Complex Inverse Trig & Inverse TrigH (H.1)

20160910

Inverse Trignometric & Hyperbolic Functions

$\sin^{-1}(z)$	$\sinh^{-1}(z)$
$\cos^{-1}(z)$	$\cosh^{-1}(z)$
$\tan^{-1}(z)$	$\tanh^{-1}(z)$
$\cot^{-1}(z)$	$\coth^{-1}(z)$
$\sec^{-1}(z)$	$\operatorname{sech}^{-1}(z)$
$\csc^{-1}(z)$	$\operatorname{csch}^{-1}(z)$

$\sin^{-1}(z)$

$$w = \sin^{-1}(z)$$

$$z = \sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - e^{-iw}$$

$$e^{iw} - 2iz - e^{-iw} = 0$$

$$\boxed{e^{2iw}} - 2iz\boxed{e^{iw}} - 1 = 0$$

$$\boxed{e^{iw}}^2 - 2iz\boxed{e^{iw}} - 1 = 0$$

$$\boxed{e^{iw}} = iz \pm \sqrt{(iz)^2 + 1}$$

$$= iz + (1 - z^2)^{1/2}$$

$$1 - z^2 = \rho e^{i\theta}$$

$$(1 - z^2)^{1/2} = \rho^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \qquad k = 0, 1$$

$$e^{\boxed{iw}} = iz + (1 - z^2)^{1/2}$$

$$\boxed{iw} = \ln\left[iz + (1 - z^2)^{1/2}\right]$$

$$w = -i \ln\left[iz + (1 - z^2)^{1/2}\right]$$

$$= \sin^{-1}(z)$$

$$\sin^{-1}(z) = -i \ln\left[iz + (1 - z^2)^{1/2}\right]$$

$\cos^{-1}(z)$

$$w = \cos^{-1}(z)$$

$$z = \cos(w) = \frac{e^{iw} + e^{-iw}}{2}$$

$$2z = e^{iw} + e^{-iw}$$

$$e^{iw} - 2z + e^{-iw} = 0$$

$$\boxed{e^{2iw}} - 2z\boxed{e^{iw}} + 1 = 0$$

$$\boxed{e^{iw}}^2 - 2z\boxed{e^{iw}} + 1 = 0$$

$$\boxed{e^{iw}} = z \pm \sqrt{z^2 - 1} = z \pm \sqrt{i^2(-z^2 + 1)}$$

$$= z + i(1 - z^2)^{1/2}$$

$$1 - z^2 = \rho e^{i\theta}$$

$$(1 - z^2)^{1/2} = \rho^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \qquad k = 0, 1$$

$$e^{\boxed{iw}} = z + i(1 - z^2)^{1/2}$$

$$\boxed{iw} = \ln[z + i(1 - z^2)^{1/2}]$$

$$w = -i \ln[z + i(1 - z^2)^{1/2}]$$

$$= \cos^{-1}(z)$$

$$\cos^{-1}(z) = -i \ln[z + i(1 - z^2)^{1/2}]$$

$\tan^{-1}(z)$

$$w = \tan^{-1}(z)$$

$$\frac{\sin^{-1}(z)}{\cos^{-1}(z)} = \frac{-i\,\ln[\,iz + (1-z^2)^{1/2}\,]}{-i\,\ln[\,z + i(1-z^2)^{1/2}\,]}$$

$$z = \tan(w) = \frac{\dfrac{e^{iw} - e^{-iw}}{2i}}{\dfrac{e^{iw} + e^{-iw}}{2}} = \frac{(e^{iw} - e^{-iw})}{i\,(e^{iw} + e^{-iw})}$$

$$z\,i\,(e^{iw} + e^{-iw}) = (e^{iw} - e^{-iw})$$

$$(iz - 1)\,e^{iw} + (iz+1)\,e^{-iw} = 0$$

$$(iz-1)\boxed{e^{iw}}^2 + (iz+1) = 0$$

$$\boxed{e^{iw}}^2 = \frac{-(iz+1)}{(iz-1)} = \frac{(1+iz)}{(1-iz)} = \frac{(i-z)}{(i+z)}$$

$$e^{\boxed{iw}} = \left(\frac{i-z}{i+z}\right)^{\frac{1}{2}}$$

$$\boxed{iw} = \frac{1}{2}\ln\left(\frac{i-z}{i+z}\right) \qquad w = \frac{-i}{2}\ln\left(\frac{i-z}{i+z}\right) = \frac{i}{2}\ln\left(\frac{i-z}{i+z}\right)^{-1}$$

$$w = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$$

$$= \tan^{-1}(z)$$

$$\tan^{-1}(z) = \frac{i}{2}\ln\left(\frac{i+z}{i-z}\right)$$

$$\sin^{-1}(z) = -i \ln\left[iz + (1-z^2)^{1/2} \right]$$

$$\cos^{-1}(z) = -i \ln\left[z + i(1-z^2)^{1/2} \right]$$

$$\tan^{-1}(z) = \frac{i}{2} \ln\left(\frac{i+z}{i-z} \right)$$

$$\frac{d}{dz}\sin^{-1}(z), \quad \frac{d}{dz}\cos^{-1}(z), \quad \frac{d}{dz}\tan^{-1}(z)$$

$$w = \sin^{-1}(z)$$

$$z = \sin(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\sin(w) = \cos(w)\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\cos(w)} = \frac{1}{(1-\sin^2(w))^{\frac{1}{2}}} = \frac{1}{(1-z^2)^{1/2}}$$

$$w = \cos^{-1}(z)$$

$$z = \cos(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\cos(w) = -\sin(w)\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{-1}{\sin(w)} = \frac{-1}{(1-\cos^2(w))^{\frac{1}{2}}} = \frac{-1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\frac{\sin(z)}{\cos(z)} = \frac{\cos^2(z)+\sin^2(z)}{\cos^2(z)} = 1+\tan^2(z)$$

$$= \frac{1}{\cos^2(z)} = \sec^2(z)$$

$$w = \tan^{-1}(z)$$

$$z = \tan(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\tan(w) = (1+\tan^2(z))\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{(1+\tan^2(z))} = \frac{1}{1+z^2}$$

$$\frac{d}{dz}\sin^{-1}(z) = \frac{1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\cos^{-1}(z) = \frac{-1}{(1-z^2)^{1/2}}$$

$$\frac{d}{dz}\tan^{-1}(z) = \frac{1}{1+z^2}$$

$\sinh^{-1}(z)$

$$w = \sinh^{-1}(z)$$

$$z = \sinh(w) = \frac{e^{w} - e^{-w}}{2}$$

$$2z = e^{w} - e^{-w}$$

$$e^{w} - 2z - e^{-w} = 0$$

$$\boxed{e^{2w}} - 2z\,\boxed{e^{w}} - 1 = 0$$

$$\boxed{e^{w}}^{2} - 2z\,\boxed{e^{w}} - 1 = 0$$

$$\boxed{e^{w}} = z \pm \sqrt{(z)^2 + 1}$$

$$= z + (z^2+1)^{1/2}$$

$$z^2 + 1 = \rho e^{i\theta}$$

$$(z^2+1)^{1/2} = \rho^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \qquad k = 0, 1$$

$$e^{\boxed{w}} = z + (z^2+1)^{1/2}$$

$$\boxed{w} = \ln\left[z + (z^2+1)^{1/2}\right]$$

$$w = \ln\left[z + (z^2+1)^{1/2}\right]$$

$$= \sinh^{-1}(z)$$

$$\sinh^{-1}(z) = \ln\left[z + (1 - z^2)^{1/2}\right]$$

$\cosh^{-1}(z)$

$$w = \cosh^{-1}(z)$$

$$z = \cosh(w) = \frac{e^{w} + e^{-w}}{2}$$

$$2z = e^{w} + e^{-w}$$

$$e^{w} - 2z + e^{-w} = 0$$

$$\boxed{e^{2w}} - 2z\boxed{e^{w}} + 1 = 0$$

$$\boxed{e^{w}}^{2} - 2z\boxed{e^{w}} + 1 = 0$$

$$\boxed{e^{w}} = z \pm \sqrt{z^{2} - 1}$$

$$= z + (z^{2}-1)^{1/2}$$

$$z^{2} - 1 = \rho e^{i\theta}$$

$$(z^{2}-1)^{1/2} = \rho^{1/2} e^{i\left(\frac{\theta + 2k\pi}{2}\right)} \qquad k = 0, 1$$

$$e^{\boxed{w}} = z + (z^{2}-1)^{1/2}$$

$$\boxed{w} = \ln\left[z + (z^{2}-1)^{1/2}\right]$$

$$w = \ln\left[z + (z^{2}-1)^{1/2}\right]$$

$$= \cosh^{-1}(z)$$

$$\cosh^{-1}(z) = \ln\left[z + (z^{2}-1)^{1/2}\right]$$

$\tanh^{-1}(z)$

$$w = \tanh^{-1}(z)$$

$$\cancel{\frac{\sinh^{-1}(z)}{\cosh^{-1}(z)}} = \frac{\ln[z + (z^2+1)^{1/2}]}{\ln[z + (z^2-1)^{1/2}]}$$

$$z = \tanh(w) = \frac{\dfrac{e^w - e^{-w}}{2}}{\dfrac{e^w + e^{-w}}{2}} = \frac{(e^w - e^{-w})}{(e^w + e^{-w})}$$

$$z\ (e^w + e^{-w}) = (e^w - e^{-w})$$

$$(z-1)\ e^w + (z+1)\ e^{-w} = 0$$

$$(z-1)\boxed{e^w}^2 + (z+1) = 0$$

$$\boxed{e^w}^2 = \frac{-(z+1)}{(z-1)} = \frac{1+z}{1-z}$$

$$e^{\boxed{w}} = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}$$

$$\boxed{w} = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$w = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$= \tanh^{-1}(z)$$

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\sinh^{-1}(z) = \ln\left[z + (1 - z^2)^{1/2}\right]$$

$$\cosh^{-1}(z) = \ln\left[z + (z^2 - 1)^{1/2}\right]$$

$$\tanh^{-1}(z) = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\frac{d}{dz}\sinh^{-1}(z), \quad \frac{d}{dz}\cosh^{-1}(z), \quad \frac{d}{dz}\tanh^{-1}(z)$$

$$W = \sinh^{-1}(z) \qquad\qquad \cosh^2 x - \sinh^2 x = 1$$

$$z = \sinh(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\sinh(w) = \cosh(w)\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\cosh(w)} = \frac{1}{(1+\sinh^2(w))^{\frac{1}{2}}} = \frac{1}{(1+z^2)^{1/2}}$$

$$W = \cosh^{-1}(z)$$

$$z = \cosh(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\cosh(w) = \sinh(w)\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{\sinh(w)} = \frac{1}{(\cos^2(w)-1)^{\frac{1}{2}}} = \frac{1}{(z^2-1)^{1/2}}$$

$$\frac{d}{dz}\frac{\sinh(z)}{\cosh(z)} = \frac{\cosh^2(z)-\sinh^2(z)}{\cosh^2(z)} = 1-\tanh(z)$$

$$W = \tan^{-1}(z) \qquad\qquad = \frac{1}{\cosh^2(z)} = \operatorname{sech}^2(z)$$

$$z = \tan(w)$$

$$\frac{d}{dz} z = \frac{d}{dz}\tan(w) = (1-\tanh^2(z))\frac{dw}{dz}$$

$$\frac{dw}{dz} = \frac{1}{(1-\tanh^2(z))} = \frac{1}{1-z^2}$$

$$\frac{d}{dz}\sinh^{-1}(z) = \frac{1}{(z^2+1)^{1/2}}$$

$$\frac{d}{dz}\cosh^{-1}(z) = \frac{1}{(z^2-1)^{1/2}}$$

$$\frac{d}{dz}\tanh^{-1}(z) = \frac{1}{1-z^2}$$