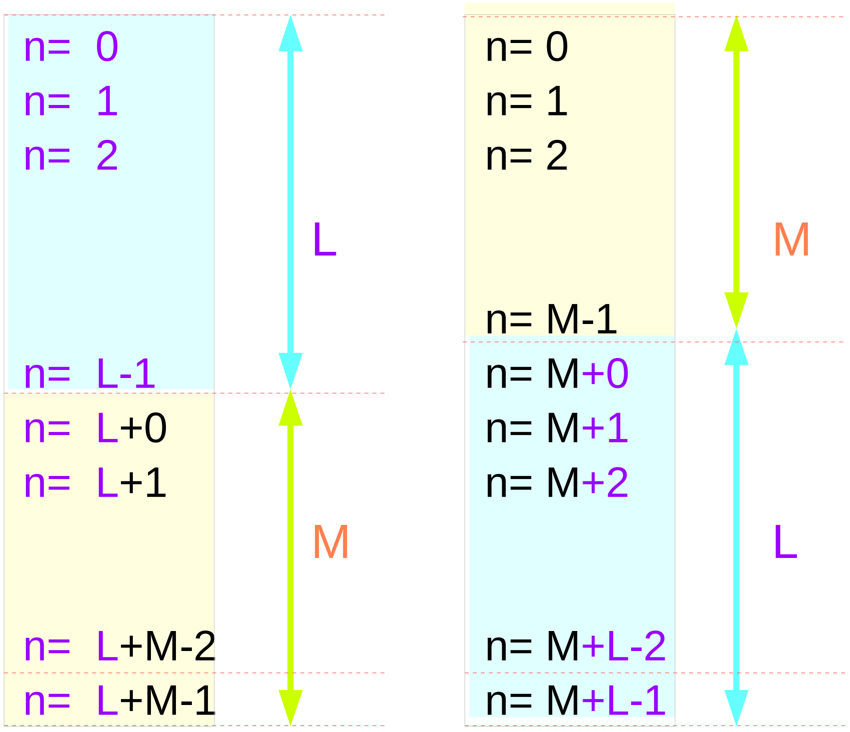


Fixed zero

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

Counting

Constraint 1 :**n ∈ [0, L+M-2]**



n= 0
n= 1
n= 2
n= L-1
n= L+0
n= L+1
n= L+M-2
n= L+M-1

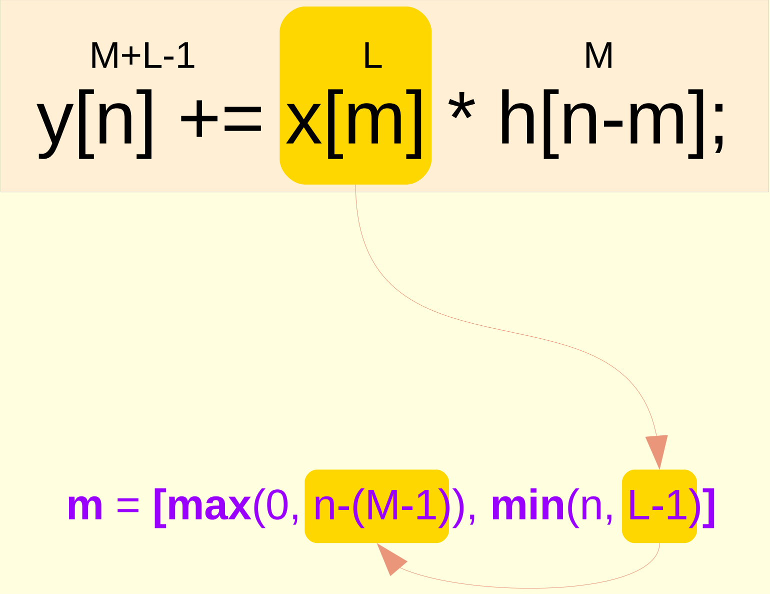
n= 0
n= 1
n= 2
n= M-1
n= M+0
n= M+1
n= M+2
n= M+L-2
n= M+L-1

Counting 1 **Counting 2**

M < L is assumed

len(filter) **<** len(input)

Case A

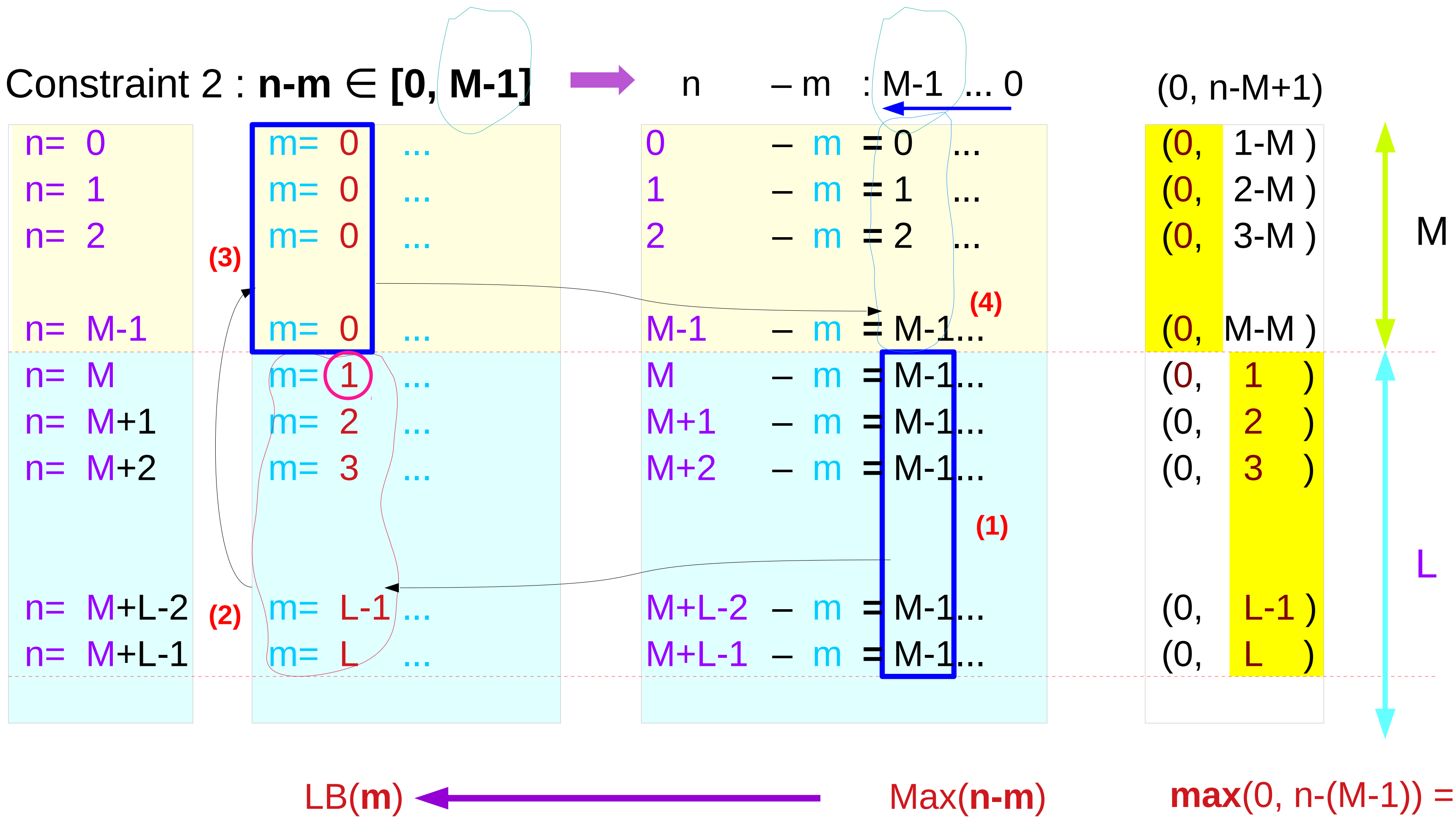


$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

$$\mathbf{m} = \mathbf{[max}(0,\ n-(M-1)),\ \mathbf{min}(n,\ L-1)\mathbf{]}$$

Constraint 1 & 2 – LB(**m**) ⟸ Max(**n-m**)

Case A



Constraint 2 : **n-m** ∈ **[0, M-1]** ⟹ n – m : M-1 ... 0

(0, n-M+1)

n= 0	m= 0 ...	0 – m = 0 ...	(0, 1-M)
n= 1	m= 0 ...	1 – m = 1 ...	(0, 2-M)
n= 2	m= 0 ...	2 – m = 2 ...	(0, 3-M)
n= M-1	m= 0 ...	M-1 – m = M-1...	(0, M-M)
n= M	m= 1 ...	M – m = M-1...	(0, 1)
n= M+1	m= 2 ...	M+1 – m = M-1...	(0, 2)
n= M+2	m= 3 ...	M+2 – m = M-1...	(0, 3)
n= M+L-2	m= L-1 ...	M+L-2 – m = M-1...	(0, L-1)
n= M+L-1	m= L ...	M+L-1 – m = M-1...	(0, L)

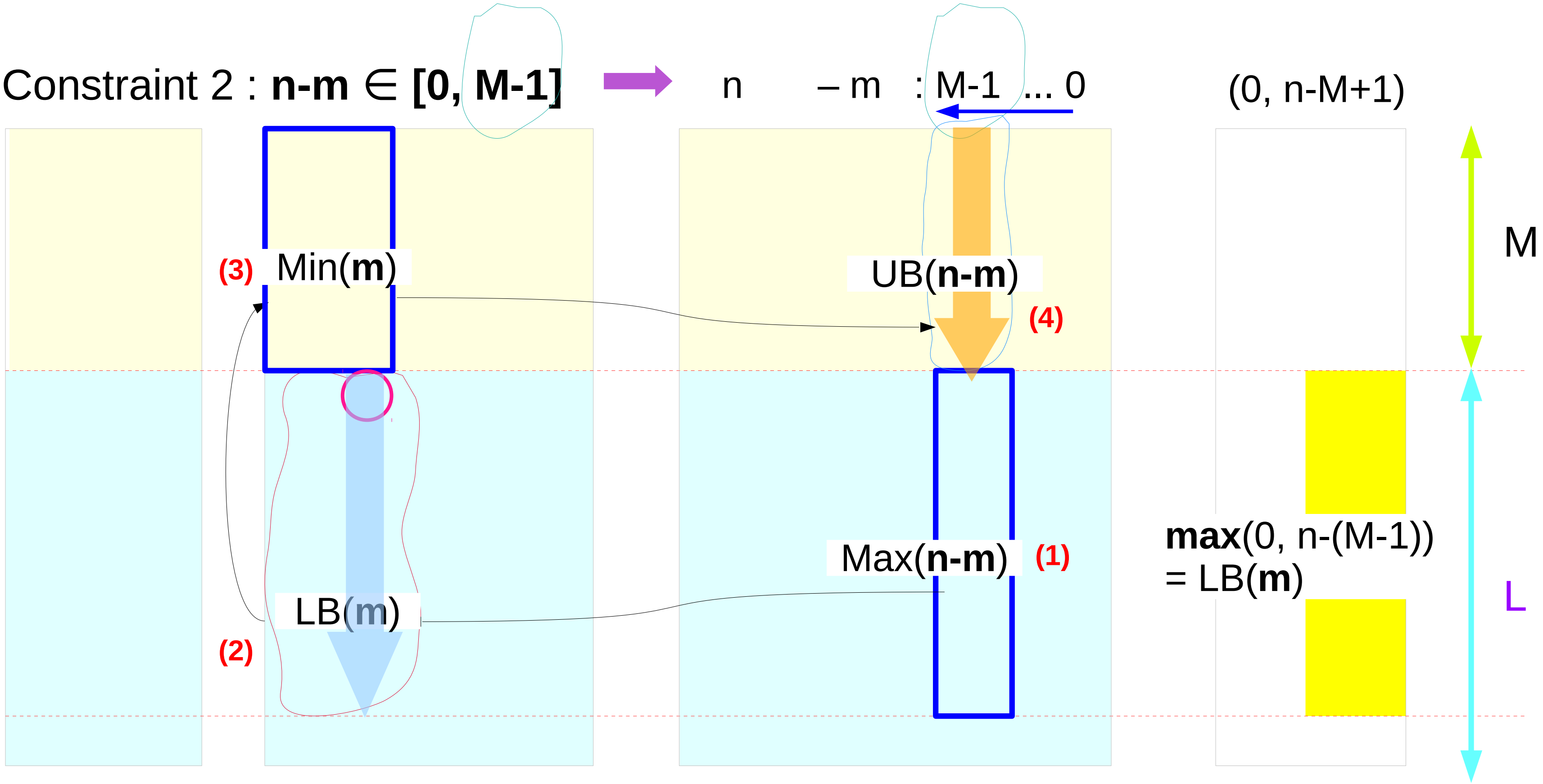
LB(**m**) ⟵ Max(**n-m**)

max(0, n-(M-1)) = LB(**m**)

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

Summary – LB(**m**) ⇐ Max(**n-m**)

Case A



Constraint 2 : **n-m** ∈ **[0, M-1]** ➡ n – m : M-1 ... 0

(0, n-M+1)

(3) Min(**m**)

UB(**n-m**)

(4)

M

Max(**n-m**) (1)

max(0, n-(M-1))
= LB(**m**)

LB(**m**)

(2)

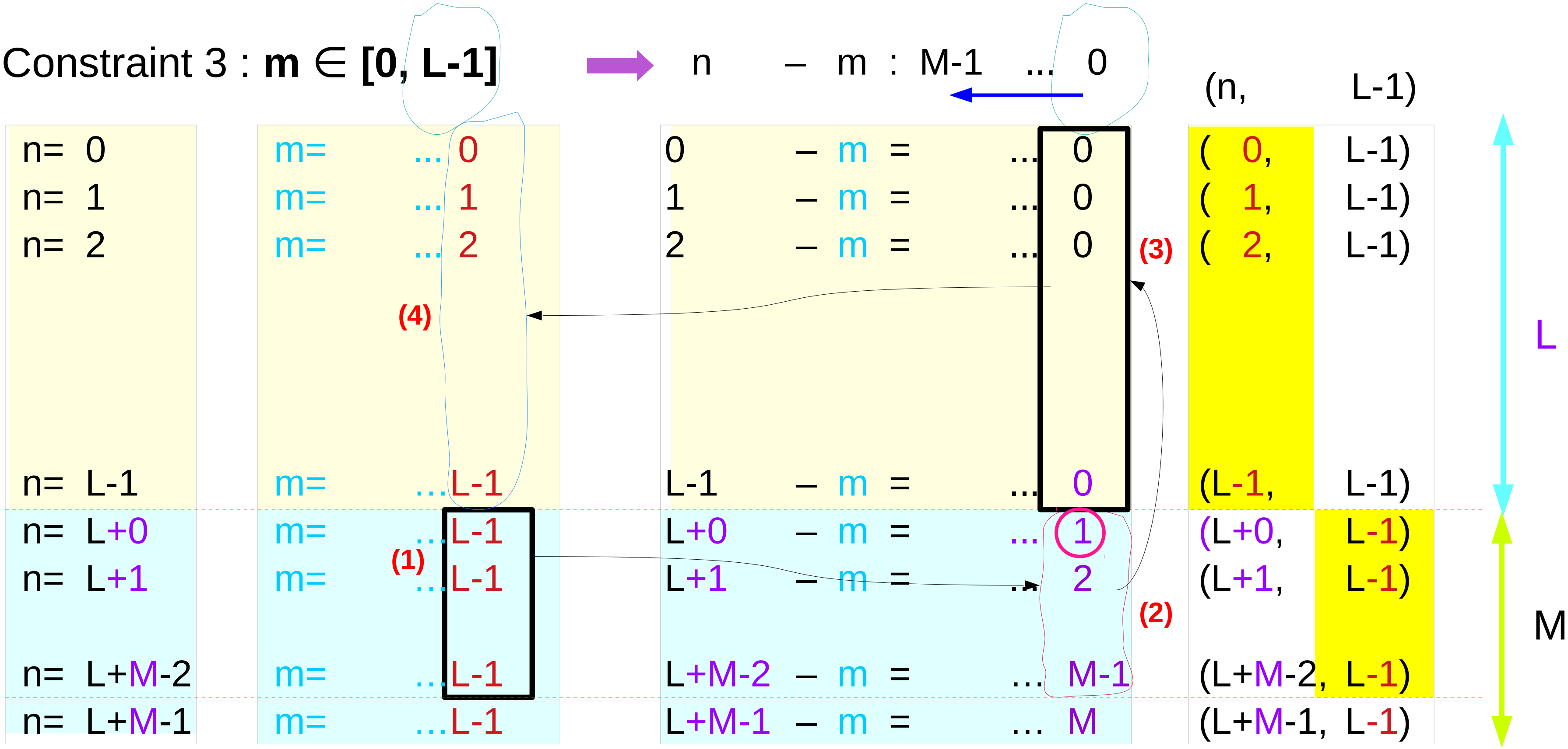
L

M+L-1 L M

y[n] += x[m] * h[n-m];

Constraint 1 & 3 – UB(**m**) ⟵ Min(**n-m**)

Case A



Constraint 3 : **m** ∈ **[0, L-1]** ⟹ n – m : M-1 ... 0

n= 0	m= ... 0	0 – m = ... 0	(0, L-1)
n= 1	m= ... 1	1 – m = ... 0	(1, L-1)
n= 2	m= ... 2	2 – m = ... 0	(2, L-1)
n= L-1	m= ...L-1	L-1 – m = ... 0	(L-1, L-1)
n= L+0	m= ...L-1	L+0 – m = ... 1	(L+0, L-1)
n= L+1	m= ...L-1	L+1 – m = ... 2	(L+1, L-1)
n= L+M-2	m= ...L-1	L+M-2 – m = ... M-1	(L+M-2, L-1)
n= L+M-1	m= ...L-1	L+M-1 – m = ... M	(L+M-1, L-1)

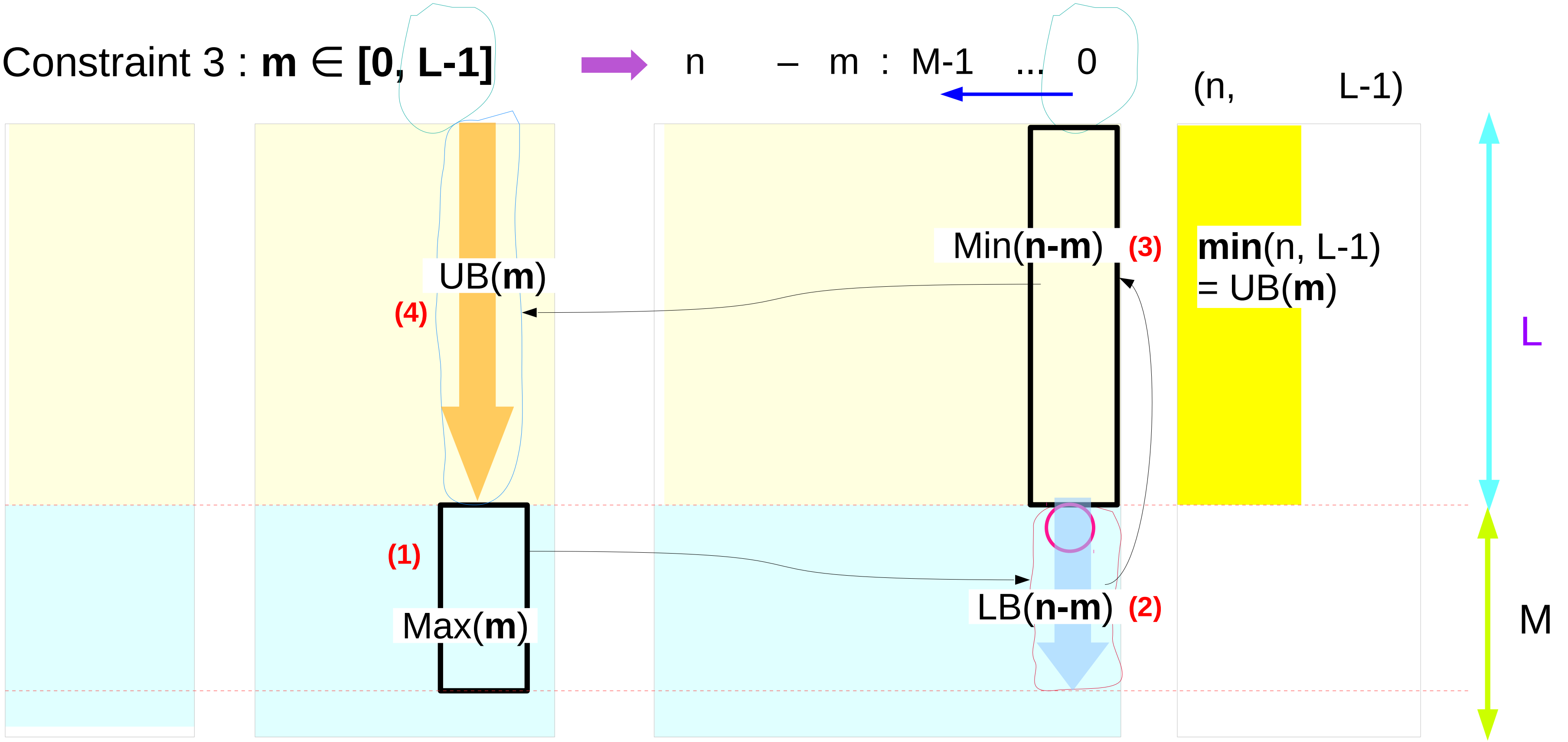
UB(**m**) ⟵ Min(**n-m**)   **min**(n, L-1) = UB(**m**)

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

Summary – UB(**m**) ⟵ Min(**n-m**)

Case A

Constraint 3 : **m** ∈ **[0, L-1]** ➡ n – m : M-1 ... 0



M+L-1 L M

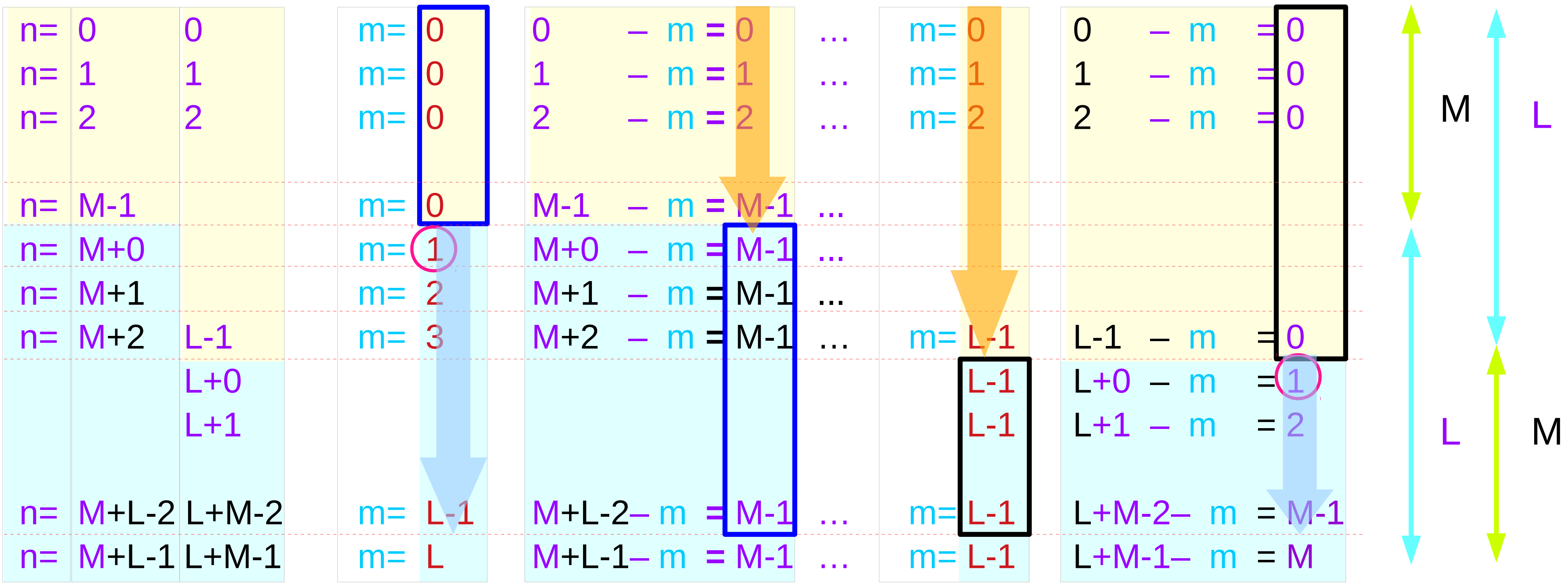
y[n] += x[m] * h[n-m];

Constraint 1, 2, 3 – UB(**m**) and LB(**m**)

Case A

Constraint 2 : **n-m ∈ [0, M-1]** ➡ n – m : M-1 … 0

Constraint 3 : **m ∈ [0, L-1]** ➡ n – m : M-1 … 0



n=	0	0	m= 0	0 – m = 0	…	m= 0	0 – m = 0
n=	1	1	m= 0	1 – m = 1	…	m= 1	1 – m = 0
n=	2	2	m= 0	2 – m = 2	…	m= 2	2 – m = 0
n=	M-1		m= 0	M-1 – m = M-1	...		
n=	M+0		m= 1	M+0 – m = M-1	...		
n=	M+1		m= 2	M+1 – m = M-1	...		
n=	M+2	L-1	m= 3	M+2 – m = M-1	…	m= L-1	L-1 – m = 0
		L+0				L-1	L+0 – m = 1
		L+1				L-1	L+1 – m = 2
n=	M+L-2	L+M-2	m= L-1	M+L-2– m = M-1	…	m= L-1	L+M-2– m = M-1
n=	M+L-1	L+M-1	m= L	M+L-1– m = M-1	…	m= L-1	L+M-1– m = M

LB(**m**) = max(0, n-(M-1))

UB(**m**) = min(n, L-1)

m = [**max**(0, n-(M-1)), **min**(n, L-1)]

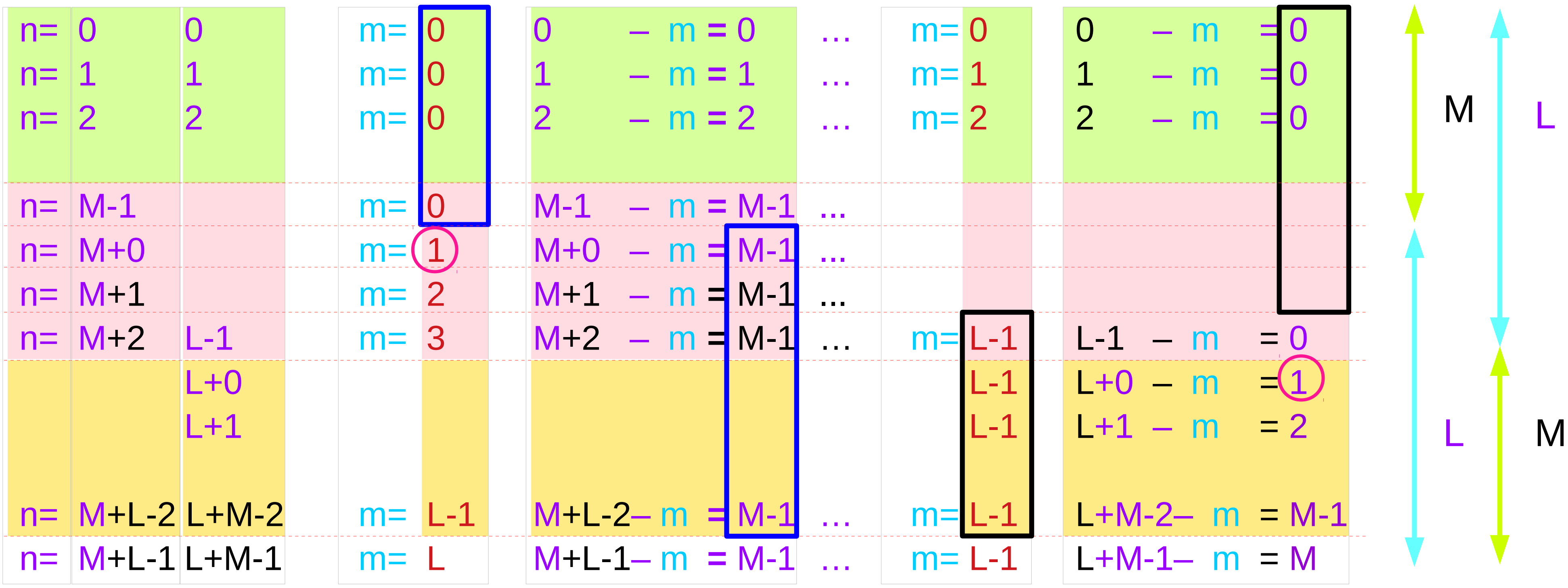
$$\overset{M+L-1}{y[n]} += \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

Constraint 1, 2, 3 – UB(**m**) and LB(**m**)

Case A

Constraint 2 : **n-m** ∈ **[0, M-1]** ➡ n – m : M-1 … 0

Constraint 3 : **m** ∈ **[0, L-1]** ➡ n – m : M-1 … 0



n=	0	0	m= 0	0 – m = 0	…	m= 0	0 – m = 0
n=	1	1	m= 0	1 – m = 1	…	m= 1	1 – m = 0
n=	2	2	m= 0	2 – m = 2	…	m= 2	2 – m = 0
n=	M-1		m= 0	M-1 – m = M-1	...		
n=	M+0		m= 1	M+0 – m = M-1	...		
n=	M+1		m= 2	M+1 – m = M-1	...		
n=	M+2	L-1	m= 3	M+2 – m = M-1	…	m= L-1	L-1 – m = 0
		L+0				L-1	L+0 – m = 1
		L+1				L-1	L+1 – m = 2
n=	M+L-2	L+M-2	m= L-1	M+L-2– m = M-1	…	m= L-1	L+M-2– m = M-1
n=	M+L-1	L+M-1	m= L	M+L-1– m = M-1	…	m= L-1	L+M-1– m = M

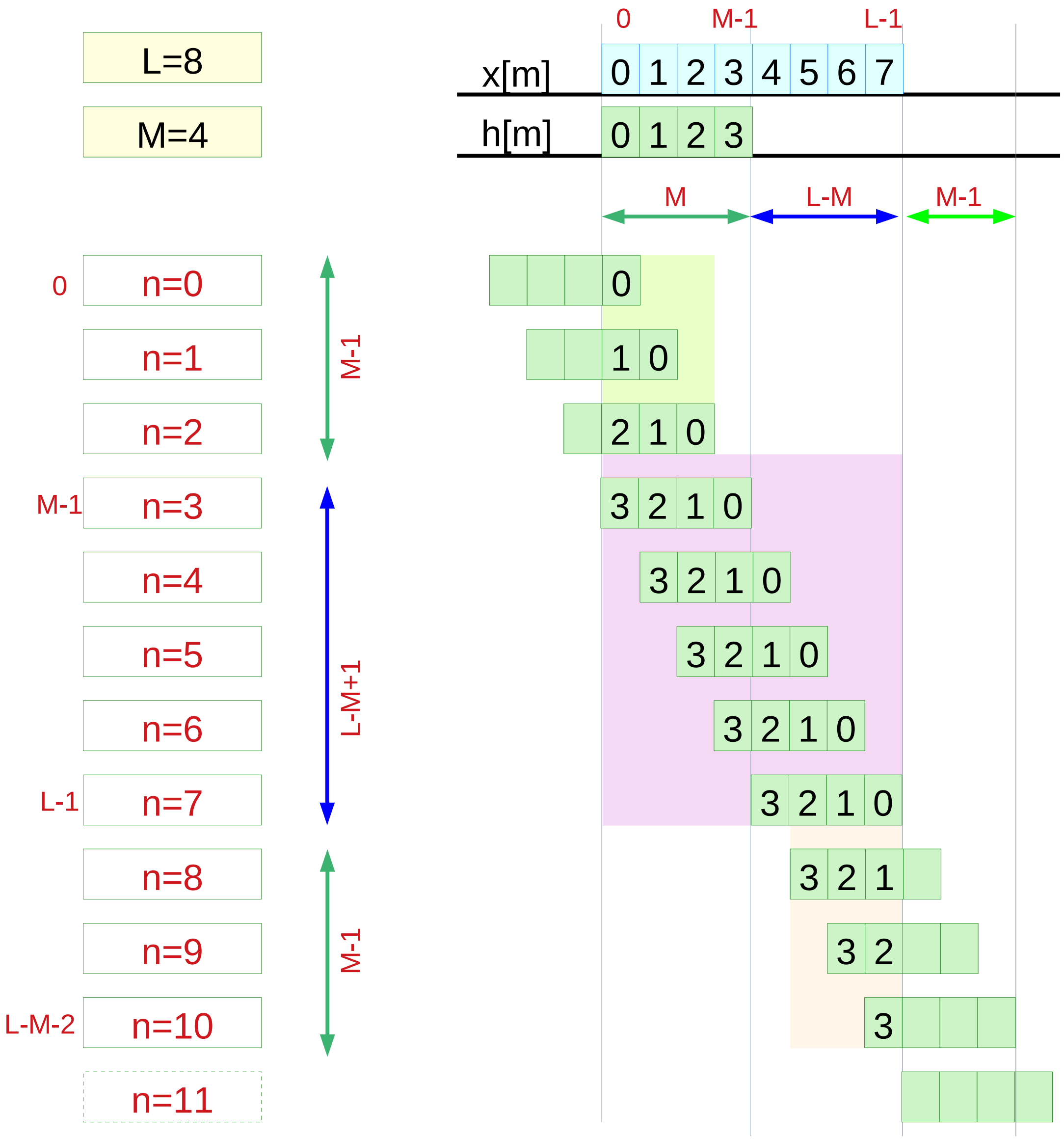
LB(**m**) = max(0, n-(M-1)) UB(**m**) = min(n, L-1)

m = **[max**(0, n-(M-1)), **min**(n, L-1)**]**

$$y[\overset{M+L-1}{n}] += x[\overset{L}{m}] * h[\overset{M}{n-m}];$$

Valid index set example

Case A



$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{L}{x[m]} * \overset{M}{h[n-m]};$$

n=0 m=0
----
n=1 m=0
n=1 m=1
----
n=2 m=0
n=2 m=1
n=2 m=2
----
n=3 m=0
n=3 m=1
n=3 m=2
n=3 m=3
----

n=4 m=0
n=4 m=1
n=4 m=2
n=4 m=3
----
n=5 m=0
n=5 m=1
n=5 m=2
n=5 m=3
----
n=6 m=0
n=6 m=1
n=6 m=2
n=6 m=3
----

n=7 m=0
n=7 m=1
n=7 m=2
n=7 m=3
----
n=8 m=1
n=8 m=2
n=8 m=3
----
n=9 m=2
n=9 m=3
----
n=10 m=3
----

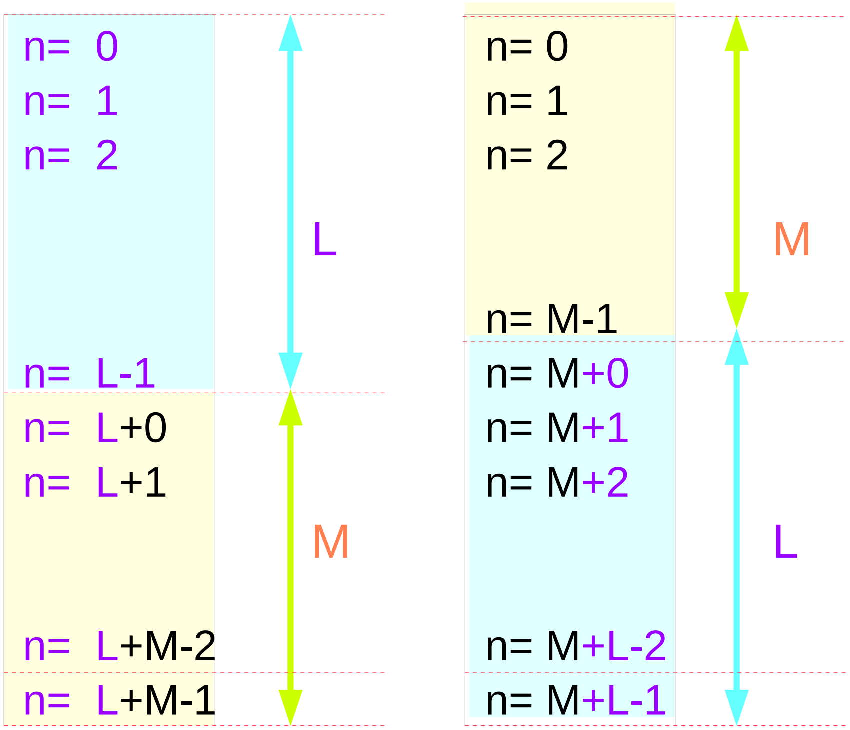
m ∈ [0, L-1]
n-m ∈ [0, M-1]

n ∈ [0, L+M-2]

Counting

Case B

Constraint 1 :**n ∈ [0, L+M-2]**



n= 0
n= 1
n= 2
n= L-1
n= L+0
n= L+1
n= L+M-2
n= L+M-1

L

M

n= 0
n= 1
n= 2
n= M-1
n= M+0
n= M+1
n= M+2
n= M+L-2
n= M+L-1

M

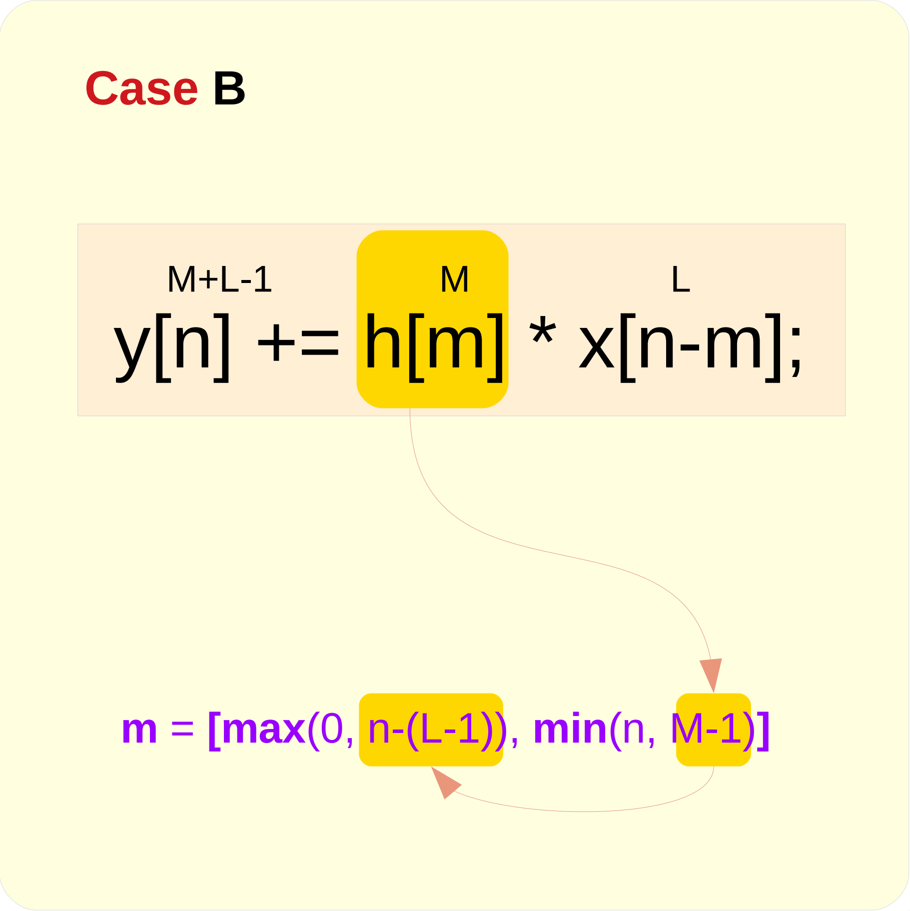
L

Counting 1

Counting 2

M < L is assumed

len(filter) **<** len(input)



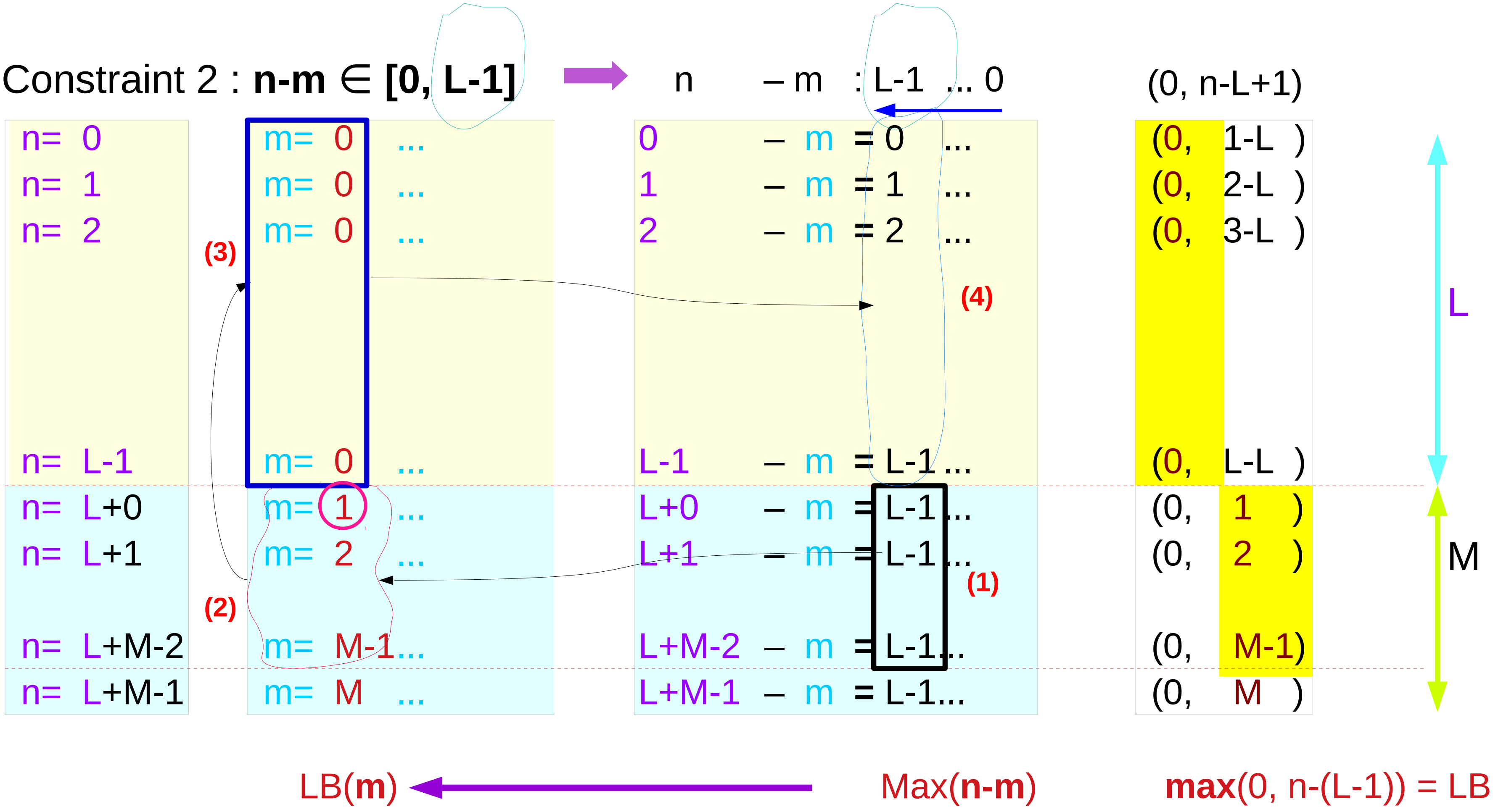
Case B

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

m = [max(0, n-(L-1)), **min**(n, M-1)**]**

Constraint 1 & 2 – LB(**m**) ⇐ Max(**n-m**)

Case B



Constraint 2 : **n-m** ∈ **[0, L-1]** ⇒ n – m : L-1 ... 0 (0, n-L+1)

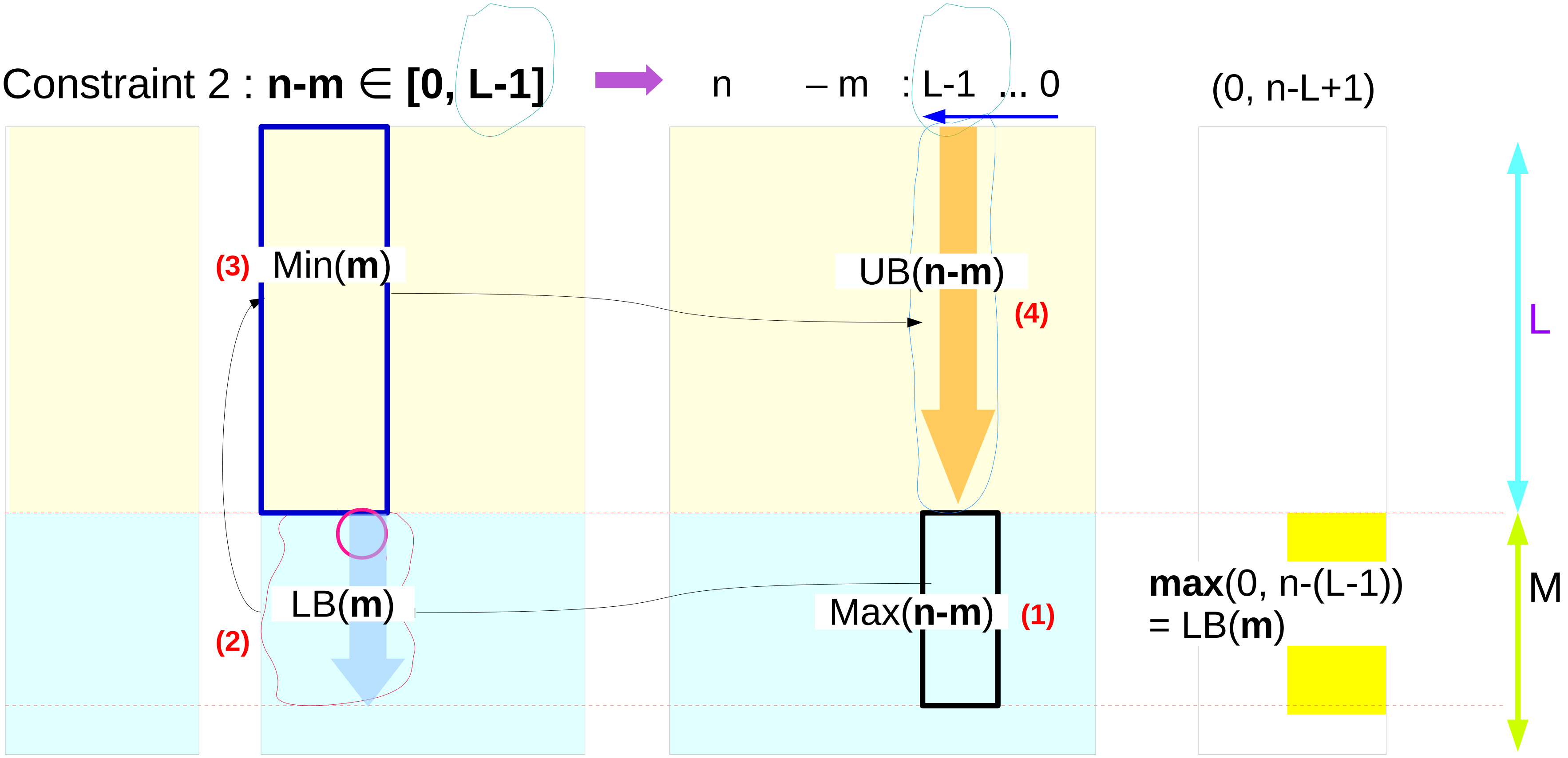
n= 0	m= 0 ...	0 – m = 0 ...	(0, 1-L)
n= 1	m= 0 ...	1 – m = 1 ...	(0, 2-L)
n= 2	m= 0 ...	2 – m = 2 ...	(0, 3-L)
n= L-1	m= 0 ...	L-1 – m = L-1 ...	(0, L-L)
n= L+0	m= 1 ...	L+0 – m = L-1 ...	(0, 1)
n= L+1	m= 2 ...	L+1 – m = L-1 ...	(0, 2)
n= L+M-2	m= M-1 ...	L+M-2 – m = L-1 ...	(0, M-1)
n= L+M-1	m= M ...	L+M-1 – m = L-1 ...	(0, M)

LB(**m**) ← Max(**n-m**) **max**(0, n-(L-1)) = LB(**m**)

$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

Summary – LB(**m**) ⟸ Max(**n-m**) Case **B**

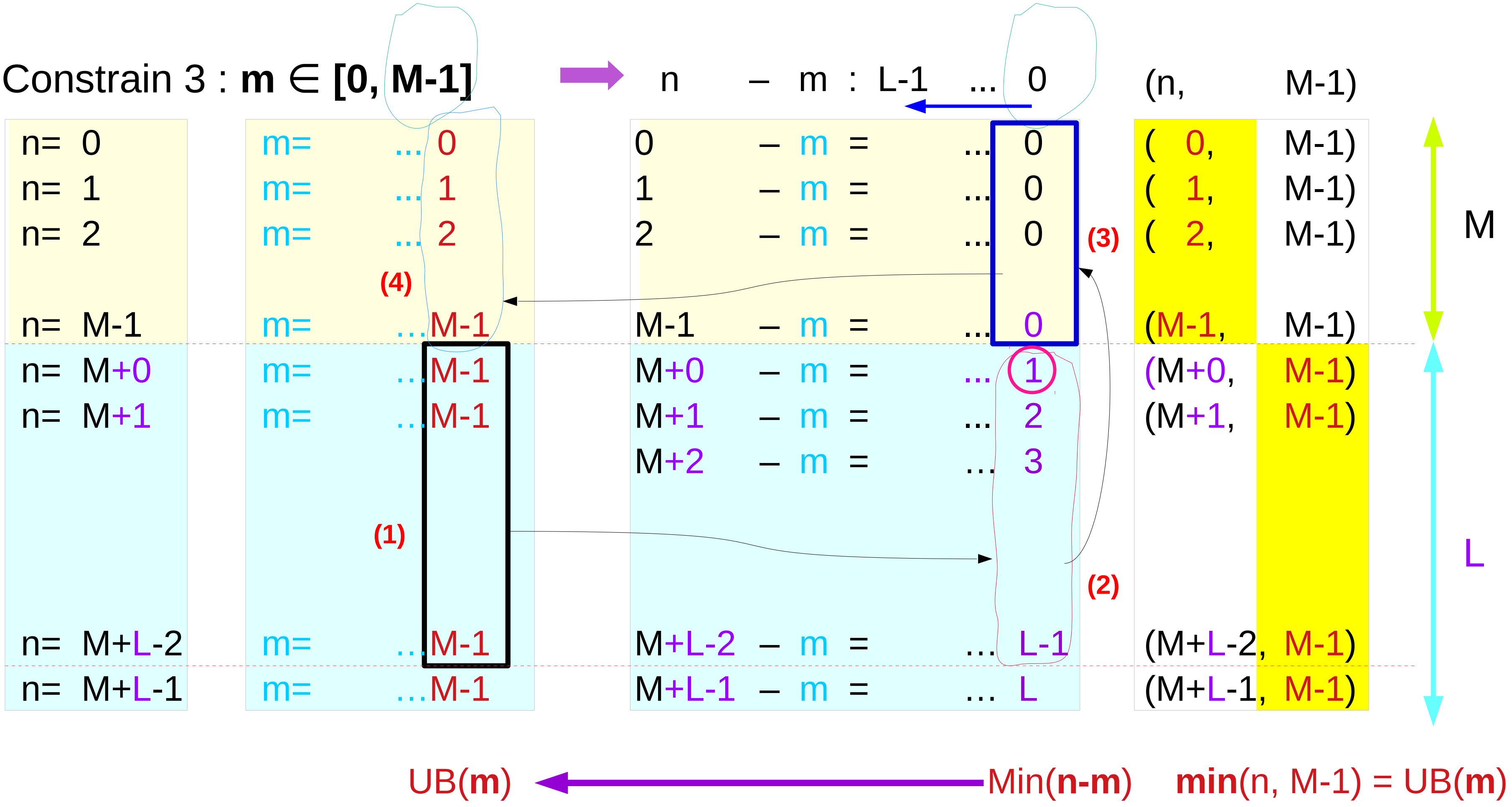
Constraint 2 : **n-m** ∈ **[0, L-1]** ⟹ n – m : L-1 ... 0 (0, n-L+1)



$$\overset{M+L-1}{y[n]} \mathrel{+}= \overset{M}{h[m]} * \overset{L}{x[n-m]};$$

Constraint 1 & 3 – UB(m) ⟵ Min(n-m)

Case B



Constrain 3 : **m ∈ [0, M-1]** ⟶

		n – m : L-1 ... 0	(n, M-1)
n= 0	m= ... 0	0 – m = ... 0	(0, M-1)
n= 1	m= ... 1	1 – m = ... 0	(1, M-1)
n= 2	m= ... 2	2 – m = ... 0	(2, M-1)
n= M-1	m= ...M-1	M-1 – m = ... 0	(M-1, M-1)
n= M+0	m= ...M-1	M+0 – m = ... 1	(M+0, M-1)
n= M+1	m= ...M-1	M+1 – m = ... 2	(M+1, M-1)
		M+2 – m = ... 3	
n= M+L-2	m= ...M-1	M+L-2 – m = ... L-1	(M+L-2, M-1)
n= M+L-1	m= ...M-1	M+L-1 – m = ... L	(M+L-1, M-1)

UB(**m**) ⟵ Min(**n-m**) **min**(n, M-1) = UB(**m**)

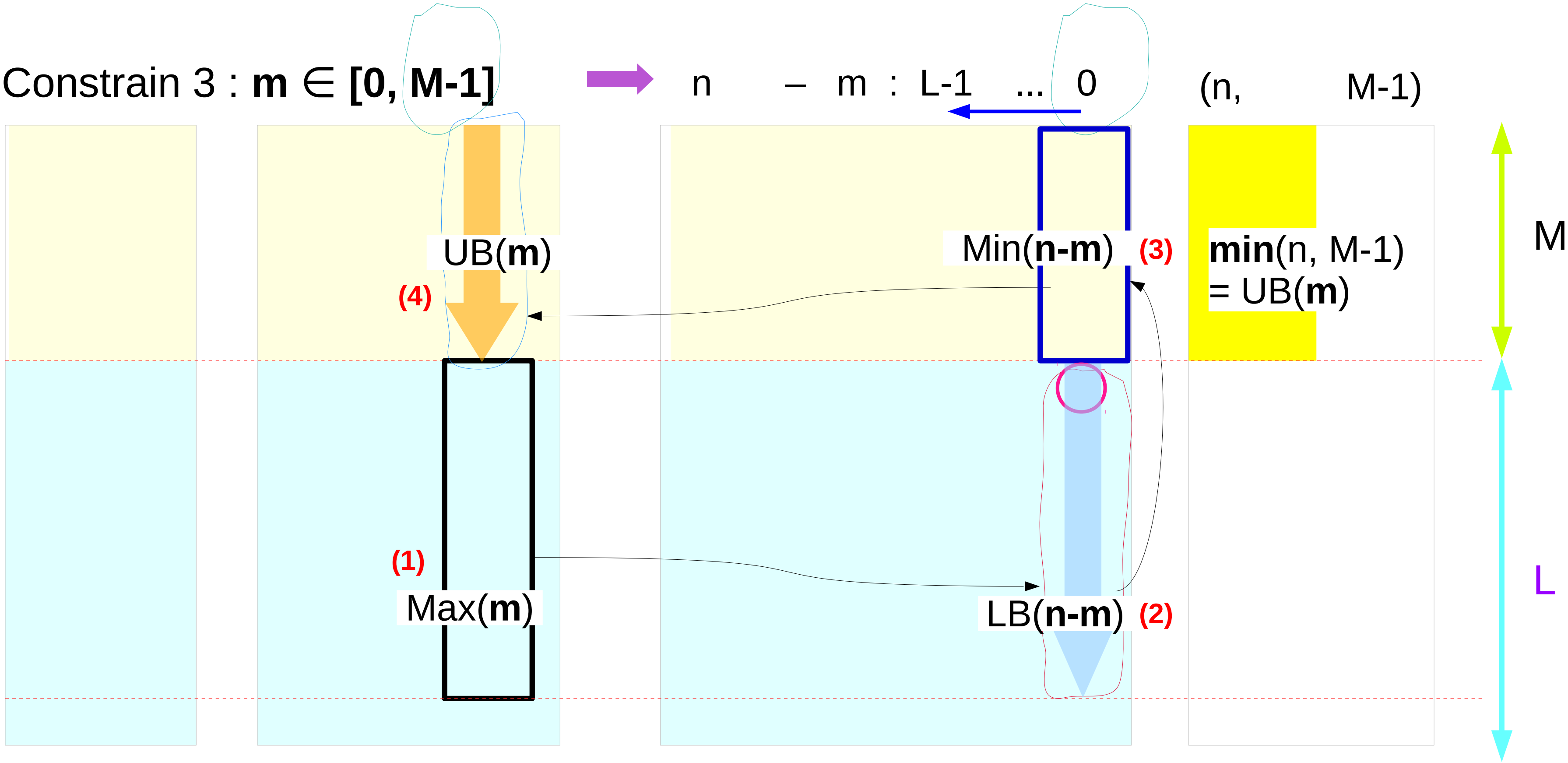
M+L-1 M L

y[n] += h[m] * x[n-m];

Summary – UB(**m**) ⇐ Min(**n-m**)

Case B

Constrain 3 : **m** ∈ **[0, M-1]** ➡ n – m : L-1 ... 0 (n, M-1)



M+L-1 M L

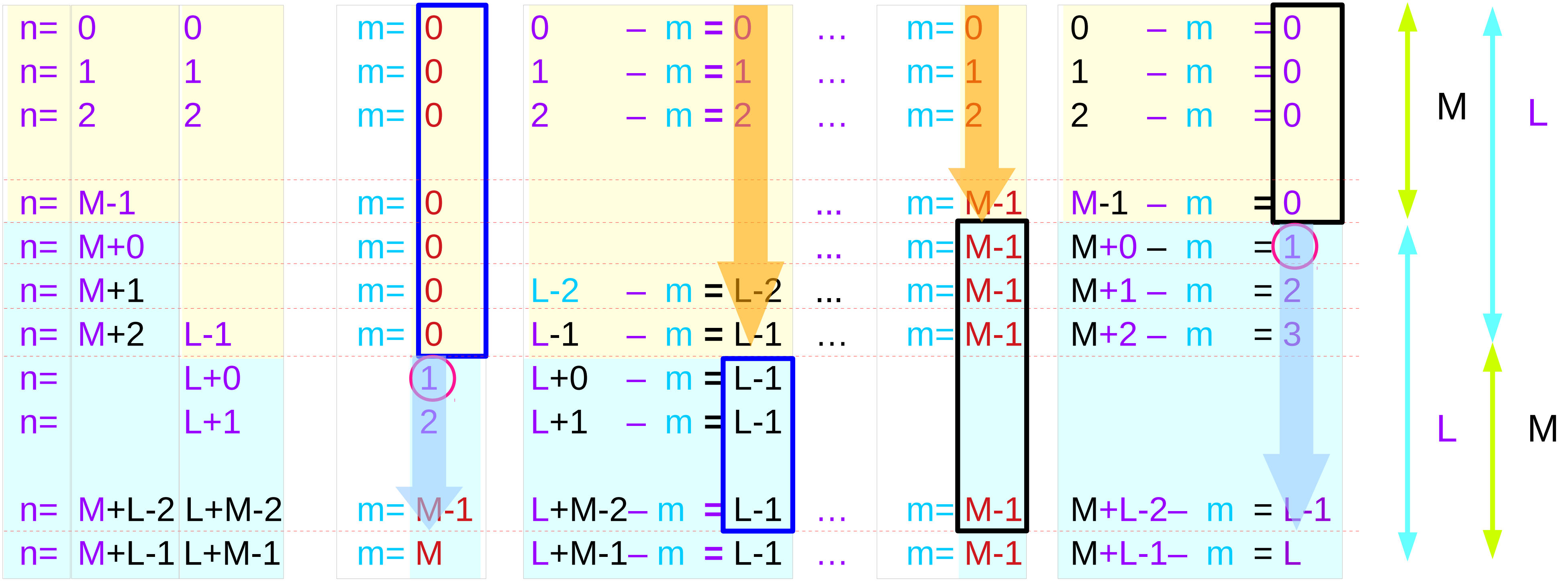
y[n] += h[m] * x[n-m];

Constraint 1, 2, 3 – max **m** and min **m**

Case B

Constraint 2 : **n-m** ∈ **[0, L-1]** ➡ n – m : L-1 … 0

Constraint 3 : **m** ∈ **[0, M-1]** ➡ n – m : L-1 … 0



n=	0	0	m= 0	0 – m = 0	…	m= 0	0 – m = 0
n=	1	1	m= 0	1 – m = 1	…	m= 1	1 – m = 0
n=	2	2	m= 0	2 – m = 2	…	m= 2	2 – m = 0
n=	M-1		m= 0		…	m= M-1	M-1 – m = 0
n=	M+0		m= 0		…	m= M-1	M+0 – m = 1
n=	M+1		m= 0	L-2 – m = L-2	…	m= M-1	M+1 – m = 2
n=	M+2	L-1	m= 0	L-1 – m = L-1	…	m= M-1	M+2 – m = 3
n=		L+0	1	L+0 – m = L-1			
n=		L+1	2	L+1 – m = L-1			
n=	M+L-2	L+M-2	m= M-1	L+M-2– m = L-1	…	m= M-1	M+L-2– m = L-1
n=	M+L-1	L+M-1	m= M	L+M-1– m = L-1	…	m= M-1	M+L-1– m = L

LB(**m**) = max(0, n-(L-1)) UB(**m**) = min(n, M-1)

m = [**max**(0, n-(L-1)), **min**(n, M-1)]

M+L-1 M L

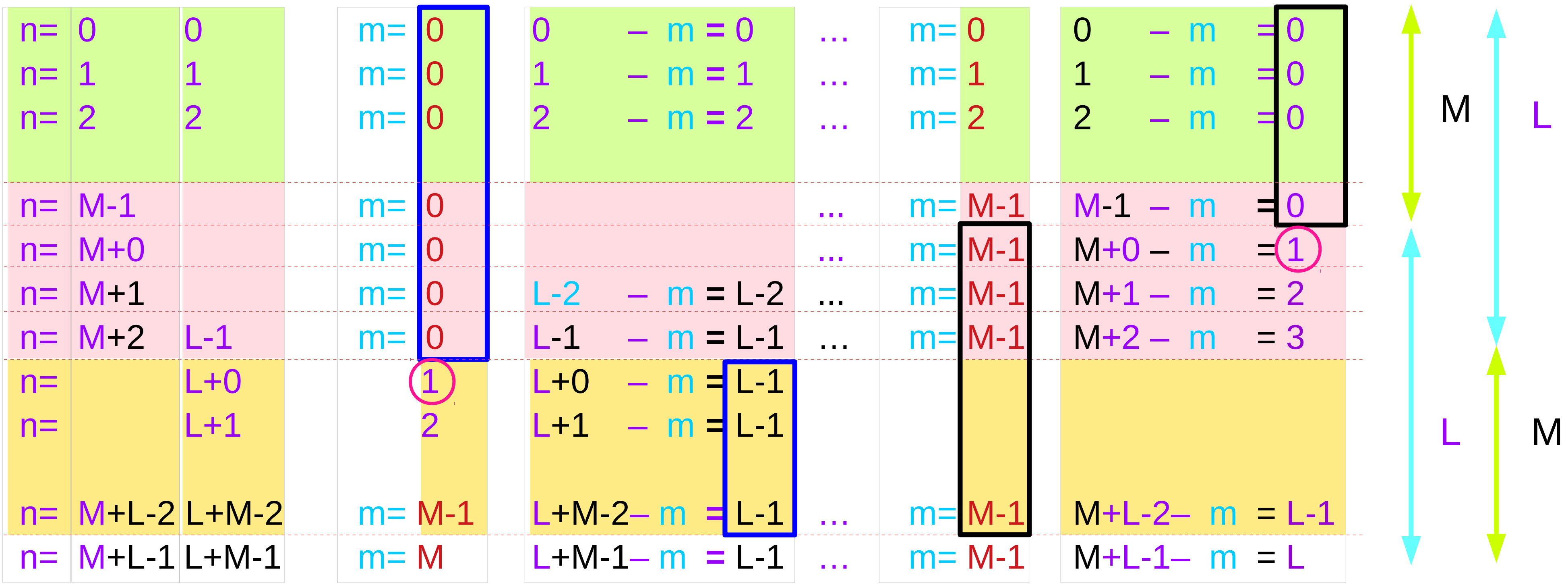
y[n] += h[m] * x[n-m];

Constraint 1, 2, 3 – max **m** and min **m**

Case B

Constraint 2 : **n-m** ∈ **[0, L-1]** ➡ n – m : L-1 … 0

Constraint 3 : **m** ∈ **[0, M-1]** ➡ n – m : L-1 … 0



n=	0	0	m= 0	0 – m = 0	…	m= 0	0 – m = 0
n=	1	1	m= 0	1 – m = 1	…	m= 1	1 – m = 0
n=	2	2	m= 0	2 – m = 2	…	m= 2	2 – m = 0
n=	M-1		m= 0		…	m= M-1	M-1 – m = 0
n=	M+0		m= 0		…	m= M-1	M+0 – m = 1
n=	M+1		m= 0	L-2 – m = L-2	…	m= M-1	M+1 – m = 2
n=	M+2	L-1	m= 0	L-1 – m = L-1	…	m= M-1	M+2 – m = 3
n=		L+0	1	L+0 – m = L-1			
n=		L+1	2	L+1 – m = L-1			
n=	M+L-2	L+M-2	m= M-1	L+M-2– m = L-1	…	m= M-1	M+L-2– m = L-1
n=	M+L-1	L+M-1	m= M	L+M-1– m = L-1	…	m= M-1	M+L-1– m = L

LB(**m**) = max(0, n-(L-1)) UB(**m**) = min(n, M-1)

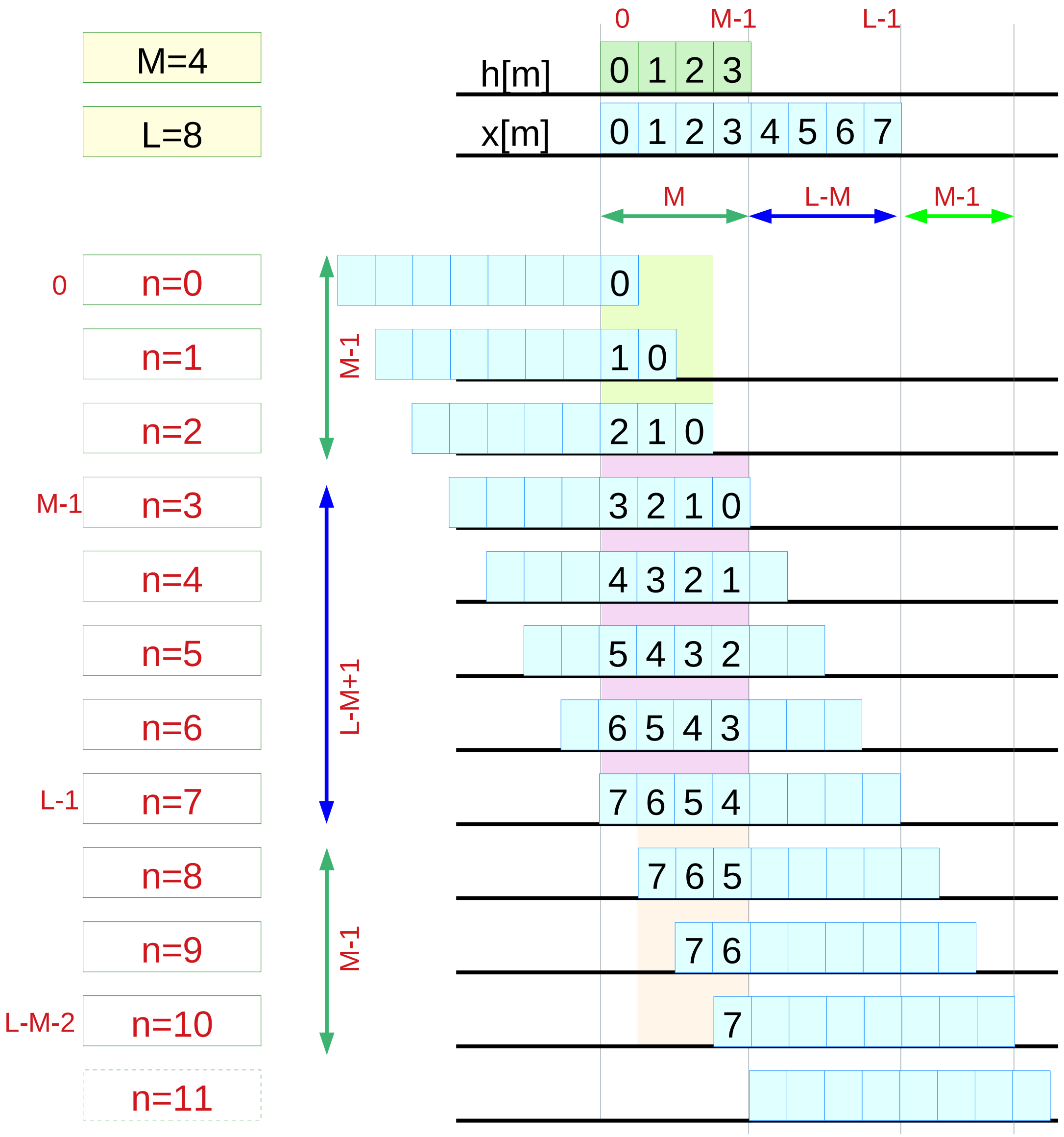
m = **[max**(0, n-(L-1)), **min**(n, M-1)**]**

M+L-1 M L

y[n] += h[m] * x[n-m];

Valid index set example

Case B



M+L-1 M L

y[n] += h[m] * x[n-m];

n=0 m=0
----
n=1 m=0
n=1 m=1
----
n=2 m=0
n=2 m=1
n=2 m=2
----
n=3 m=0
n=3 m=1
n=3 m=2
n=3 m=3
----

n=4 m=0
n=4 m=1
n=4 m=2
n=4 m=3
----
n=5 m=0
n=5 m=1
n=5 m=2
n=5 m=3
----
n=6 m=0
n=6 m=1
n=6 m=2
n=6 m=3
----

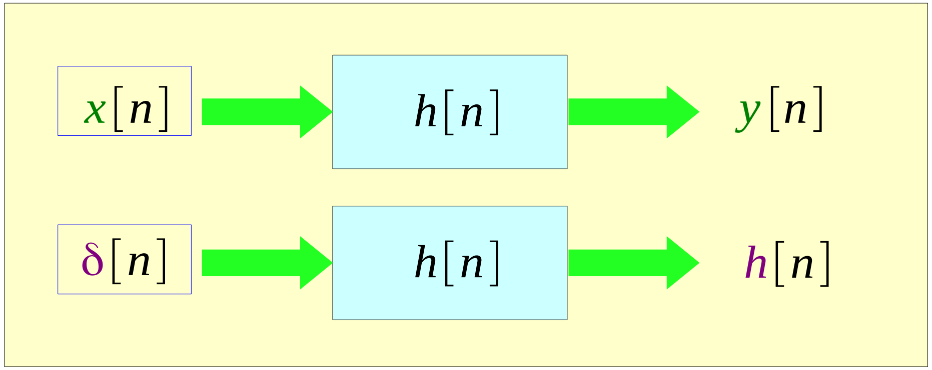
n=7 m=0
n=7 m=1
n=7 m=2
n=7 m=3
----
n=8 m=1
n=8 m=2
n=8 m=3
----
n=9 m=2
n=9 m=3
----
n=10 m=3

$\mathbf{m \in [0, M-1]}$
$\mathbf{n-m \in [0, L-1]}$

$\mathbf{n \in [0, L+M-2]}$

Impulse Response

Discrete Time LTI System

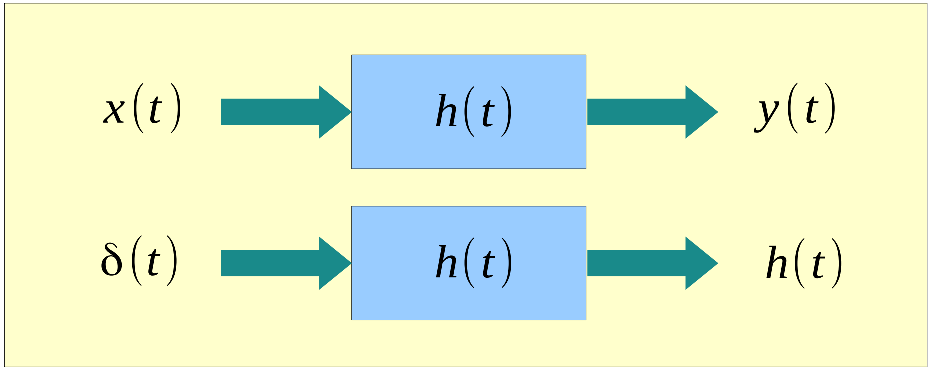


$$\boxed{x[n]} \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$\boxed{\delta[n]} \rightarrow \boxed{h[n]} \rightarrow h[n]$$

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m]$$

Continuous Time LTI System



$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

$$\delta(t) \rightarrow \boxed{h(t)} \rightarrow h(t)$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\,x(t-\tau)\,d\tau$$

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = \boxed{x[n]}$$

$$a_n h[n] + a_{n-1} h[n-1] + \cdots + a_{n-N} h[n-N] = \boxed{\delta[n]}$$

The most general form of
a Discrete Time LTI System

$$h[n] = \frac{1}{a_n}(\delta[n] - a_{n-1} h[n-1] - \cdots - a_{n-N} h[n-N])$$

Convolution Sum

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$

$$y[n] = \cdots\ x[\overset{m=-1}{\underline{-1}}]\,h[\underline{n+1}] + x[\overset{m=0}{\underline{0}}]\,h[\underline{n}] + x[\overset{m=1}{\underline{1}}]\,h[\underline{n-1}] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m]$$

$-1+n+1=n$ $\quad 0+n-0=n$ $\quad +1+n-1=n$

$$y[n] = \cdots\ x[\underline{n+1}]\,h[\overset{m=-1}{\underline{-1}}] + x[\underline{n}]\,h[\overset{m=0}{\underline{0}}] + x[\underline{n-1}]\,h[\overset{m=1}{\underline{1}}] = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$-1+n+1=n$ $\quad 0+n-0=n$ $\quad +1+n-1=n$

Difference Equation

$$a_n \underline{y[n]} + \underbrace{a_{n-1}\underline{y[n-1]} + \cdots + a_{n-N}\underline{y[n-N]}} = b_n x[n] + b_{n-1}x[n-1] + \cdots + b_{n-M}x[n-M]$$

present
output

past outputs
Feedback
recursive

$$a_n y[n] = b_n x[n] + b_{n-1}x[n-1] + \cdots + b_{n-M}x[n-M]$$

$$a_i = 0 \quad for\ all\ i$$

Non-recursive
Finite Impulse Response (FIR) filter

$$a_n y[n] = b_n x[n] + b_{n-1}x[n-1] + \cdots + b_{n-M}x[n-M] - a_{n-1}y[n-1] - \cdots - a_{n-N}y[n-N]$$

$$a_i \neq 0 \quad for\ some\ i$$

Recursive
Infinite Impulse Response (IIR) filter

Infinite Impulse Response (IIR)

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = b_n x[n] + b_{n-1} x[n-1] + \cdots + b_{n-M} x[n-M]$$

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = b_n x[n] \quad \Rightarrow \quad b_n h[n]$$

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = b_{n-1} x[n-1] \quad \Rightarrow \quad b_{n-1} h[n-1]$$

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = \cdots$$

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = b_{n-M} x[n-M] \quad \Rightarrow \quad b_{n-M} h[n-M]$$

$$h_{all}[n] = b_n h[n] - b_{n-1} h[n-1] + \cdots + b_{n-N} h[n-N]$$

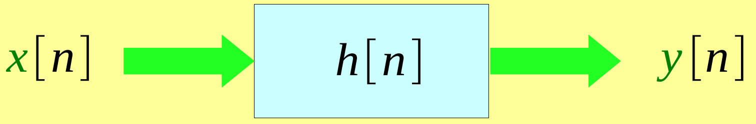
$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = x[n]$$

$$h[n] = \frac{1}{a_n}(\delta[n] - a_{n-1} h[n-1] - \cdots - a_{n-N} h[n-N])$$

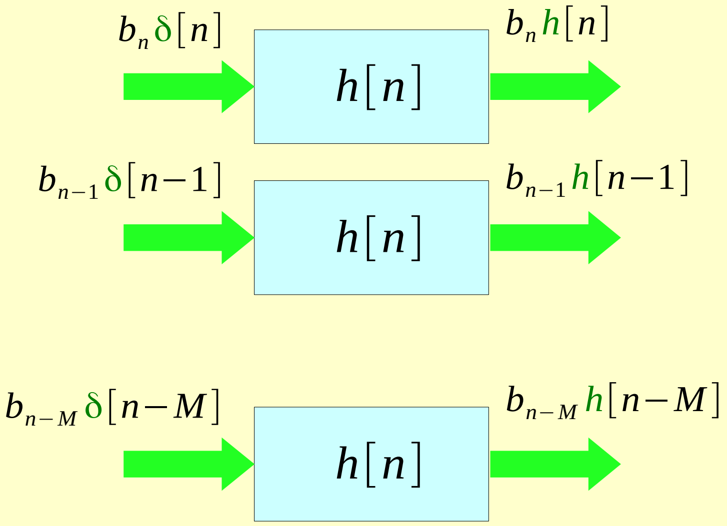
IIR and a Superposition of Inputs

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = x[n]$$

$$h[n] = \frac{1}{a_n}(\delta[n] - a_{n-1} h[n-1] - \cdots - a_{n-N} h[n-N])$$



$x[n]$ $h[n]$ $y[n]$



$b_n \delta[n]$ $h[n]$ $b_n h[n]$

$b_{n-1} \delta[n-1]$ $h[n]$ $b_{n-1} h[n-1]$

$b_{n-M} \delta[n-M]$ $h[n]$ $b_{n-M} h[n-M]$

$$a_n h[n] + a_{n-1} h[n-1] + \cdots + a_{n-N} h[n-N] = b_n x[n]$$

$$a_n h[n] + a_{n-1} h[n-1] + \cdots + a_{n-N} h[n-N] = b_{n-1} x[n-1]$$

$$a_n h[n] + a_{n-1} h[n-1] + \cdots + a_{n-N} h[n-N] = b_{n-M} x[n-M]$$

IIR as an Sum of All Impulse Responses

$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = \boxed{b_n x[n]} + \boxed{b_{n-1} x[n-1]} + \cdots + \boxed{b_{n-M} x[n-M]}$$

$$x_{all}[n] \longrightarrow \boxed{h_{all}[n]} \longrightarrow y[n]$$

$$x_{all}[n] = \boxed{b_n x[n]} + \boxed{b_{n-1} x[n-1]} + \cdots + \boxed{b_{n-M} x[n-M]}$$

$$h_{all}[n] = b_n h[n] - b_{n-1} h[n-1] + \cdots + b_{n-N} h[n-N]$$

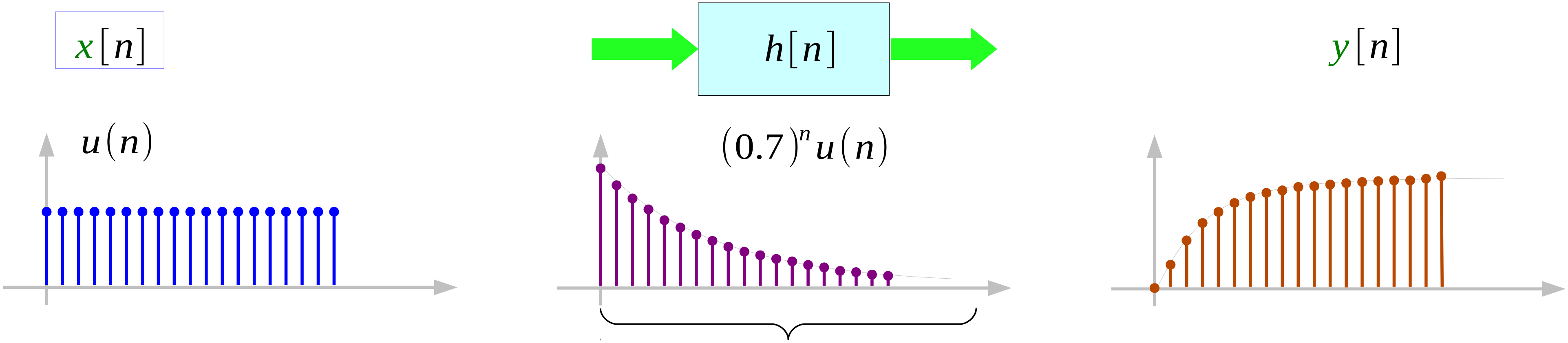
$$a_n y[n] + a_{n-1} y[n-1] + \cdots + a_{n-N} y[n-N] = \boxed{x[n]}$$

$$h[n] = \frac{1}{a_n}(\delta[n] - a_{n-1} h[n-1] - \cdots - a_{n-N} h[n-N])$$

IIR Example

Discrete Time LTI System

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m]$$



$x[n]$

$h[n]$

$y[n]$

$u(n)$

$(0.7)^n u(n)$

Infinite length

$$y[n] = \sum_{m=-\infty}^{+\infty} (0.7)^5 u(m) u(n-m)$$

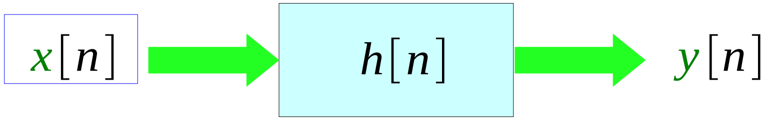
$$= \sum_{m=0}^{n} (0.7)^5 = \frac{1-(0.7)^{n+1}}{1-0.7}$$

$$= \frac{10}{3}\left(1-(0.7)^{n+1}\right)$$

Discrete Time Exponential γ^n

Discrete Time LTI System

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m]$$



$$e^{\lambda t} = \gamma^t$$

$$e^{\lambda n} = \gamma^n$$

$$\lambda = \ln \gamma$$

$$\gamma = e^{\lambda}$$

$$\gamma = e^{\lambda}$$

$$\lambda = \ln \gamma$$

$$\lambda = -0.3$$

$$\gamma = e^{-0.3} = 0.7408$$

$$e^{-0.3t} = 0.7408^t$$

$$\gamma = 4$$

$$\lambda = \ln 4 = 1.386$$

$$e^{1.386t} = 4^t$$

Finite Impulse Response (FIR)

$$a_n y[n] = b_n x[n] + b_{n-1} x[n-1] + \cdots + b_{n-M} x[n-M]$$

Tapped Delay Line

Transversal Filter

$$a_n h[n] = b_n \delta[n] + b_{n-1} \delta[n-1] + \cdots + b_{n-M} \delta[n-M]$$

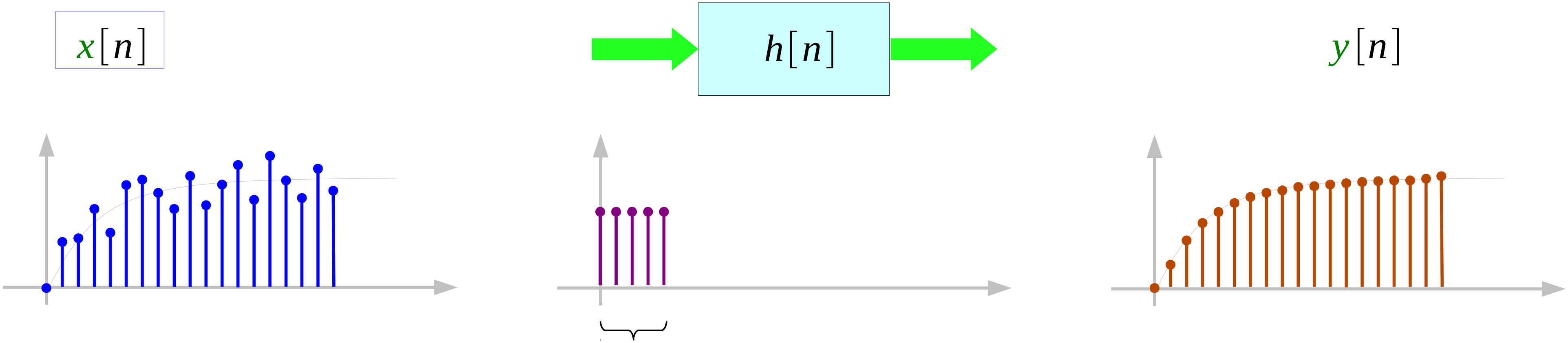
$$h[k] = \begin{cases} 0 & (k \le 0) \\ b_k / a_n & (0 \le k \le M) \\ 0 & (k > M) \end{cases}$$

FIR Example

Discrete Time LTI System

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m]$$

moving average filter



finite length

Convolution Sums in FIR Systems

$$a_n y[n] = b_n x[n] + b_{n-1} x[n-1] + \cdots + b_{n-M} x[n-M]$$

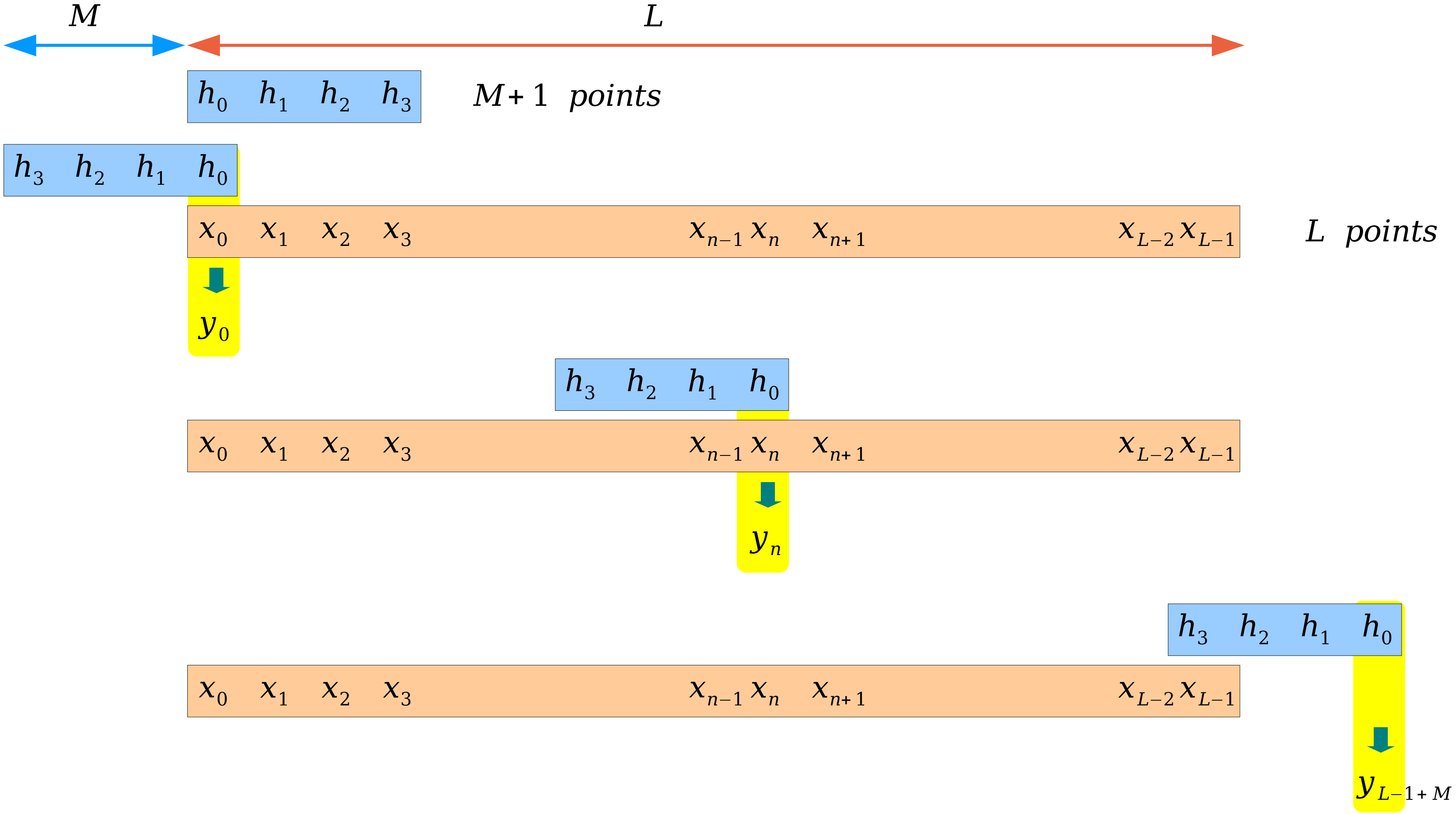
Computing Convolution Sums of an FIR

- Flip and Slide Form
- LTI Form
- Convolution Table
- Direct Form

Flip and Slide Form

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

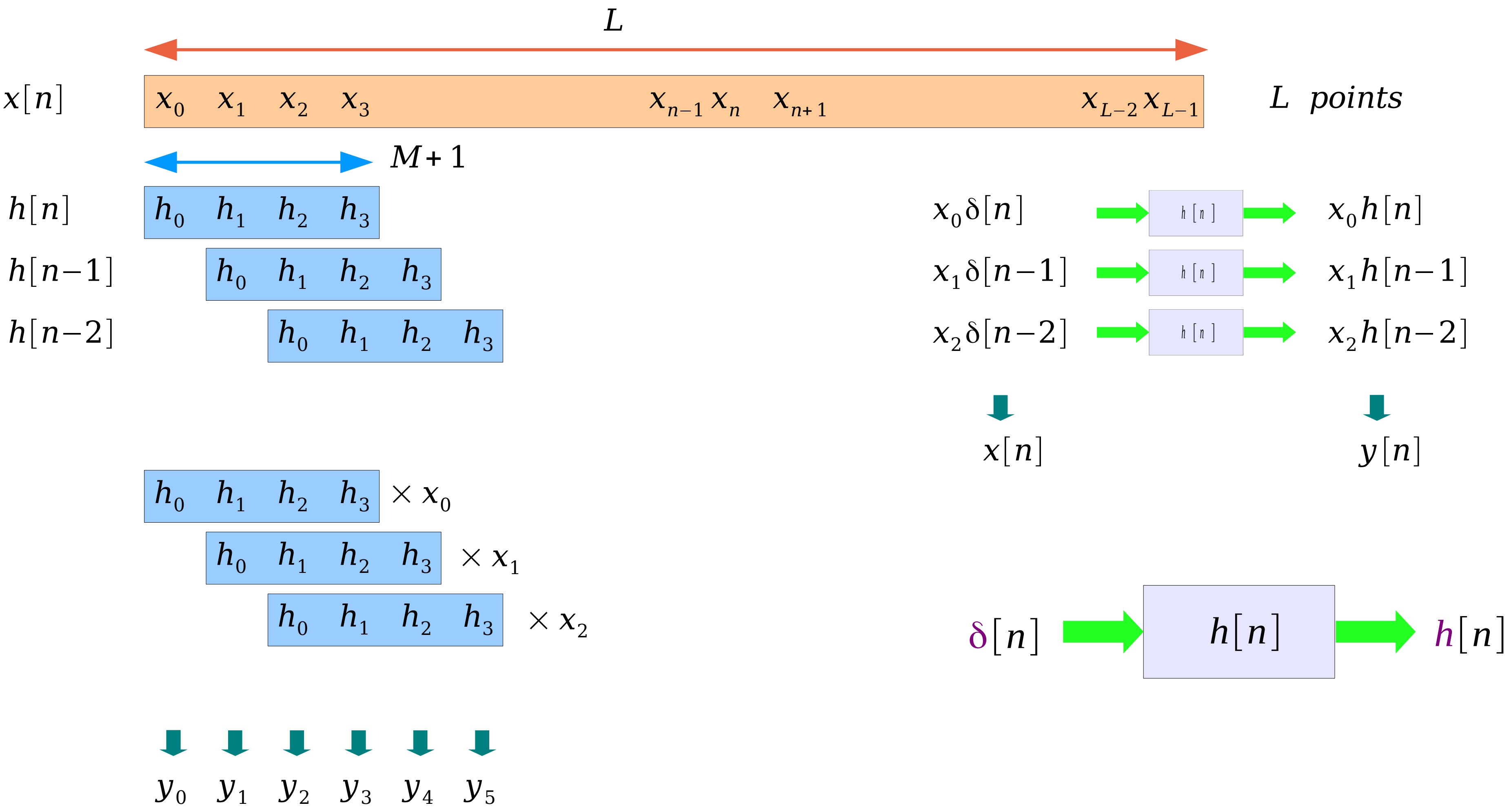
$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$



LTI Form

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$



L

$x[n]$ $x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_{n-1}\, x_n \quad x_{n+1} \quad x_{L-2}\, x_{L-1}$ L points

$M+1$

$h[n]$ $h_0 \quad h_1 \quad h_2 \quad h_3$

$h[n-1]$ $h_0 \quad h_1 \quad h_2 \quad h_3$

$h[n-2]$ $h_0 \quad h_1 \quad h_2 \quad h_3$

$h_0 \quad h_1 \quad h_2 \quad h_3 \times x_0$

$h_0 \quad h_1 \quad h_2 \quad h_3 \times x_1$

$h_0 \quad h_1 \quad h_2 \quad h_3 \times x_2$

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5$

$x_0\delta[n] \rightarrow h[n] \rightarrow x_0 h[n]$

$x_1\delta[n-1] \rightarrow h[n] \rightarrow x_1 h[n-1]$

$x_2\delta[n-2] \rightarrow h[n] \rightarrow x_2 h[n-2]$

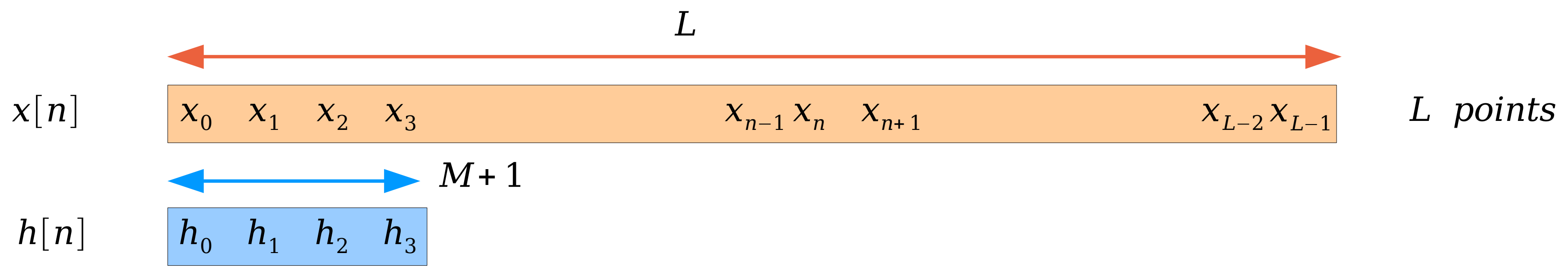
$x[n]$ $y[n]$

$\delta[n] \rightarrow \boxed{h[n]} \rightarrow h[n]$

Convolution Table

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] \qquad = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$


L

$x[n]$ x_0 x_1 x_2 x_3 x_{n-1} x_n x_{n+1} x_{L-2} x_{L-1} L points

$M+1$

$h[n]$ h_0 h_1 h_2 h_3

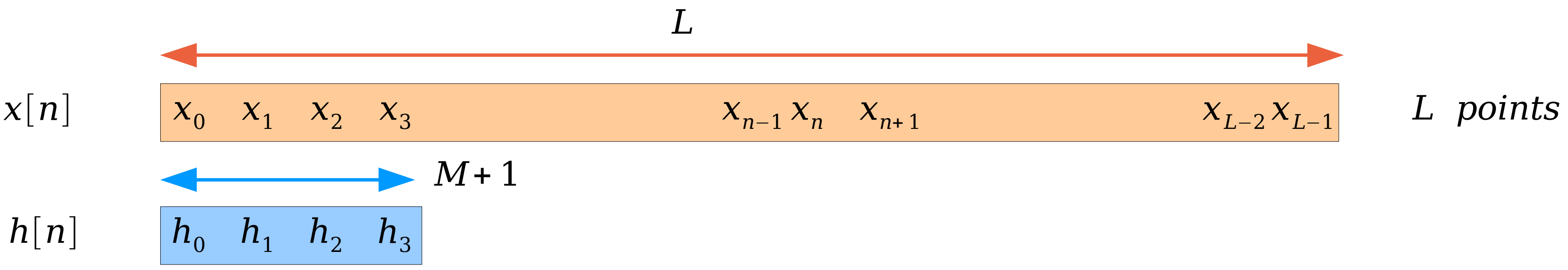
	x_0	x_1	x_3	x_4	x_5	x_6
h_0	$h_0 x_0$	$h_0 x_1$	$h_0 x_3$	$h_0 x_4$	$h_0 x_5$	$h_0 x_6$
h_1	$h_1 x_0$	$h_1 x_1$	$h_1 x_3$	$h_1 x_4$	$h_1 x_5$	$h_1 x_6$
h_2	$h_2 x_0$	$h_2 x_1$	$h_2 x_3$	$h_2 x_4$	$h_2 x_5$	$h_2 x_6$
h_3	$h_3 x_0$	$h_3 x_1$	$h_3 x_3$	$h_3 x_4$	$h_3 x_5$	$h_3 x_6$

$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$

Direct Form

$$y[n] = \sum_{m=-\infty}^{+\infty} x[m]\,h[n-m] \qquad = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$$y[n] = \sum_{\substack{i,j \\ i+j=n}} x[i]\,y[j]$$



$$y[n] = \sum_{m=-\infty}^{+\infty} x[n-m]\,h[m]$$

$$0 \le m \le M$$

$$0 \le n-m \le L-1$$

$$m \le n \le L-1+m$$

$$\downarrow \qquad \downarrow$$

$$0 \le n \le L-1+M$$

$$y[n] = \sum_{m=max[n-(L-1),0]}^{min[n,M]} x[n-m]\,h[m]$$

$$0 \le m \le M$$

$$-(L-1) \le m-n \le 0$$

$$n-(L-1) \le m \le n$$

$$max[n-(L-1),0] \le m \le min[n,M]$$

Convolution Property

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{+\infty} x[m]\, h[n-m]$$

$$y[n] = \delta[n] * h[n] = \sum_{m=-\infty}^{+\infty} \delta[m]\, h[n-m]$$

$$h[n] = \delta[n] * h[n]$$

$$x[n] * A\delta[n-n_0] = A x[n-n_0]$$

Frequency Response

$$y(t) = h(t) * x(t) \qquad = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) = A\, e^{j\Phi}\, e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = H(jw) \cdot A\, e^{j\Phi}\, e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, A\, e^{j\Phi}\, e^{jw(t-\tau)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)\, A\, e^{j\Phi}\, e^{jwt}\, e^{-j\omega\tau}\, d\tau$$

$$= \underline{A\, e^{j\Phi}\, e^{jwt}} \cdot \underline{\int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}$$

$$= \underline{x(t)} \quad \cdot \quad \underline{H(jw)}$$

Direct Form
Convolution Table
LTI Form
Matrix Form
Flip-and-side form
Overlap-Add Block Convolution

Block Processing Method
Sample Processing Method

Orfanidis intro to signal processing

References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] M. J. Roberts, Fundamentals of Signals and Systems
[4] S. J. Orfanidis, Introduction to Signal Processing
[5] B. P. Lathi, Signals and Systems