Complex Exp & Log (H.1)

20160721

$$f(x) = e^x$$

$$\boxed{\begin{aligned} f'(x) &= f(x) \\ f(x_1 + x_2) &= f(x_1) \cdot f(x_2) \end{aligned}}$$

$$\boxed{e^{iy} = \cos y + i \sin y}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right)$$

$$= \cos(y) + i \sin(y)$$

$$z = x + iy$$

$$e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$$

$$\boxed{e^z = e^x(\cos y + i \sin y)}$$

$$e^{z} = e^{x}(\cos y + i \sin y)$$
$$= e^{x}\cos y + i\, e^{x}\sin y$$
$$= u(x,y) + i\, v(x,y)$$

$$\begin{cases} u(x,y) = e^{x}\cos y & \text{Continuous} \\ v(x,y) = e^{x}\sin y & \text{Continuous} \end{cases}$$

$$\begin{cases} \dfrac{\partial u}{\partial x} = e^{x}\cos y & \text{Continuous} \\ \dfrac{\partial u}{\partial y} = -e^{x}\sin y & \text{Continuous} \end{cases}$$

$$\begin{cases} \dfrac{\partial v}{\partial x} = e^{x}\sin y & \text{Continuous} \\ \dfrac{\partial v}{\partial y} = e^{x}\cos y & \text{Continuous} \end{cases}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} = e^{x}\cos y$$
$$\boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} = -e^{x}\sin y$$

Caucy-Rieman Eq

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x}}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Big\downarrow \qquad \Big\downarrow$$

apply Cauchy-Rieman Eq

$$\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x}$$

Properties

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial x}}$$

$$= e^x \cos y + i e^x \sin y$$
$$= e^x \cos y - i(-e^x \sin y)$$
$$= e^z$$

$$\frac{d}{dz} e^z = e^z$$

$$z_1 = x_1 + i y_1$$
$$z_2 = x_2 + i y_2$$

$$f(z_1) f(z_2) = e^{x_1}(\cos y_1 + i \sin y_1)\, e^{x_2}(\cos y_2 + i \sin y_2)$$
$$= e^{x_1 + x_2}\left[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \cos y_2)\right]$$
$$= e^{x_1 + x_2}\left(\cos(y_1 + y_2) + i \sin(y_1 + y_2)\right)$$
$$= f(z_1 + z_2)$$

$$e^{z_1} e^{z_2} = e^{(z_1 + z_2)}$$

$$f(z_1)/f(z_2) = e^{x_1}(\cos y_1 + i \sin y_1)/e^{x_2}(\cos y_2 + i \sin y_2)$$
$$= \frac{e^{x_1}}{e^{x_2}} \frac{(\cos y_1 + i \sin y_1)(\cos y_2 - i \sin y_2)}{(\cos y_2 + i \sin y_2)(\cos y_2 - i \sin y_2)}$$
$$= e^{x_1 - x_2}\left[(\cos y_1 \cos y_2 + \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 - \cos y_1 \cos y_2)\right]$$
$$= e^{x_1 - x_2}\left(\cos(y_1 - y_2) + i \sin(y_1 - y_2)\right)$$
$$= f(z_1 - z_2)$$

$$e^{z_1}/e^{z_2} = e^{(z_1 - z_2)}$$

Periodicity

e^x ~~periodic~~

e^z periodic

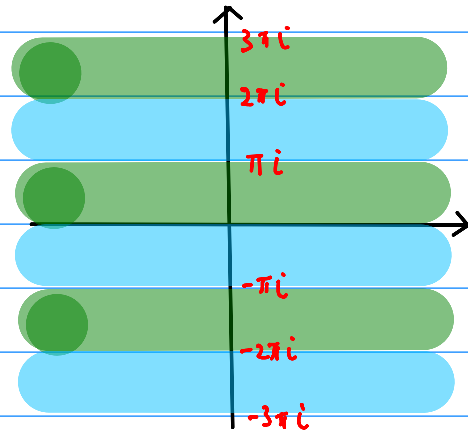
$$e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z\left(\cos(2\pi) + i\sin(2\pi)\right) = e^z$$

$$f(z+2\pi i) = f(z)$$

horizontal strip

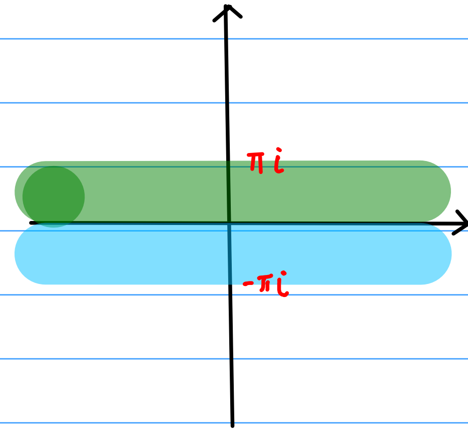
$(2n-1)\pi < y \leq (2n+1)\pi$

$n = 0, \pm 1, \pm 2, \cdots$



Fundamental Region

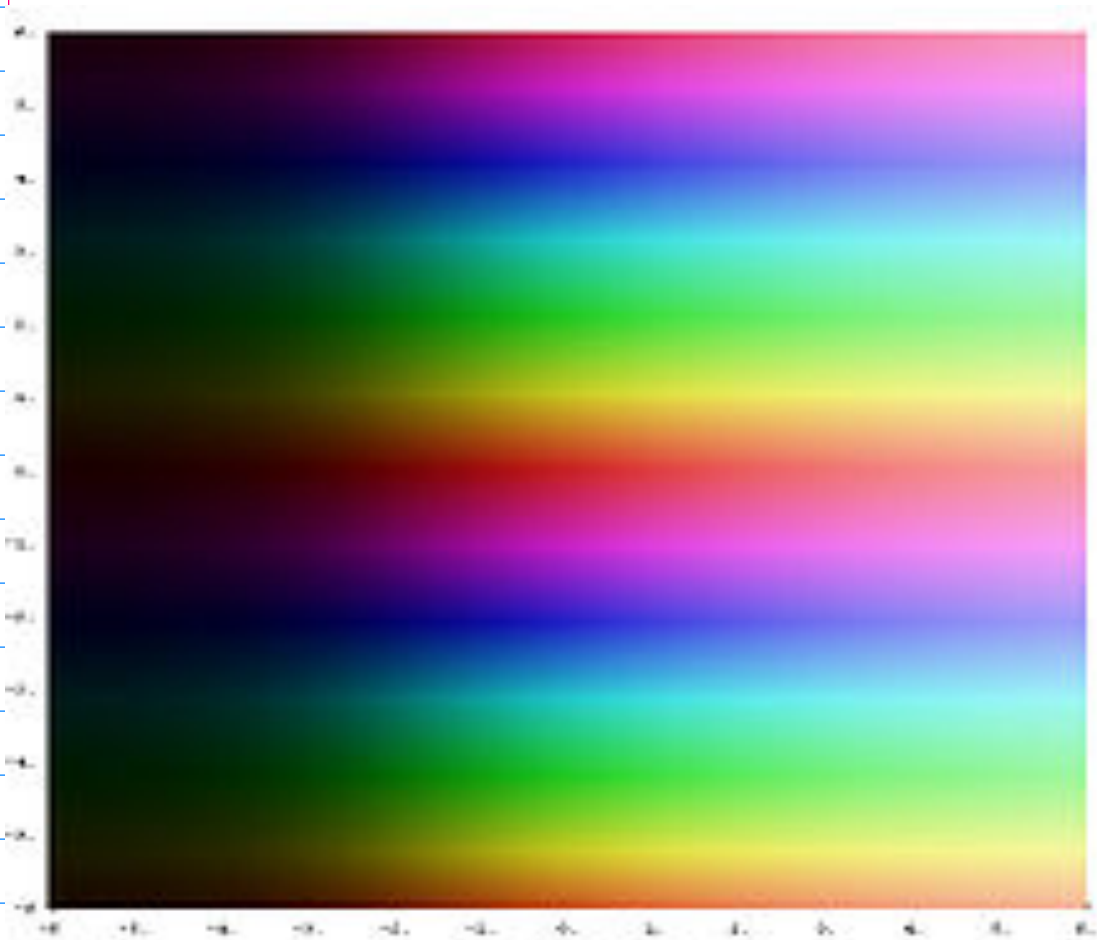
$-\pi < y < +\pi$



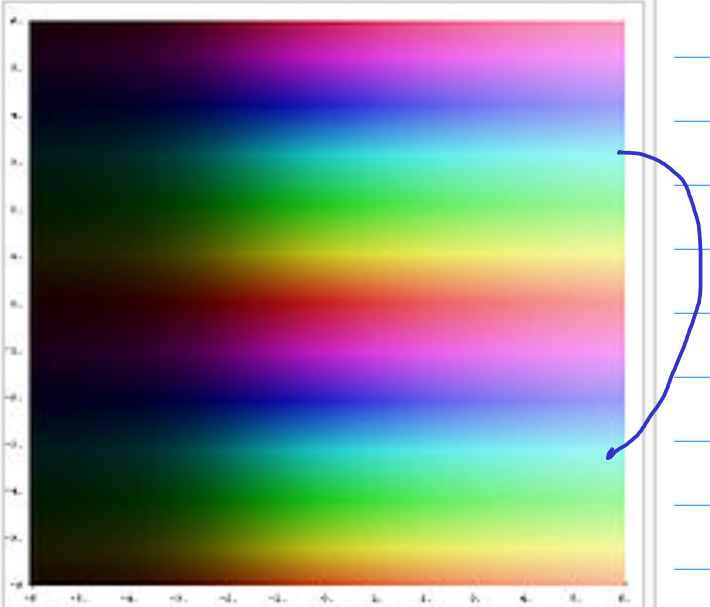
$$f(z) = f(z+2\pi i) = f(z+4\pi i) = \cdots$$

$$= f(z-2\pi i) = f(z-4\pi i) = \cdots$$

https://en.wikipedia.org/wiki/Exponential_function

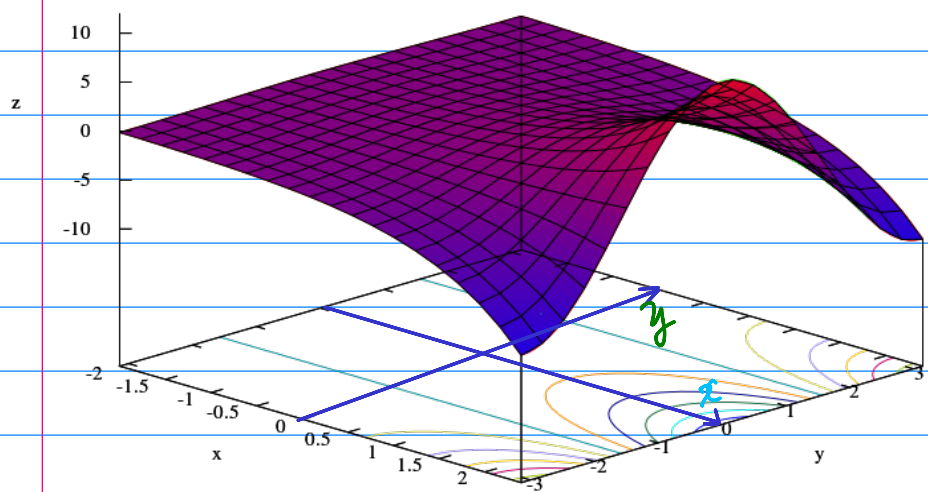


https://en.wikipedia.org/wiki/Exponential_function



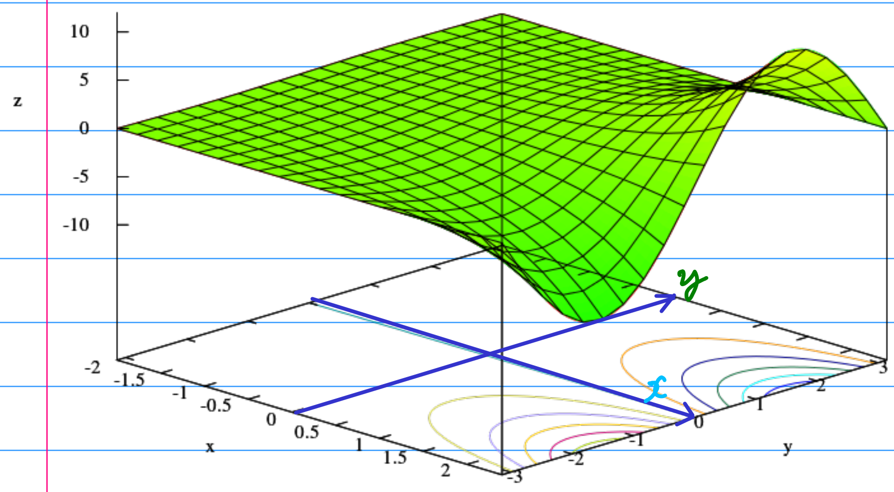
Exponential function on the complex plane. The transition from dark to light colors shows that the magnitude of the exponential function is increasing to the right. The periodic horizontal bands indicate that the exponential function is periodic in the imaginary part of its argument.

https://en.wikipedia.org/wiki/Exponential_function



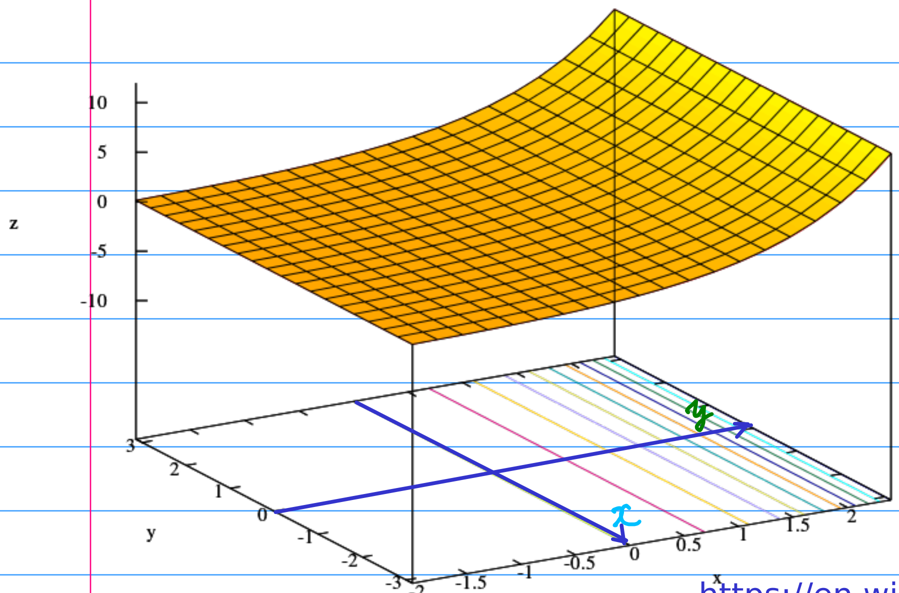
$\mathrm{Re}\{e^{x+iy}\}$

$= e^x \cos y$



$\mathrm{Im}\{e^{x+iy}\}$

$= e^x \sin y$



$|e^{x+iy}| = e^x$

https://en.wikipedia.org/wiki/Exponential_function

$$z = x + iy$$

$$= e^{z}$$

$$z = x + iy$$

$$w = u + iv$$

$$w = e^{z}$$

$$u + iv = e^{x+iy}$$

$$= e^{x}(\cos y + i\sin y)$$

$$u = e^{x}\cos y$$

$$v = e^{x}\sin y$$

$$e^{z} = e^{x+iy}$$

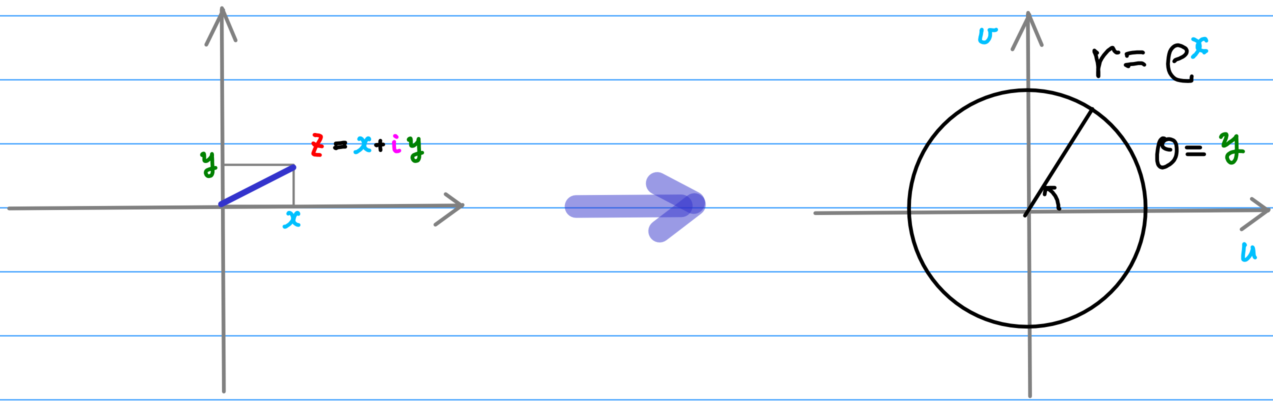
x-y plane

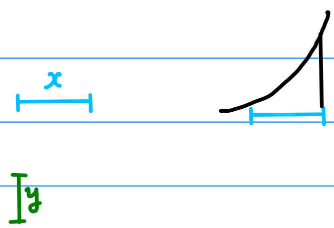
$z = x + iy$

$$\boxed{w = e^{z}}$$

u-v plane

$w = u + iv$





$e^{x} \longrightarrow r$

$y \longrightarrow \theta$

$$u = e^{x}\cos y$$

$$v = e^{x}\sin y$$

$$u + iv = e^{x}(\cos y + i\sin y)$$

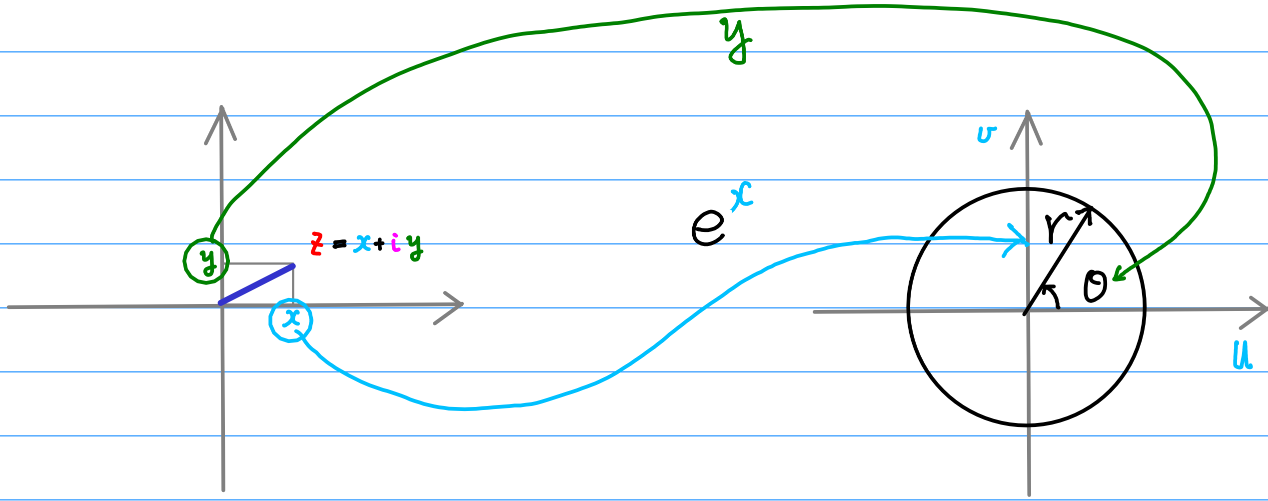
$$= e^{x+iy} = e^{z}$$

$$e^{z} = e^{x+iy}$$

$$\boxed{\begin{array}{lll} x \to r & (=e^{x}) \\ y \to \theta & (=y) \end{array}}$$

$x \to r = e^{x}$

y



x-y plane

$z = x + iy$

u-v plane

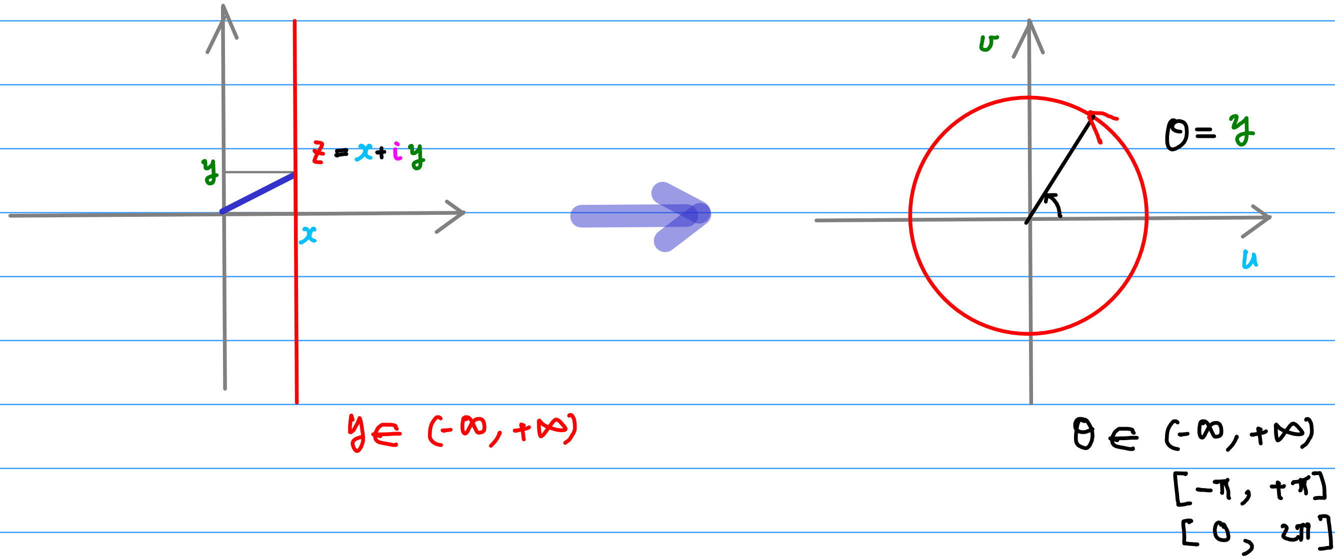
$w = u + iv$

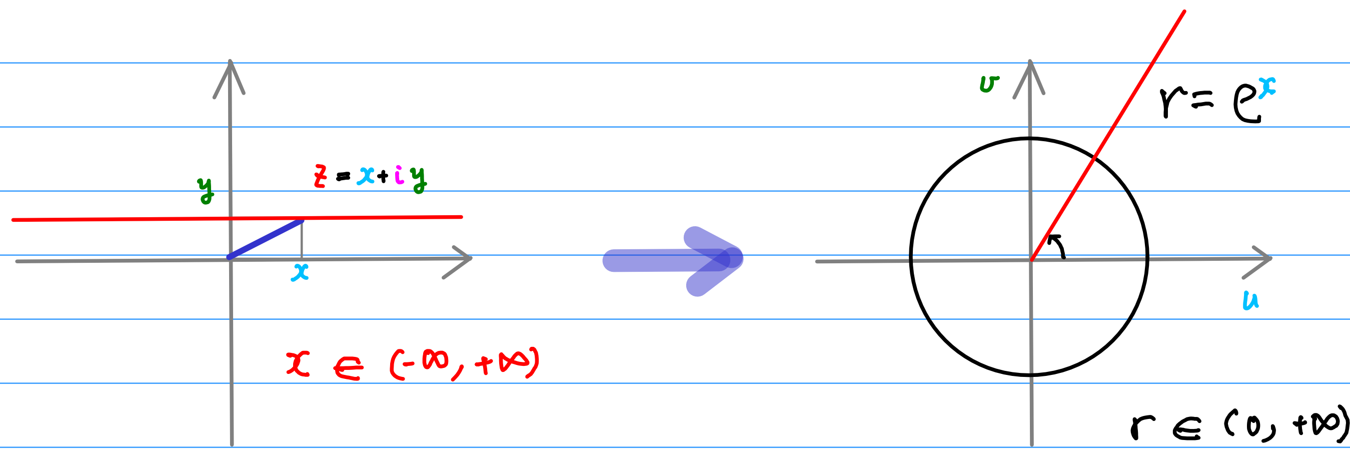
$$\boxed{w = e^{z}}$$

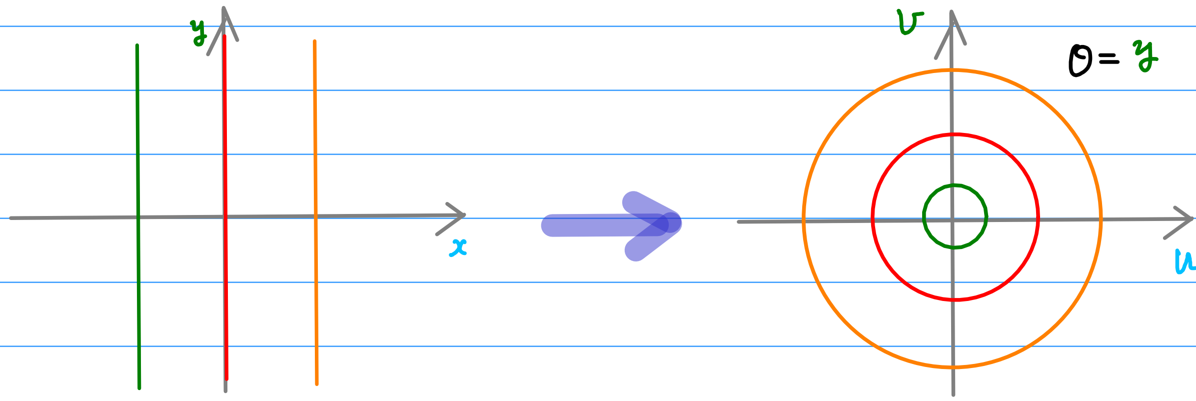
$$= e^{x}(\cos y + i \sin y)$$

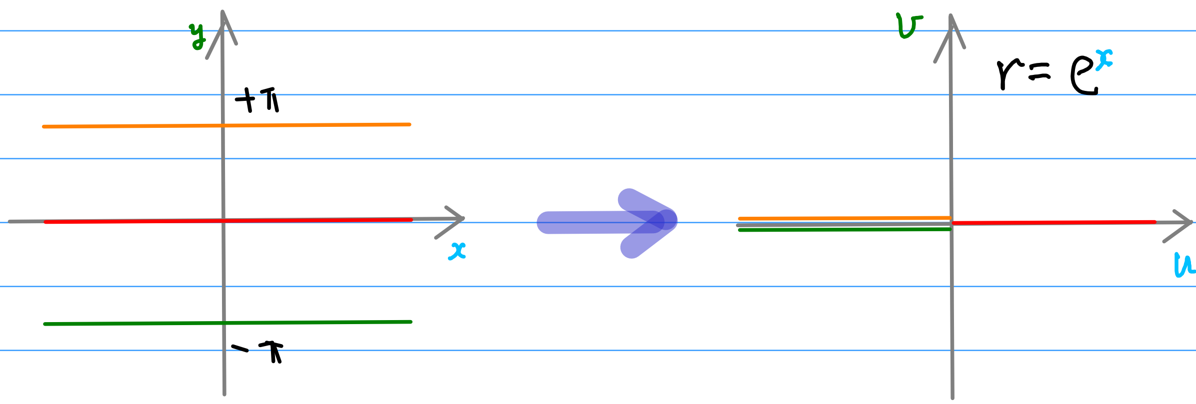
$$= \boxed{e^{x}\cos y} + i\,\boxed{e^{x}\sin y}$$

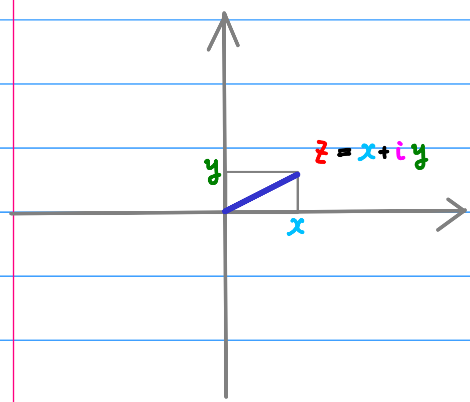
$$= \boxed{\begin{array}{c} u \\ + iv \end{array}} = \boxed{\begin{array}{c} \boxed{e^{x}\cos y} \\ + i\,\boxed{e^{x}\sin y} \end{array}}$$

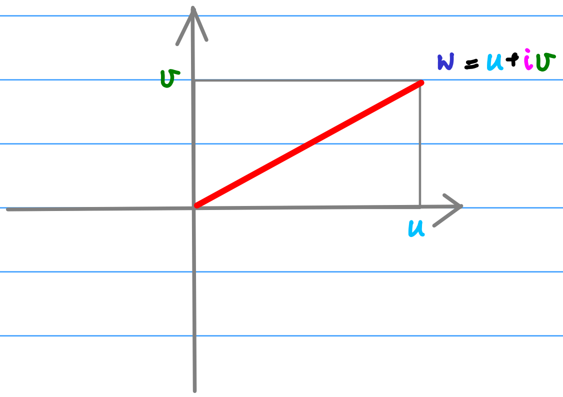

$z = x + iy$
y
x
u
v
$\theta = y$
$y \in (-\infty, +\infty)$
$\theta \in (-\infty, +\infty)$
$[-\pi, +\pi]$
$[0, 2\pi]$


$z = x + iy$
y
x
u
v
$r = e^{x}$
$x \in (-\infty, +\infty)$
$r \in (0, +\infty)$


y
x
v
u
θ = y


y
+π
x
−π
v
r = e^x
u


z = x + i y
y
x


w = u + i v
v
u

Logarithmic Function

$z = x + iy$

$$\begin{cases} w = \ln z & \text{if } z = e^w \quad (z \neq 0) \\ w = \text{undefined} & \text{if } z = 0 \quad \leftarrow \quad 0 \neq e^w \end{cases}$$

$z = x + iy$

$w = u + iv$

$z = e^w$

$x + iy = e^{u+iv}$

$= e^u (\cos v + i \sin v)$

z can be negative real

$x = e^u \cos v$

$y = e^u \sin v$

$$\boxed{\begin{aligned} z &= e^{w} \\ w &= \ln z \end{aligned}}$$

$$\boxed{\begin{aligned} z &= x + iy \\ w &= u + iv \end{aligned}}$$

$$\boxed{\begin{aligned} x &= e^{u} \cos v \\ y &= e^{u} \sin v \end{aligned}}$$

$$x = e^{u} \cos v$$
$$y = e^{u} \sin v$$

$$x^2 = e^{2u} \cos^2 v$$
$$y^2 = e^{2u} \sin^2 v$$

$$x^2 + y^2 = e^{2u}$$
$$|z|^2 = r^2 = e^{2u}$$

$$\boxed{u = \ln |z|}$$ real natural logarithm

$$\frac{x}{y} = \frac{e^{u} \cos v}{e^{u} \sin v} = \frac{1}{\tan v}$$ $$\left(\frac{\sin v}{\cos v} = \tan v \right)$$

$$\tan v = \frac{y}{x} = \arg z$$

$$\boxed{v = \arg z} = \theta$$ no unique argument

$$\theta + 2n\pi \quad n = 0, \pm 1, \pm 2 \ldots$$

$$\boxed{\begin{aligned} z &= e^{w} \\ w &= \ln z \end{aligned}}$$

$$\boxed{\begin{aligned} x &= e^{u} \cos v \\ y &= e^{u} \sin v \end{aligned}}$$

$$\boxed{\begin{aligned} z &= x + iy \\ w &= u + iv \end{aligned}}$$

$$\boxed{\begin{aligned} u &= \ln (x^2+y^2)^{1/2} \\ v &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}} + 2n\pi$$

$$u = \ln |z| = \ln r$$

$$v = \arg z = \theta$$

$$\ln z = \log_e |z| + i(\theta + 2n\pi)$$

$$z = |z| e^{i \arg z}$$

$$\ln z = \ln\left[|z| e^{i \arg z} \right]$$

$$= \ln |z| + \ln e^{i \arg z}$$

$$\ln z = \ln |z| + i \arg z$$

for $z \neq 0$, $\theta = \arg z$

$$\ln z = \log_e |z| + i(\theta + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \cdots$$

$$\ln z = \ln(x + iy)$$

$$\boxed{\begin{aligned} r &\to u \quad (= \ln r) \\ \theta &\to v \quad (= \theta + 2n\pi) \end{aligned}}$$

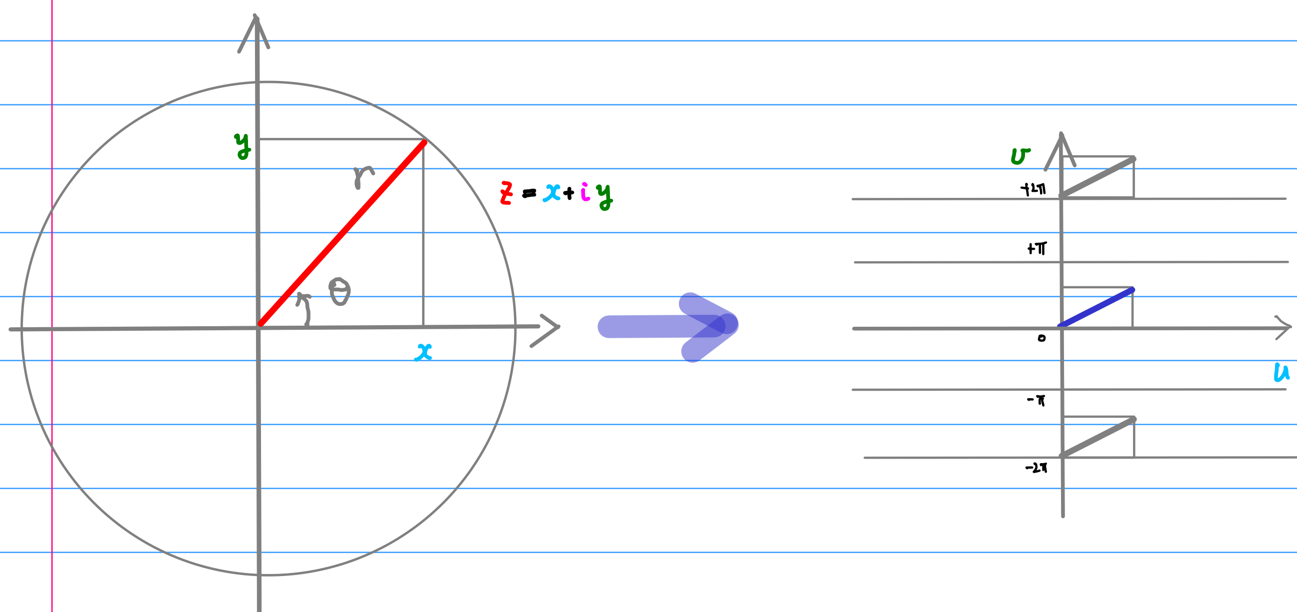
x-y plane

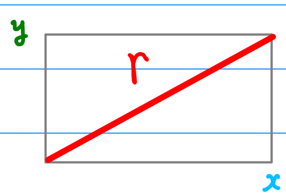
$z = x + iy$

$$\boxed{W = \ln z}$$

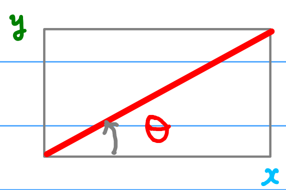
u-v plane

$W = u + iv$





$\ln r \longrightarrow u$

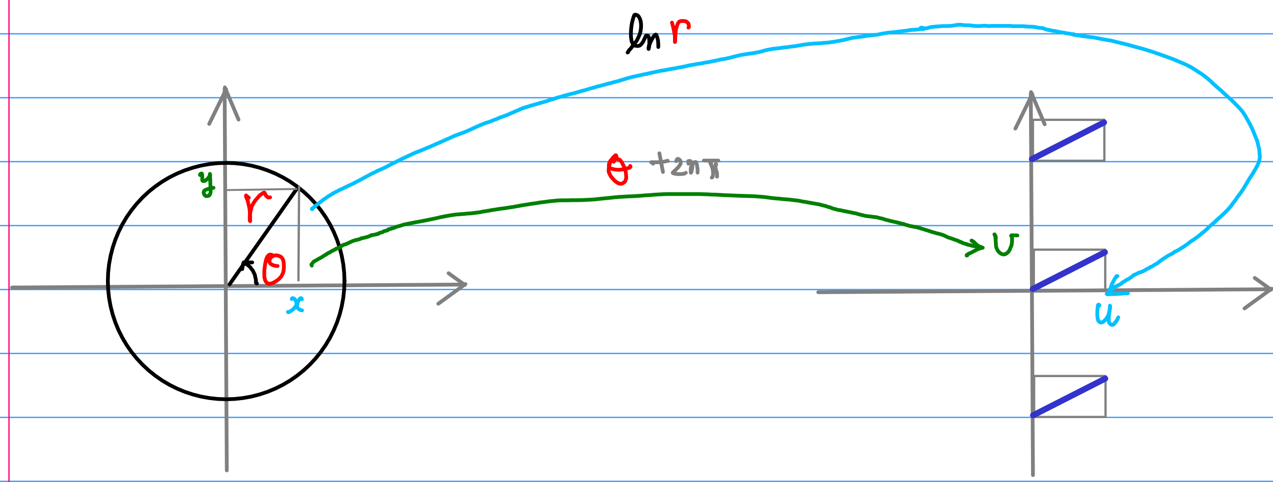


$\theta + 2n\pi \longrightarrow v$

$$u = \ln(x^2 + y^2)^{\frac{1}{2}}$$

$$v = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$$

$$\ln z = \ln(x + iy)$$



x-y plane

$z = x + iy$

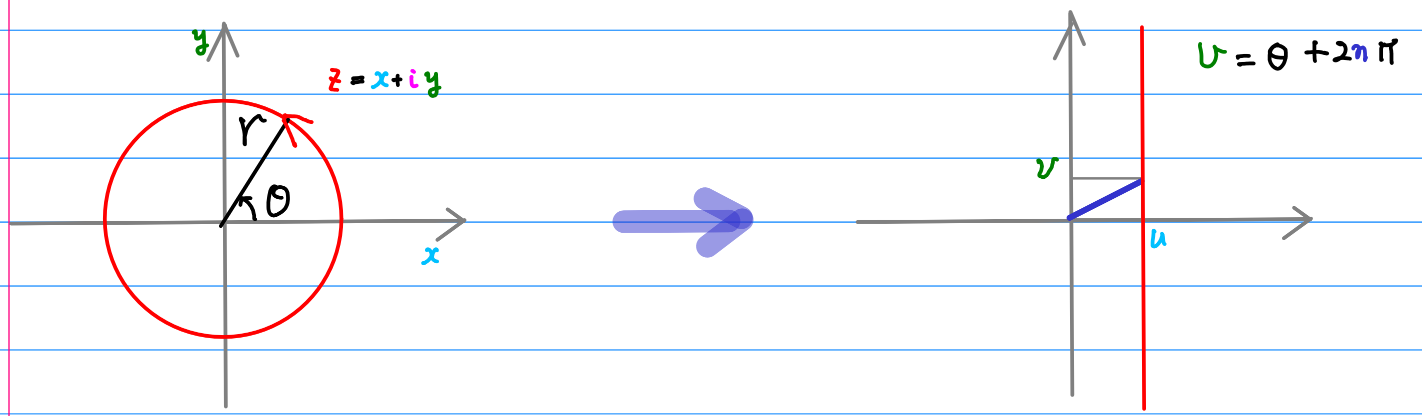
$$\boxed{w = \ln z}$$

u-v plane

$w = u + iv$

$$= \boxed{\ln(x^2+y^2)^{\frac{1}{2}}} + i \boxed{\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi}$$

$$= \boxed{\begin{matrix} u \\ + iv \end{matrix}} = \boxed{\begin{matrix} \boxed{\ln(x^2+y^2)^{\frac{1}{2}}} \\ + i \boxed{\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi} \end{matrix}}$$

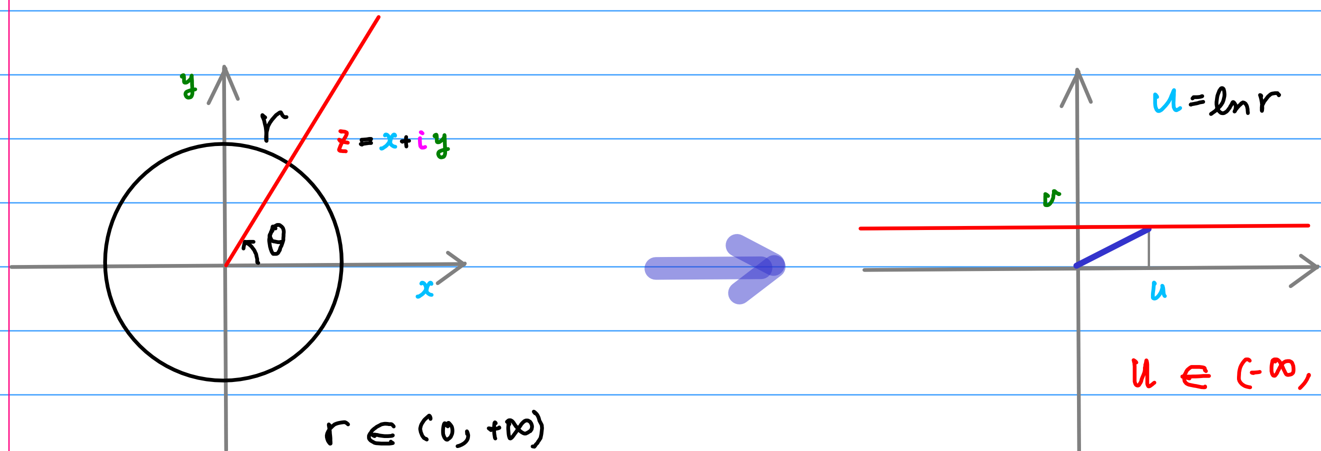


$\theta \in (-\infty, +\infty)$

$[-\pi, +\pi]$

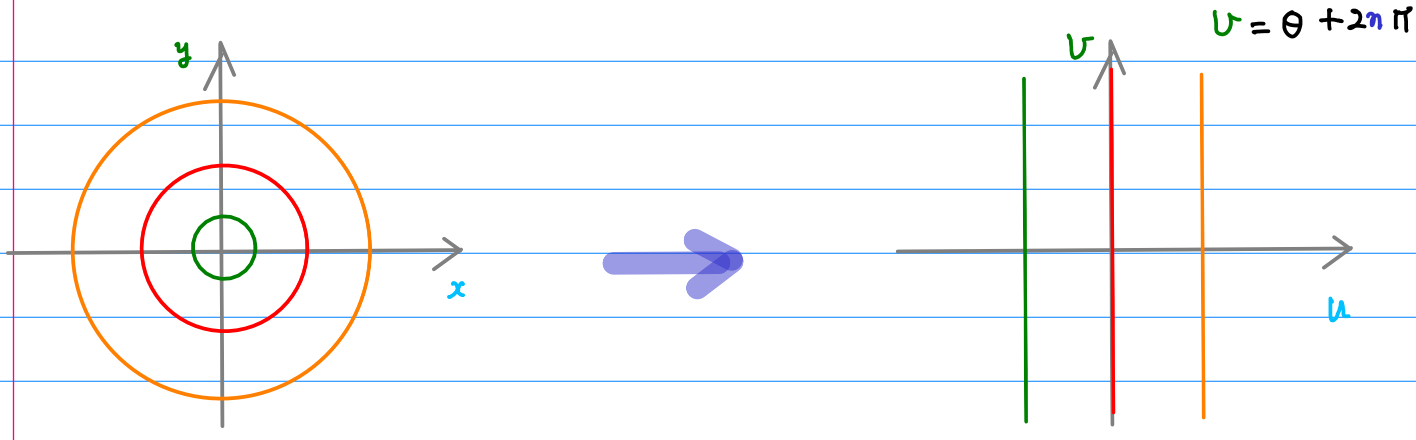
$[0, 2\pi]$

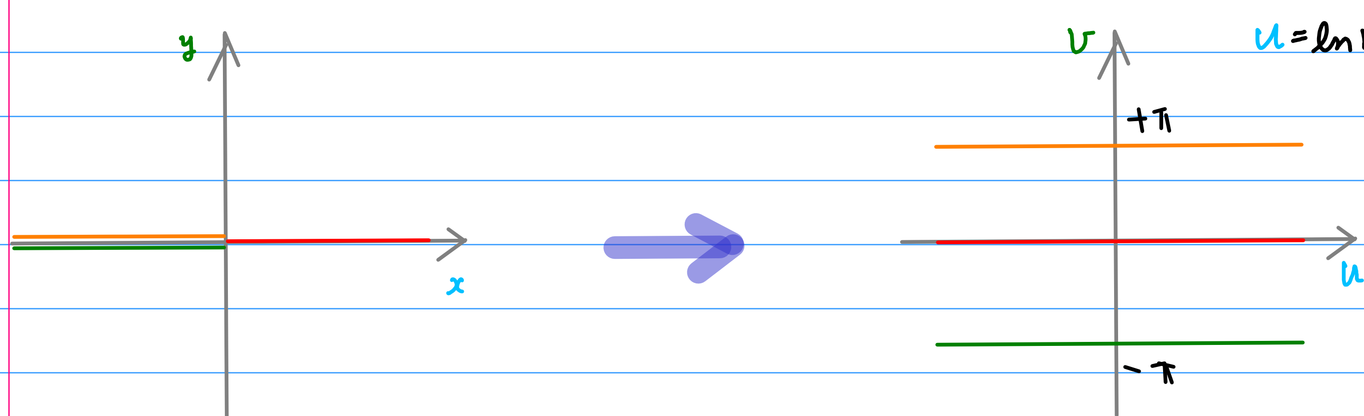
$v \in (-\infty, +\infty)$



$u \in (-\infty, +\infty)$

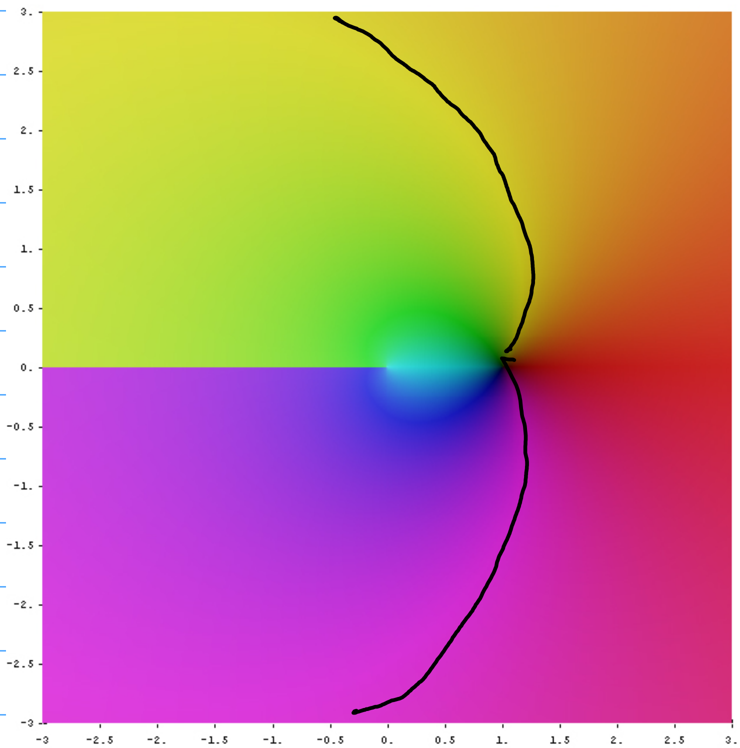
$r \in (0, +\infty)$


y
x
v
u
v = θ + 2nπ


y
x
v
u = ln r
+π
−π
u

$$|w| = |\ln z| , \qquad \arg(w) = \arg(\ln z)$$

saturation & value $\qquad$ hue



$$\ln z = \ln|z| + i(\theta + 2n\pi)$$

$$|\ln z| = \sqrt{(\ln|z|)^2 + (\theta + 2n\pi)^2}$$

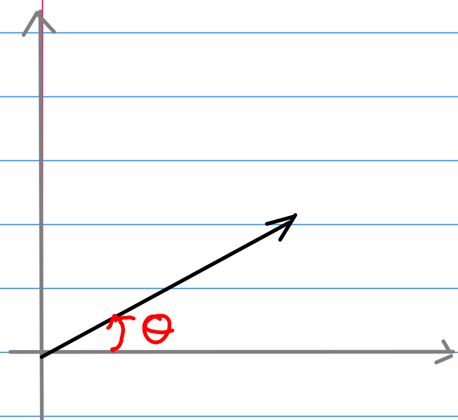
$$\tan \theta = \frac{(\theta + 2n\pi)}{\ln|z|} \quad \begin{matrix} +, - \\ + \end{matrix}$$

$$\arg(\ln z) = \theta_0 + 2n\pi$$

https://en.wikipedia.org/wiki/Complex_logarithm

A single branch of the complex logarithm. The hue of the color is used to show the ***arg*** (polar coordinate angle) of the complex logarithm. The saturation and value (intensity and brightness) of the color is used to show the ***modulus*** of the complex logarithm. The image file's page shows the encoding of colors as a function of their complex values.

Argument, Principal Argument



the argument of z $\theta = \arg z$

the angle θ of inclination of a vector z

measured in radians from the positive real axis

positive angle – counterclockwise
negative angle – clockwise

$$\tan\theta = \frac{y}{x} = \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}$$

θ : not unique

θ_0 : an argument → $\theta_0 + 2n\pi$ arguments also

the Principal Argument $\mathrm{Arg}\, z = \theta_0 \quad -\pi < \theta_0 \leq \pi$

Complex Exp & Log

$$e^{z+2n\pi i} = e^{z} e^{i2\pi n}$$
$$= e^{z}(\cos(2\pi n) + i\sin(2\pi n))$$
$$= e^{z}$$

$$w = \ln z \qquad \boxed{z = e^{w}}$$
$$x + iy = e^{u}(\cos v + i\sin v)$$
$$u = \sqrt{x^2+y^2} \qquad \tan v = \frac{y}{x}$$

$$\boxed{e^{z+2n\pi i} = e^{z}}$$

$$\boxed{\ln z = \ln|z| + i(\theta + 2\pi n)}$$

periodic

Infinitely many values

$n = 0, \pm 1, \pm 2, \pm 3 \cdots$

$n = 0$

principal argument

$\mathrm{Arg}\, z = \theta$

$n = 0$

Principal value

$\mathrm{Ln}\, z = \ln|z| + i\theta$

$$\underbrace{\mathrm{Ln}\, z}_{\text{P.V.}} = \ln|z| + i\,\underbrace{\mathrm{Arg}\, z}_{\text{P.A.}}$$

Branch

$$\ln z = \ln|z| + i(\theta + 2\pi n)$$

$\downarrow$

Infinitely many values

$n = 0, \pm 1, \pm 2, \pm 3 \cdots$

$\downarrow$

strictly speaking, not a function

- multiple valued function
- infinite collection of logarithmic functions

$\vdots$

$n=-2$	$\ln\lvert z\rvert + i(\theta - 4\pi)$	⇢ a branch
$n=-1$	$\ln\lvert z\rvert + i(\theta - 2\pi)$	⇢ a branch
$n=0$	$\ln\lvert z\rvert + i(\theta + 0)$	⇢ a branch
$n=+1$	$\ln\lvert z\rvert + i(\theta + 2\pi)$	⇢ a branch
$n=+2$	$\ln\lvert z\rvert + i(\theta + 4\pi)$	⇢ a branch

$\vdots$

the principal logarithmic function $\quad \mathrm{Ln}(z) = \ln|z| + i\theta$ ⇢ defined on the principal branch real number

$\boxed{\ln 5} = 1.6094$

complex number

$\boxed{\ln 5} = 1.6094 + i\,2\pi n$

$\boxed{\ln(-2)}$

$\arg(-2) = -\pi$

$\ln|-2| = 0.6932$

$\ln(-2) = \ln|-2| + i(\arg(-2) + 2\pi n)$

$= 0.6932 + i(\pi + 2\pi n)$

$\boxed{\ln(i)}$

$\arg(i) = \pi/2$

$\ln|i| = \ln 1 = 0$

$\ln(i) = \ln|i| + i(\arg(i) + 2\pi n)$

$= (\pi/2 + 2\pi n)$

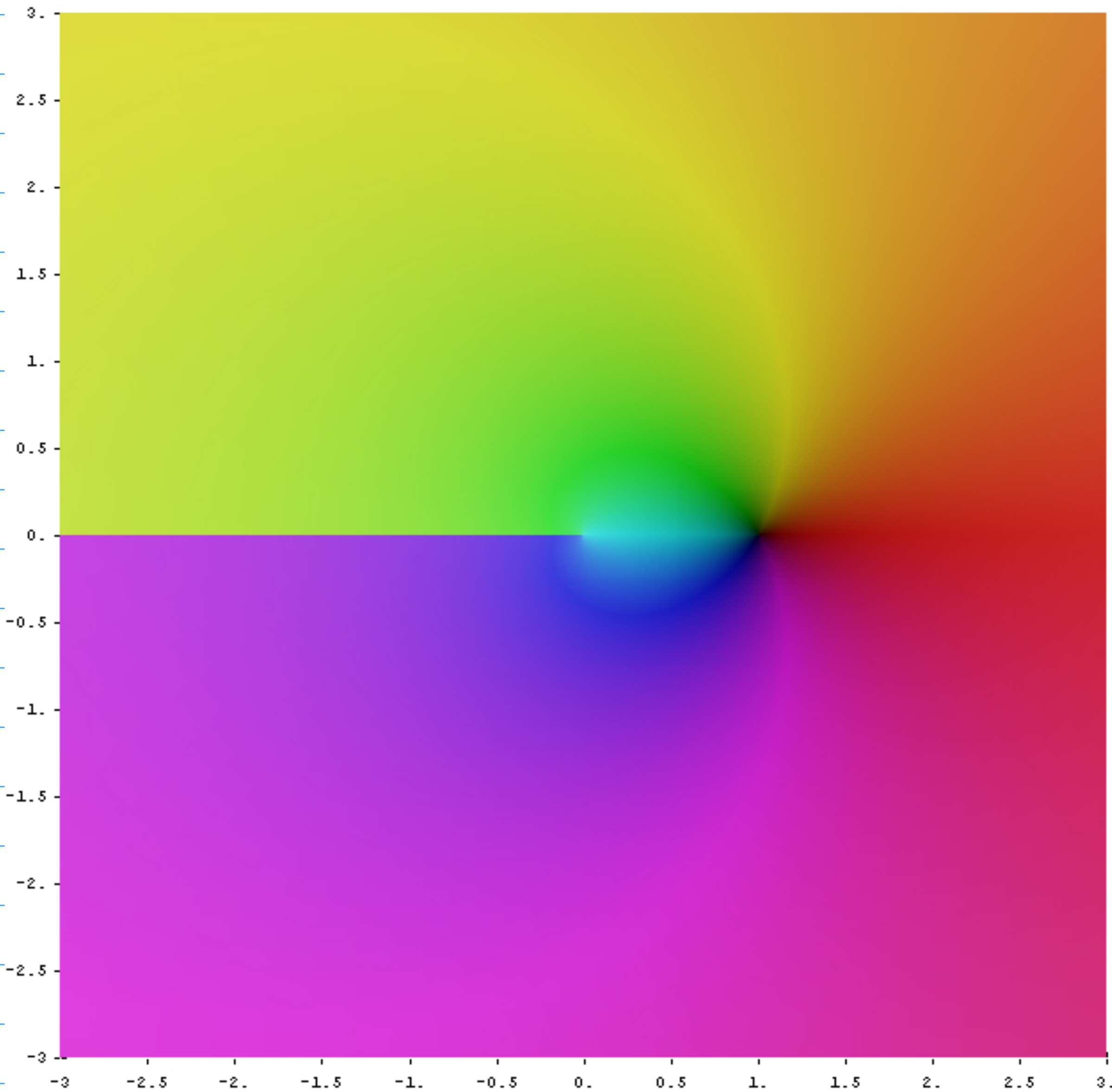
$\boxed{\ln(-1-i)}$

$\arg(-1-i) = 5\pi/4$

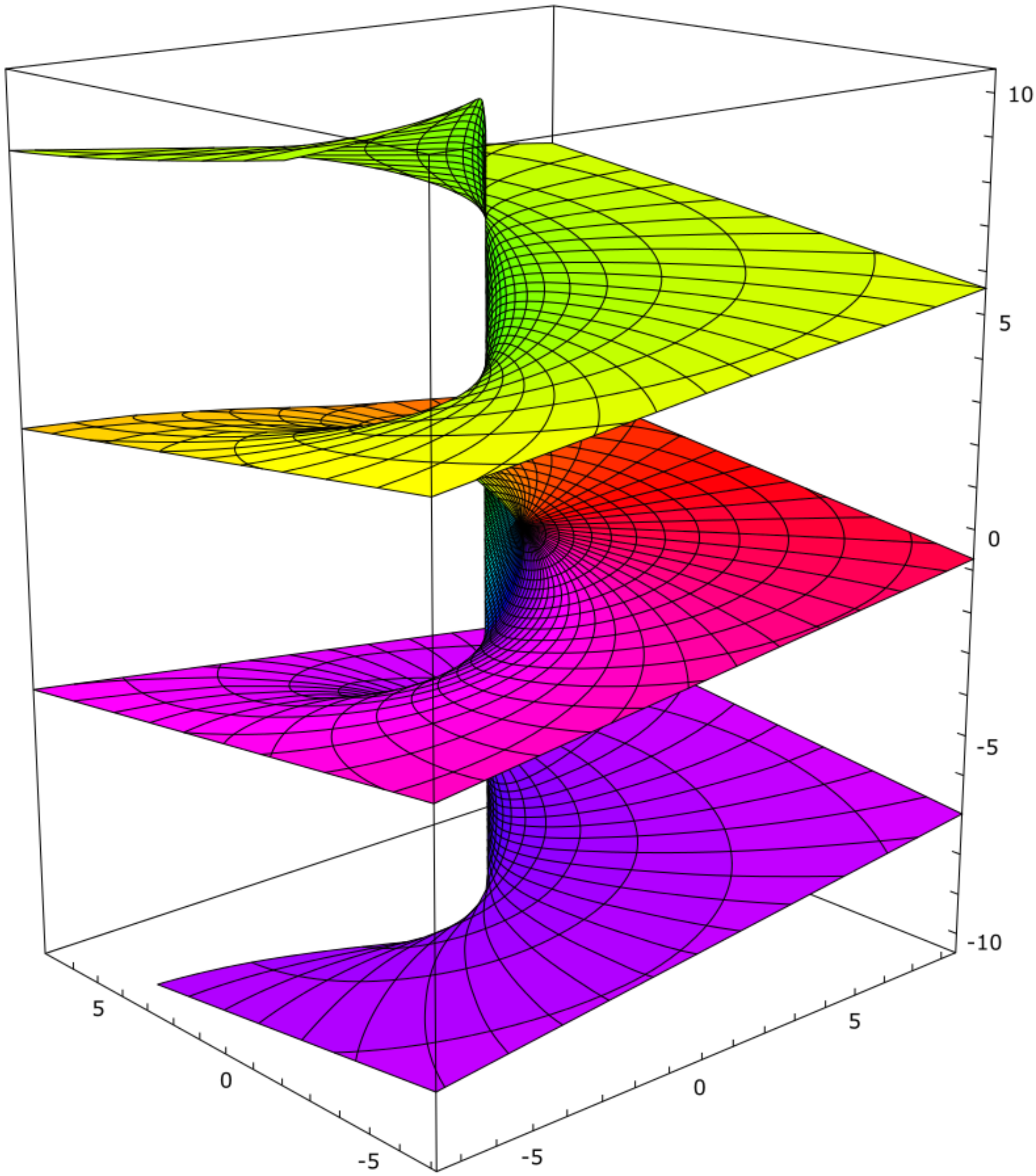
$\ln|-1-i| = \ln\sqrt{2} = 0.3466$

$\ln(-1-i) = \ln|i| + i(\arg(i) + 2\pi n)$

$= 0.3466 + i\left(\frac{5\pi}{4} + 2\pi n\right)$

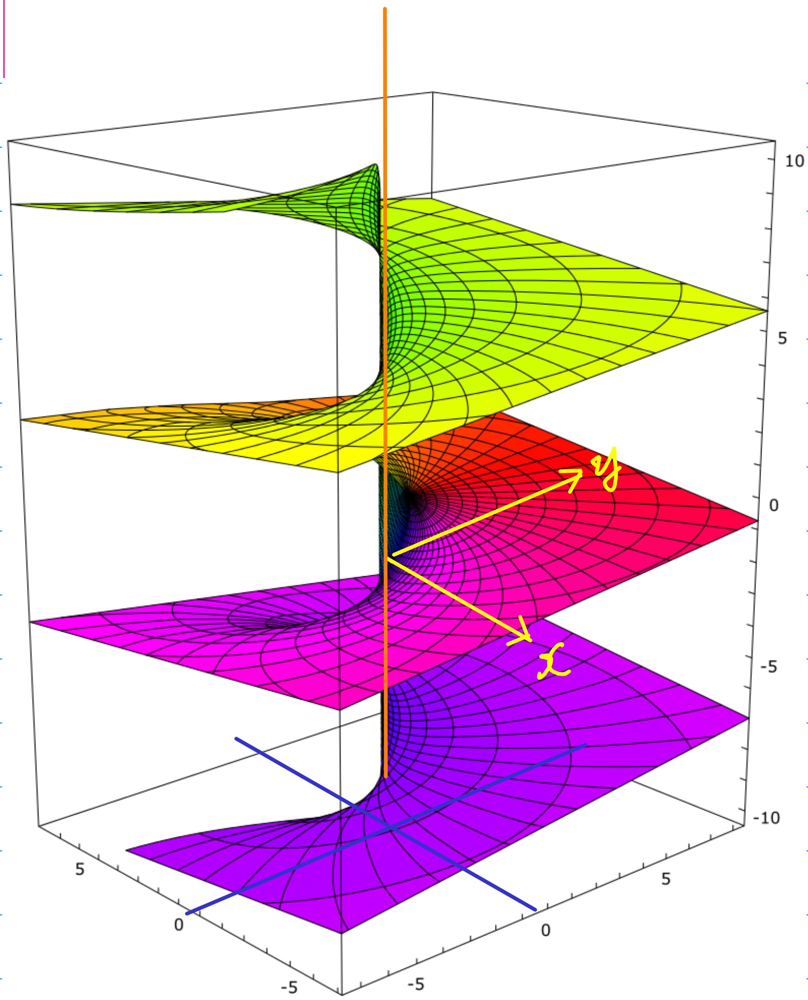


3.
2.5
2.
1.5
1.
0.5
0.
-0.5
-1.
-1.5
-2.
-2.5
-3
-3
-2.5
-2.
-1.5
-1.
-0.5
0.
0.5
1.
1.5
2.
2.5
3.



https://en.wikipedia.org/wiki/Complex_logarithm

A plot of the multi-valued imaginary part of the complex logarithm function, which shows the branches. As a complex number z goes around the origin, the imaginary part of the logarithm goes up or down. This makes the origin a *branch point* of the function.



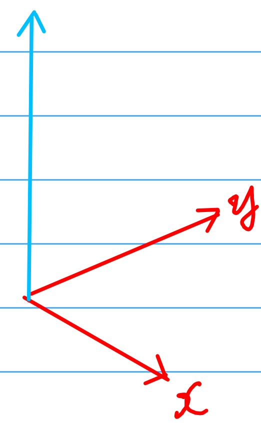
a branch

a branch

the principle branch

a branch

$\mathrm{Im}\{w\} = \arg z$



$u = \ln|z| = \ln r$

$v = \arg z = \theta$

$\ln z = \log_e|z| + i(\theta + 2n\pi)$

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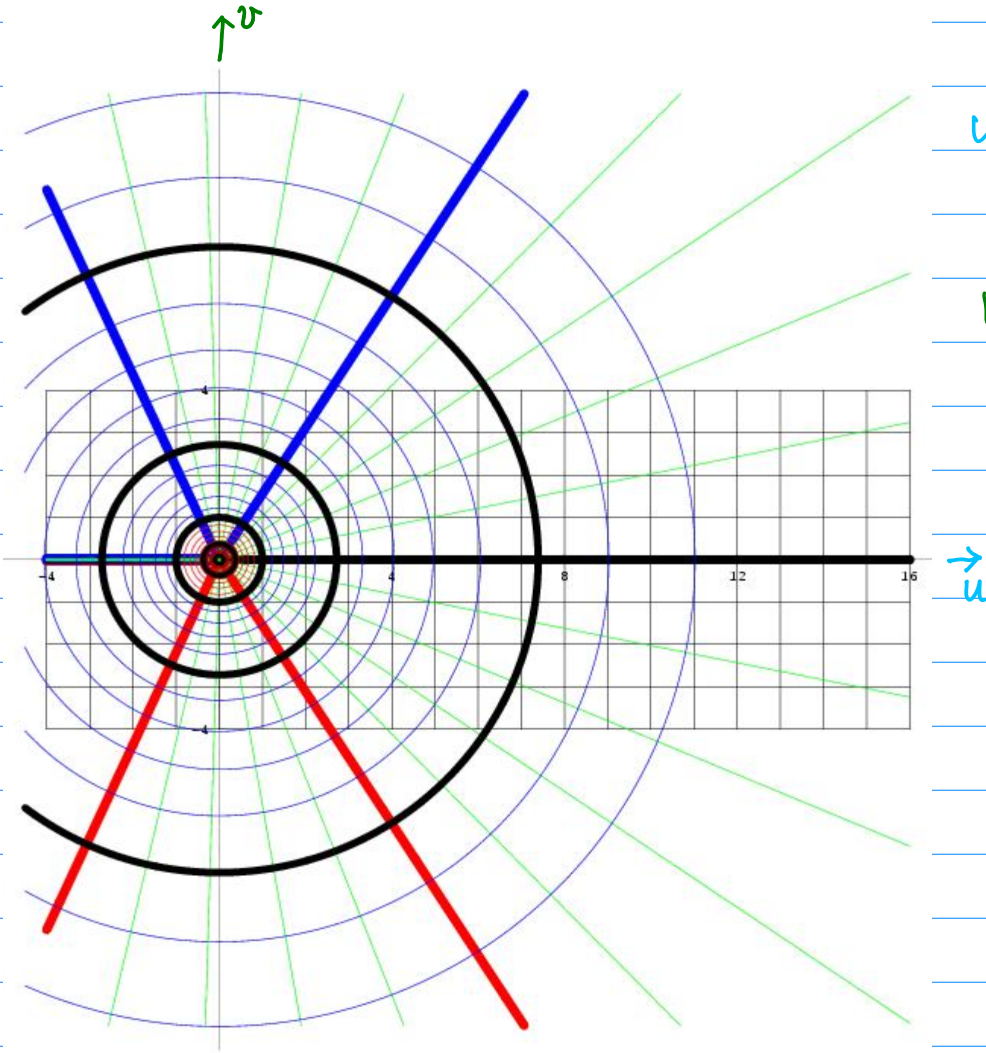
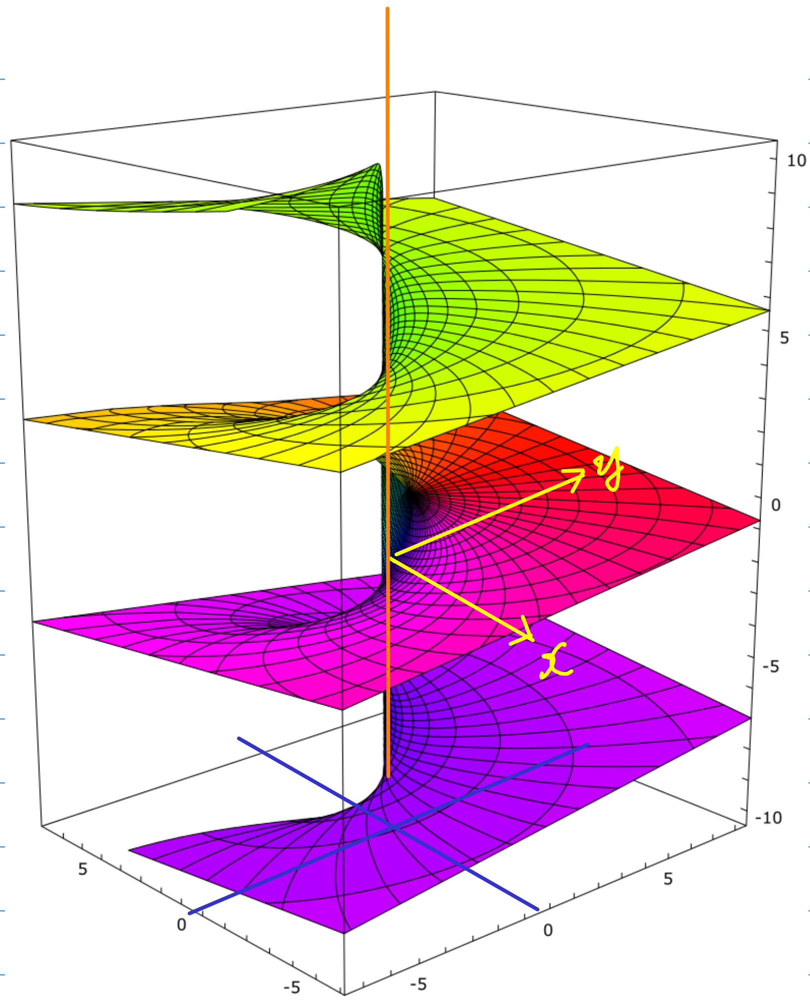


$$u = \ln|z| \qquad \mathrm{Re}\{\ln z\}$$
$$= \ln r$$

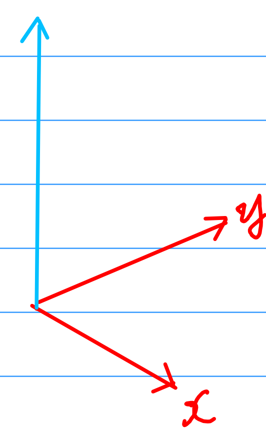
$$v = \arg z \qquad \mathrm{Im}\{\ln z\}$$
$$= \theta$$

The circles Re(Log *z*) = constant and the rays Im(Log *z*) = constant in the complex *z*-plane.

https://en.wikipedia.org/wiki/Complex_logarithm



$$\arg w = \arg(\ln z)$$



A visualization of the Riemann surface of log *z*. The surface appears to spiral around a vertical line corresponding to the origin of the complex plane. The actual surface extends arbitrarily far both horizontally and vertically, but is cut off in this image.

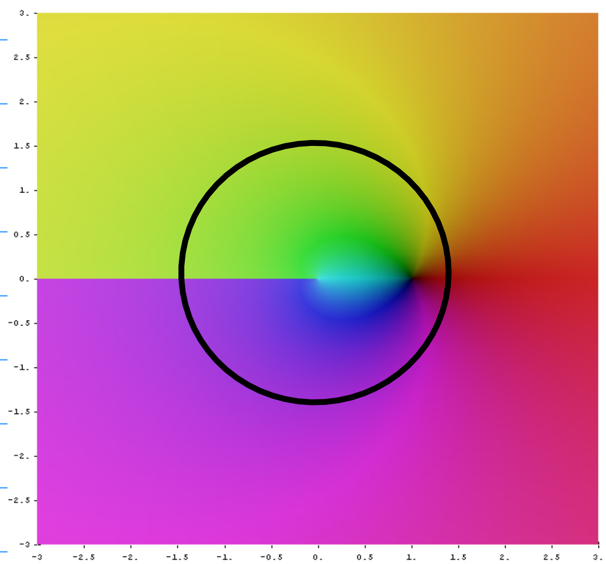
https://en.wikipedia.org/wiki/Complex_logarithm

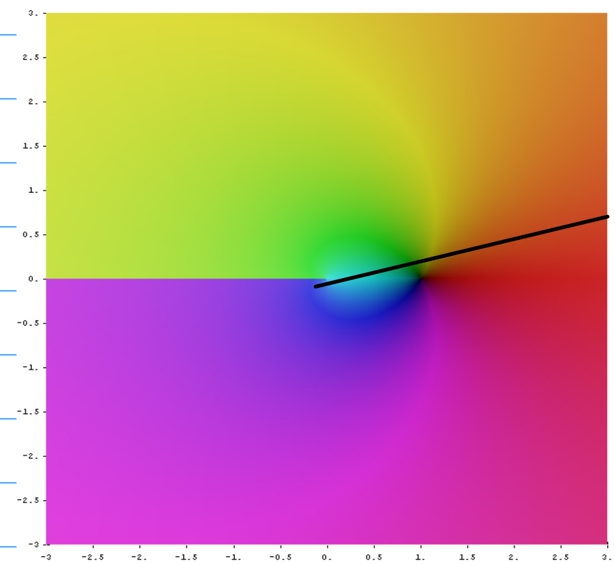
$|w| = |\ln z|$

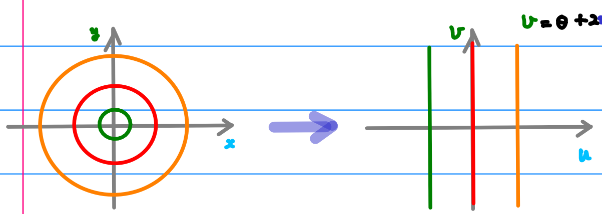
Covers all the hue changes

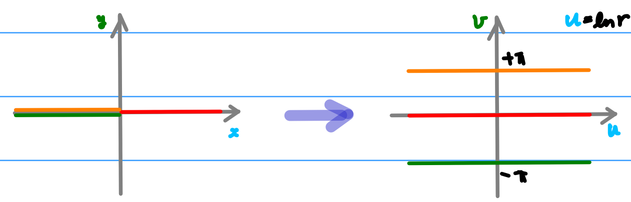
$\arg w = \arg(\ln z)$

Covers all the intensity change





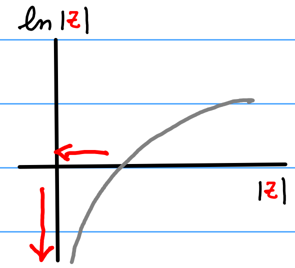




Analiticity

$f(z) = \text{Ln } z$

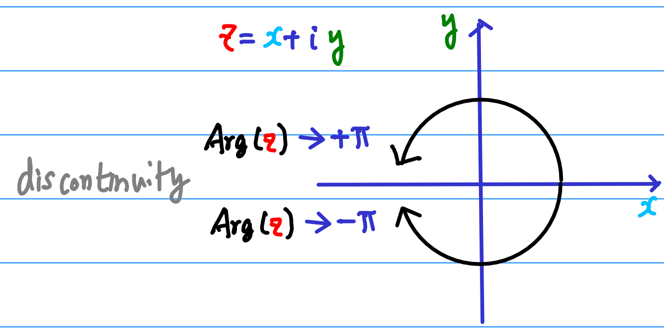
$\ln|z| + i\,\text{Arg}(z)$ $\qquad |z| > 0$



$f(0) \to -\infty$ undefined

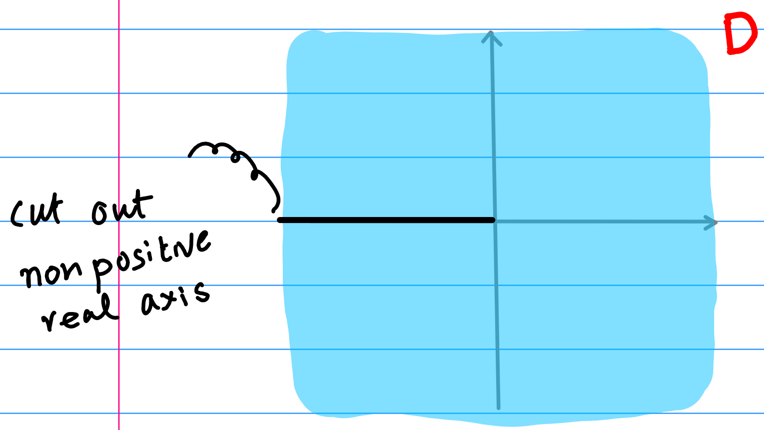
f(z) is not analytic at $z = 0$ ← f(z) is not continuous at $z = 0$

$z = x + iy$



f(z) is not analytic throughout negative real axis ← f(z) is not continuous throughout negative real axis

$f(z) = \mathrm{Ln}\, z$ is analytic in the domain D



$f(z) = \mathrm{Ln}\, z$ is defined on the principal branch

this cut out is a branch cut

$$\frac{d}{dz} \mathrm{Ln}\, z = \frac{1}{z}$$

$$\frac{d}{dz} \operatorname{Ln} z = \frac{1}{z}$$

$$\begin{aligned}
\operatorname{Ln} z &= \ln |z| + i \operatorname{Arg} z \\
&= \ln (x^2+y^2)^{1/2} + i \tan^{-1} \frac{y}{x} \\
&= \frac{1}{2} \ln (x^2+y^2) \quad + i \tan^{-1} \frac{y}{x}
\end{aligned}$$

$$\frac{d}{dz} \arcsin(z) = \frac{1}{\sqrt{1-z^2}}; \qquad z \neq -1, +1$$

$$\frac{d}{dz} \arccos(z) = -\frac{1}{\sqrt{1-z^2}}; \qquad z \neq -1, +1$$

$$\frac{d}{dz} \arctan(z) = \frac{1}{1+z^2}; \qquad z \neq -\mathrm{i}, +\mathrm{i}$$

$$\frac{d}{dz} \operatorname{arccot}(z) = -\frac{1}{1+z^2}; \qquad z \neq -\mathrm{i}, +\mathrm{i}$$

$$\frac{d}{dz} \operatorname{arcsec}(z) = \frac{1}{z^2\sqrt{1-\frac{1}{z^2}}}; \qquad z \neq -1, 0, +1$$

$$\frac{d}{dz} \operatorname{arccsc}(z) = -\frac{1}{z^2\sqrt{1-\frac{1}{z^2}}}; \qquad z \neq -1, 0, +1$$

$$\begin{aligned}
\frac{d}{dz} \operatorname{Ln} z &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\
&= \frac{\frac{1}{2}}{(x^2+y^2)} \frac{\partial}{\partial x}(x^2+y^2) \quad + i \frac{1}{1+\frac{y^2}{x^2}} \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \\
&= \frac{\frac{1}{2}}{(x^2+y^2)} \quad 2x \quad + i \frac{1}{1+\frac{y^2}{x^2}} \frac{-y}{x^2} \\
&= \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2} \\
&= \frac{(x-iy)}{(x-iy)} \frac{1}{(x+iy)} \\
&= \frac{1}{(x+iy)} = \frac{1}{z}
\end{aligned}$$

the domain D excludes a branch cut (the non-positive real axis)

$$\begin{cases} 0 + i\,0 & \times \\ x + i \cdot 0 \quad (x<0) & \times \end{cases}$$

Complex Powers

$$x^a = e^{a \ln x} = e^{\ln x^a}$$

$$\boxed{} = e^{\bigcirc} \qquad \bigcirc = \ln \boxed{}$$

$$\boxed{x} = e^{\ln x} \leftarrow \ln x = \ln \boxed{x}$$

$$x^a = e^{\ln x^a}$$

$$x^a = e^{a \ln x}$$

$$z^\alpha = e^{\alpha \ln z} \qquad (z \neq 0)$$

In general, $\ln z \to$ multivalued

$z^\alpha \to$ multivalued

But if $\alpha = n$ integer

$z^\alpha = z^n \to$ single valued

$\because z^{-1}, z^0, z^{+1}, z^{+2}$: only one value

$$z^2 = e^{2 \ln z} = e^{2(\ln r + \theta + 2\pi n)} = e^{\ln r^2} \cdot e^{\theta + 2\pi n} \cdot e^{\theta + 2\pi n}$$

$$= (re^{i\theta})(re^{i\theta})$$

principal value of $z^\alpha = e^{\alpha \operatorname{Ln} z}$

$$i^{2i} = e^{2i(\ln 1 + i(\frac{\pi}{2} + 2\pi n))} = e^{i(\pi + 4\pi n)}$$