Root Locus (2A)

Young Won Lim
10/15/14

Please send corrections (or suggestions) to youngwlim@hotmail.com.

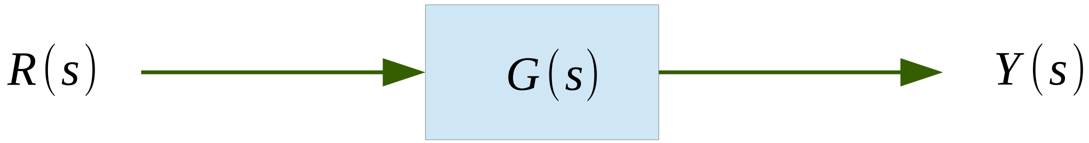
This document was produced by using OpenOffice and Octave.

The Original Transfer Function

$$G(s) = \frac{N(s)}{D(s)}$$

$$= \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$= \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$



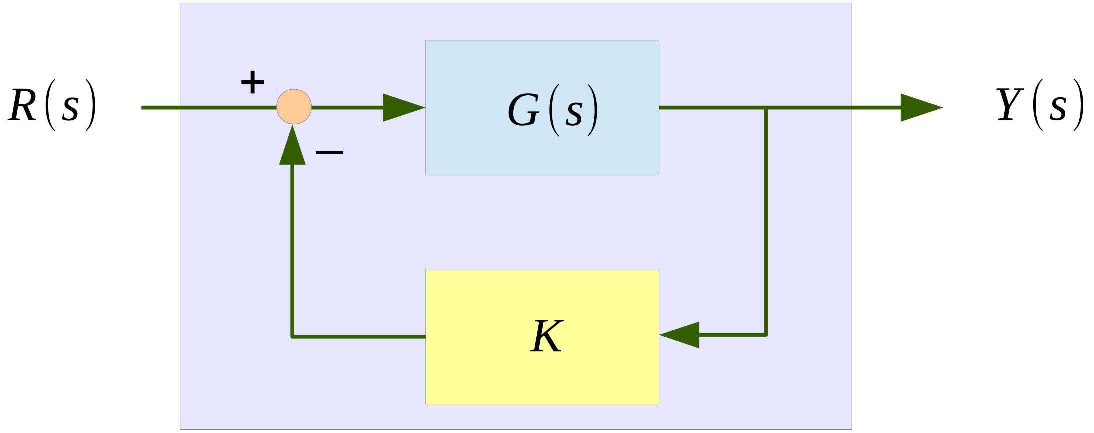
http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

The New Transfer Function

$$\frac{KG(s)}{1+KG(s)} = \frac{K\,N(s)/D(s)}{1+K\,N(s)/D(s)}$$

$$= \frac{K\,N(s)}{D(s)+K\,N(s)}$$

$$D(s) + K\,N(s) = 0$$



http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Conditions for new roots (1)

Find roots s

$$D(s) + K\,N(s) = 0$$

Conditions for a root s_0

$$(K = 0) \wedge (D(s_0) = 0) \quad \Rightarrow \quad 0 + 0\,N(s) = 0$$

$$(K \neq 0) \wedge (D(s_0) \neq 0) \wedge \left(\frac{D(S_0)}{N(s_0)} = real\right) \quad \Rightarrow \quad D(s_0) + \left(-\frac{D(s_0)}{N(s_0)}\right)N(s_0) = 0$$

$$(D(s_0) \neq 0) \wedge \left(K = -\frac{D(s_0)}{N(s_0)} = real \neq 0\right)$$

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Conditions for new roots (2)

Find roots s

$$D(s) + K\,N(s) \;=\; 0$$

$N(s_0) = 0$ ⇨ $D(s_0) \neq 0$ ($\Leftarrow$ no common root)

$D(s_0) + K\,N(s_0) \;=\; D(s_0) \;\neq\; 0$ ⇨ cannot be a root

$N(s_0) \neq 0$ for a root s_0

$K = 0$ ⇨ $D(s_0) + 0\,N(s_0) \;=\; 0$ ⇨ $D(s_0) \;=\; 0$: original poles

$K \neq 0$ ⇨ $D(s_0) + K\,N(s_0) \;=\; 0$ ⇨ $D(s_0) \;\neq\; 0$

$D(s_0) \;=\; -K\,N(s_0) \;\neq\; 0$ $\qquad K \;=\; -\dfrac{D(s_0)}{N(s_0)} \;\neq\; 0$

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Check for new roots

Find roots s

$$D(s) + K\,N(s) = 0$$

$D(s_0) = 0 \Rightarrow K = 0 \Rightarrow D(s_0) + K\,N(s_0) = 0$

s_0 is on the root locus of the pair (N, D)

$D(s_0) \neq 0 \Rightarrow K = -\dfrac{D(s_0)}{N(s_0)} \Rightarrow D(s_0) + K\,N(s_0) = 0$

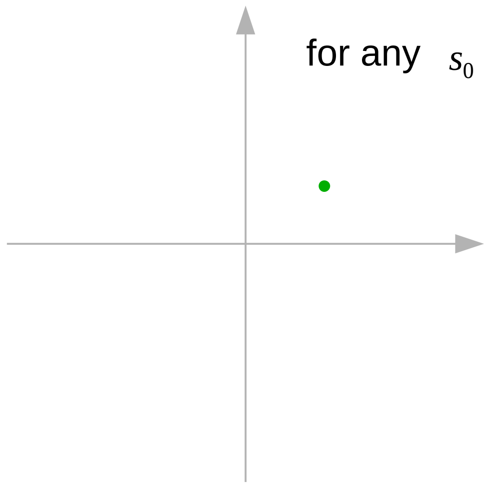
$\dfrac{N(s_0)}{D(s_0)} = real$ s_0 is on the root locus of the pair (N, D)

$K > 0 \quad -\frac{1}{G(s_0)} = -\frac{D(s_0)}{N(s_0)} > 0$ positive root locus of the pair (N, D)

$K < 0 \quad -\frac{1}{G(s_0)} = -\frac{D(s_0)}{N(s_0)} < 0$ negative root locus of the pair (N, D)

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Rule: Uniqueness of K



either s_0 is not on the root locus

or s_0 is on the root locus

then there is a unique K

$$D(s_0) + K\,N(s_0) = 0$$

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Rule: Real number condition of K

$$D(s_0) \neq 0$$

$$\frac{N(s_0)}{D(s_0)} = real$$

$$\Rightarrow \quad K = -\frac{D(s_0)}{N(s_0)} \quad \Rightarrow \quad D(s_0) + K\,N(s_0) = 0$$

s_0 is on the root locus of the pair (N, D)

$$\Updownarrow$$

$$arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = q\pi$$

$$\frac{N(s_0)}{D(s_0)} = -\frac{1}{K}$$

$K > 0$	$Arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = \pi$
$K < 0$	$Arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = 0$

$$arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = arg\left\{\frac{(s_0-z_1)(s_0-z_2)\cdots(s_0-z_m)}{(s_0-p_1)(s_0-p_2)\cdots(s_0-p_n)}\right\} = arg(s_0-z_1) + arg(s_0-z_2) + \cdots + arg(s_0-z_m) - [arg(s_0-p_1) + arg(s_0-p_2) + \cdots + arg(s_0-p_n)]$$

$$arg(s_0-z_1) + arg(s_0-z_2) + \cdots + arg(s_0-z_m) - [arg(s_0-p_1) + arg(s_0-p_2) + \cdots + arg(s_0-p_n)] =$$

$(2q+1)\pi$	$K > 0$	positive root locus
$(2q)\pi$	$K < 0$	negative root locus

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Number of Roots

$$D(s) + K\,N(s) = 0$$

n roots

$$G(s) = \frac{N(s)}{D(s)}$$

$$= \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$= \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$K = -1$

n-1 roots
1 root at infinity

n-2 roots
2 root at infinity

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Roots when K=0

$$K = 0 \quad \Rightarrow \quad D(s_0) + 0\,N(s_0) = 0 \quad \Rightarrow \quad D(s_0) = 0$$

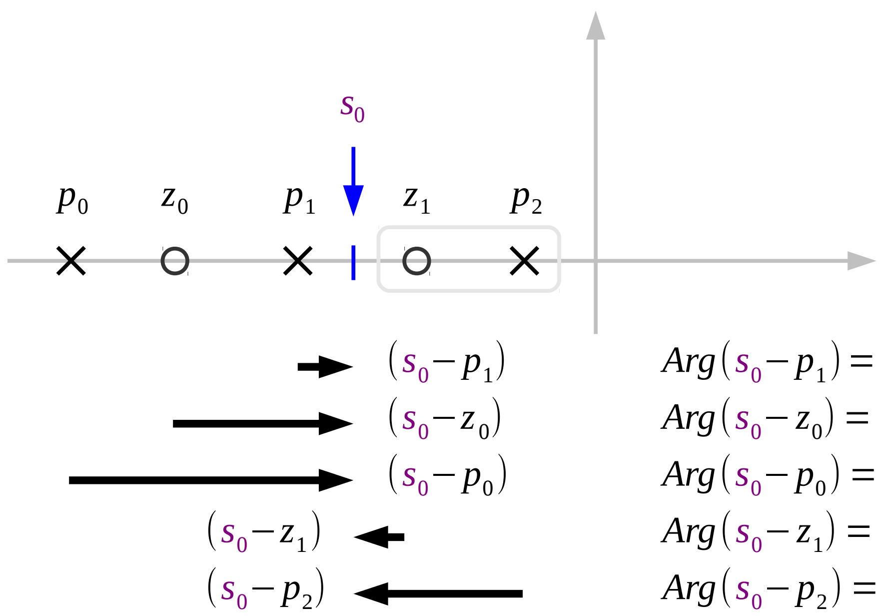
The roots of D(s) :
The original poles

$$G(s) = \frac{N(s)}{D(s)}$$

$$= \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$= \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf



$Arg(s_0 - p_1) = 0$

$Arg(s_0 - z_0) = 0$

$Arg(s_0 - p_0) = 0$

$Arg(s_0 - z_1) = \pi$

$Arg(s_0 - p_2) = \pi$

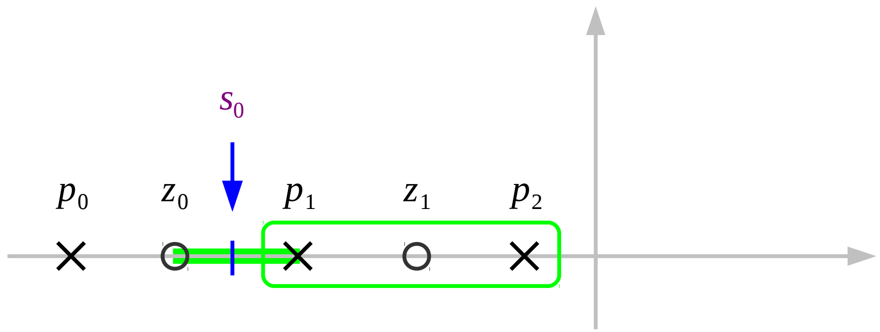
positive root locus

$$arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = (2q+1)\pi$$

$$Arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = Arg(s_0 - z_0) + Arg(s_0 - z_1)$$
$$- [Arg(s_0 - p_0) + Arg(s_0 - p_1) + Arg(s_0 - p_2)] = 0 \quad \neq (2q+1)\pi$$

Not on the positive root locus

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf



positive root locus

$$arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = (2q+1)\pi$$

$(s_0 - z_0)$	$Arg(s_0 - z_0) = 0$
$(s_0 - p_0)$	$Arg(s_0 - p_0) = 0$
$(s_0 - p_1)$	$Arg(s_0 - p_1) = \pi$
$(s_0 - z_1)$	$Arg(s_0 - z_1) = \pi$
$(s_0 - p_2)$	$Arg(s_0 - z_1) = \pi$

$$Arg\left\{\frac{N(s_0)}{D(s_0)}\right\} = Arg(s_0 - z_0) + Arg(s_0 - z_1)$$
$$- [Arg(s_0 - p_0) + Arg(s_0 - p_1) + Arg(s_0 - p_2)] = \pi \quad = (2q+1)\pi$$

on the positive root locus

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Roots when K → ∞ (1)

$$K \to \infty \quad \Rightarrow \quad D(s) + K\,N(s) \;\to\; K\,N(s) \quad \Rightarrow \quad N(s) = 0$$

The roots of N(s) :
The original zeros

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$-K = \frac{D(s)}{N(s)} = \frac{s^n + a_1 s^{n-1} + \cdots + a_n}{s^m + b_1 s^{m-1} + \cdots + b_m}$$

$$= s^{n-m}\left[1 + \frac{a_1 - b_1}{s} + \cdots \right]$$

$$\begin{array}{r|l} & s^{n-m} + (a_1 - b_1)\, s^{n-m-1} + \cdots \\ \hline s^m + b_1 s^{m-1} + \cdots + b_m & s^n + a_1 s^{n-1} + \cdots + a_n \\ & s^n + b_1 s^{n-1} + \cdots \\ \cline{2-2} & (a_1 - b_1)\, s^{n-1} + \cdots \end{array}$$

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Roots when K → ∞ (2)

$$-K = \frac{D(s)}{N(s)} = \frac{s^n + a_1 s^{n-1} + \cdots + a_n}{s^m + b_1 s^{m-1} + \cdots + b_m}$$

$$= s^{n-m}\left[1 + \frac{a_1 - b_1}{s} + \cdots \right]$$

$$= s^{n-m}\left[1 + \frac{a_1 - b_1}{s} + \cdots \right]$$

$$(-K)^{\frac{1}{n-m}} \approx s\left[1 + \frac{a_1 - b_1}{s}\right]^{\frac{1}{n-m}}$$

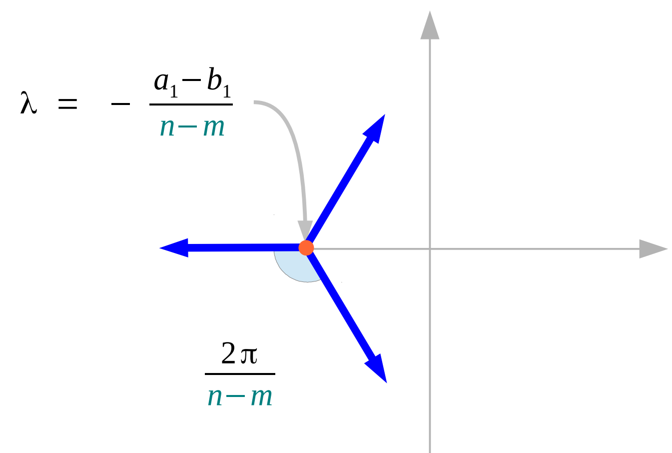
$$(1+x)^k \approx 1 + kx \quad (x \ll 1)$$

$$(-K)^{\frac{1}{n-m}} \approx s\left[1 + \frac{a_1 - b_1}{(n-m)\,s}\right]$$

$$(K e^{j\pi})^{\frac{1}{n-m}} \approx s + \frac{a_1 - b_1}{n-m}$$

$$K^{\frac{1}{n-m}} e^{j\left(\frac{\pi + 2q\pi}{n-m}\right)} \approx s + \frac{a_1 - b_1}{n-m}$$

$$s_0 \equiv K^{\frac{1}{n-m}} e^{j\left(\frac{\pi + 2q\pi}{n-m}\right)} - \frac{a_1 - b_1}{n-m}$$ centroid



http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Centroid (1)

$$G(s) = \frac{N(s)}{D(s)}$$

$$= \frac{s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

$$= \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$b_1 = -\sum_i z_i = -\sum_i \Re\{z_i\}$$

$$a_1 = -\sum_j p_j = -\sum_j \Re\{p_j\}$$

$$s_0 \equiv K^{\frac{1}{n-m}} e^{j\left(\frac{\pi+2q\pi}{n-m}\right)} - \frac{a_1 - b_1}{n-m}$$

$$\lambda = -\frac{a_1 - b_1}{n-m} = \frac{\sum p_j - \sum z_i}{n-m} = \frac{\sum \Re\{p_j\} - \sum \Re\{z_i\}}{n-m}$$ Centroid

$$\theta = \frac{(2q+1)\pi}{n-m} \qquad q = 0, 1, \cdots, n-m+1$$ Angle of asymptotes

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Centroid (2)

$$G(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$\lambda = -\frac{a_1-b_1}{n-m} = \frac{\sum p_j - \sum z_i}{n-m} = \frac{\sum \Re\{p_j\} - \sum \Re\{z_i\}}{n-m}$$ Centroid

$$G(s) = \frac{N(s)}{D(s)} = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

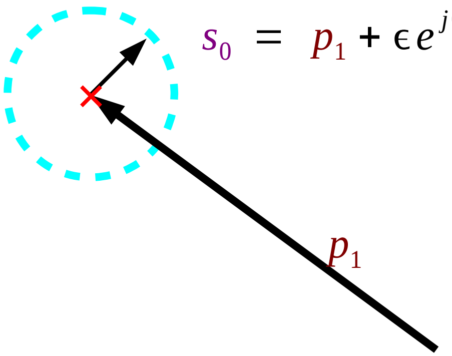
$$\lambda = +\frac{a_1-b_1}{n-m} = \frac{-\sum p_j + \sum z_i}{n-m} = \frac{-\sum \Re\{p_j\} + \sum \Re\{z_i\}}{n-m}$$ Centroid

http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf

Angle of departure from a pole

$$arg(s_0 - z_1) + arg(s_0 - z_2) + \cdots + arg(s_0 - z_m)$$
$$-[arg(s_0 - p_1) + arg(s_0 - p_2) + \cdots + arg(s_0 - p_n)] =$$

$(2q+1)\pi$ $K > 0$ positive root locus



$$s_0 = p_1 + \epsilon e^{j\theta_d}$$

$$arg(p_1 + \epsilon e^{j\theta_d} - z_1) + arg(p_1 + \epsilon e^{j\theta_d} - z_2) + \cdots + arg(p_1 + \epsilon e^{j\theta_d} - z_m)$$
$$-[arg(p_1 + \epsilon e^{j\theta_d} - p_1) + arg(p_1 + \epsilon e^{j\theta_d} - p_2) + \cdots + arg(p_1 + \epsilon e^{j\theta_d} - p_n)] \approx$$

$$arg(p_1 - z_1) + arg(p_1 - z_2) + \cdots + arg(p_1 - z_m)$$
$$-[arg(\epsilon e^{j\theta_d}) + arg(p_1 - p_2) + \cdots + arg(p_1 - p_n)] = (2q+1)\pi$$

$$arg(p_1 - z_1) + arg(p_1 - z_2) + \cdots + arg(p_1 - z_m)$$
$$-[\theta_d + arg(p_1 - p_2) + \cdots + arg(p_1 - p_n)] = (2q+1)\pi$$

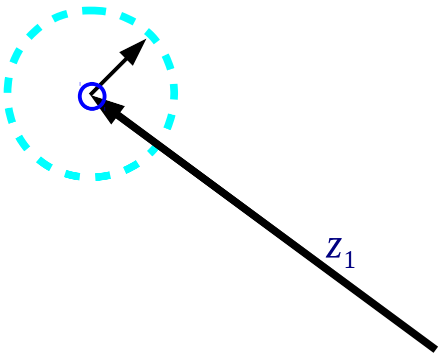
$$\pi + arg(p_1 - z_1) + arg(p_1 - z_2) + \cdots + arg(p_1 - z_m)$$
$$-[arg(p_1 - p_2) + \cdots + arg(p_1 - p_n)] = \theta_d$$

Angle of arrival from a zero

$$\arg(s_0 - z_1) + \arg(s_0 - z_2) + \cdots + \arg(s_0 - z_m)$$
$$-[\arg(s_0 - p_1) + \arg(s_0 - p_2) + \cdots + \arg(s_0 - p_n)] =$$

$(2q+1)\pi$ $K > 0$ positive root locus

$$s_0 = z_1 + \epsilon e^{j\theta_a}$$



$$\arg(z_1 + \epsilon e^{j\theta_a} - z_1) + \arg(z_1 + \epsilon e^{j\theta_a} - z_2) + \cdots + \arg(z_1 + \epsilon e^{j\theta_a} - z_m)$$
$$-[\arg(z_1 + \epsilon e^{j\theta_a} - p_1) + \arg(z_1 + \epsilon e^{j\theta_a} - p_2) + \cdots + \arg(z_1 + \epsilon e^{j\theta_a} - p_n)] \approx$$

$$\arg(\epsilon e^{j\theta_a}) + \arg(z_1 - z_2) + \cdots + \arg(z_1 - z_m)$$
$$-[\arg(z_1 - p_1) + \arg(z_1 - p_2) + \cdots + \arg(z_1 - p_n)] = (2q+1)\pi$$

$$\theta_a + \arg(z_1 - z_2) + \cdots + \arg(z_1 - z_m)$$
$$-[\arg(z_1 - p_1) + \arg(z_1 - p_2) + \cdots + \arg(z_1 - p_n)] = (2q+1)\pi$$

$$\pi - [\arg(z_1 - z_2) + \cdots + \arg(z_1 - z_m)]$$
$$+ [\arg(z_1 - p_1) + \arg(z_1 - p_2) + \cdots + \arg(z_1 - p_n)] = \theta_a$$

Breakaway Points (Repeated Roots)

$$D(s) + K\,N(s) = (s - s_1)^r (s - s_2)\cdots$$

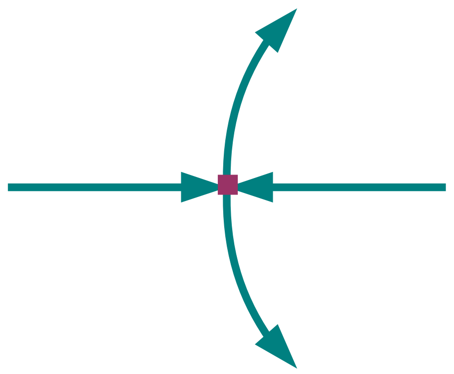
$$\big[D(s) + K\,N(s)\big]_{s=s_0} = 0$$

$$\left[\frac{d}{ds}\big(D(s) + K\,N(s)\big)\right]_{s=s_0} = 0$$

$$\left[\frac{d}{ds}D(s) + K\frac{d}{ds}N(s)\right]_{s=s_0} = 0$$

$$\frac{d}{ds}D(s_0) - \frac{D(s_0)}{N(s_0)}\frac{d}{ds}N(s_0) = 0$$

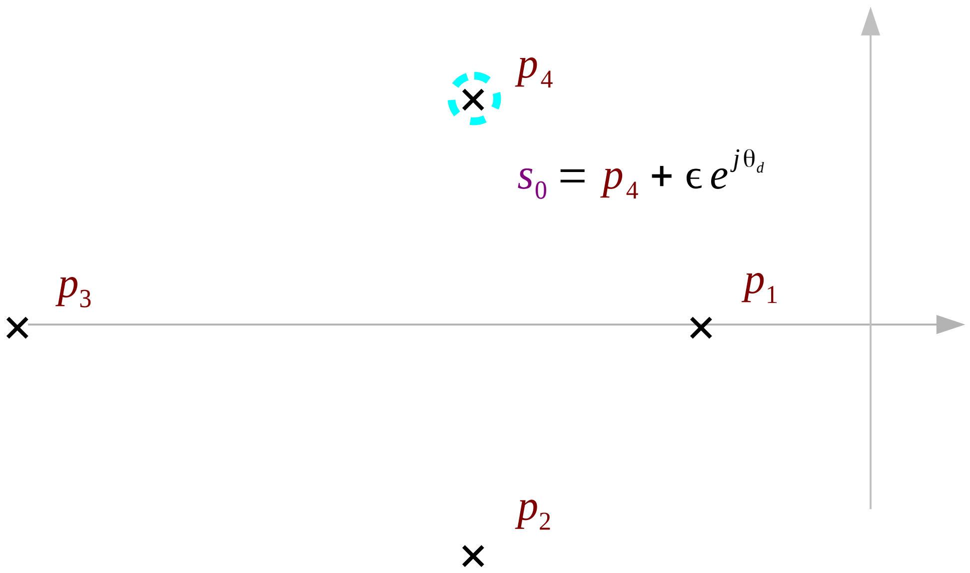
$$N(s_0)\frac{d}{ds}D(s_0) - D(s_0)\frac{d}{ds}N(s_0) = 0$$

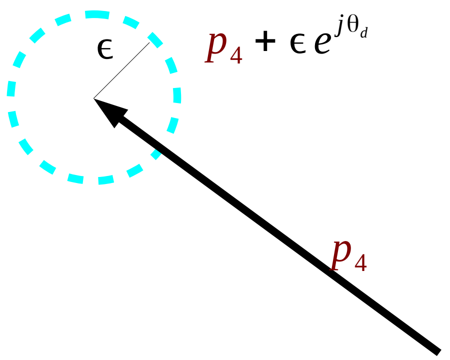


Check this for repeated roots

$$N(s)\frac{d}{ds}D(s) - D(s)\frac{d}{ds}N(s) = 0$$

$$\frac{d}{ds}K = \frac{d}{ds}\left[-\frac{D(s)}{N(s)}\right] = 0$$



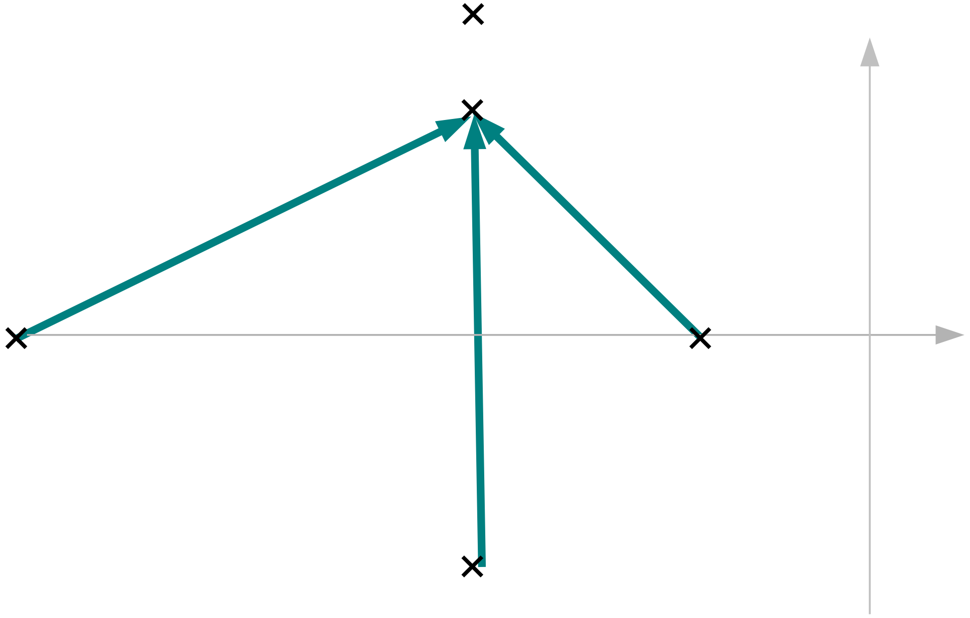


$$arg(s_0 - z_1) + arg(s_0 - z_2) + \cdots + arg(s_0 - z_m)$$
$$-[arg(s_{0s} - p_1) + arg(s_0 - p_2) + \cdots + arg(s_0 - p_n)] =$$

$(2q+1)\pi$ $K > 0$ positive root locus

$$= -[arg(s_{0s} - p_1) + arg(s_0 - p_2) + \cdots + arg(s_0 - p_n)] =$$

A



References

[1] http://en.wikipedia.org/
[2] http://www.cds.caltech.edu/~murray/courses/cds101/fa03/caltech/pph02_ch29.pdf