CLTI Impulse Matching (6B)

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Finding Impulse Response from Diff Equations

- Impulse Matching Method
- Simplified Impulse Matching Method
- Green's Function

- Impulse Matching

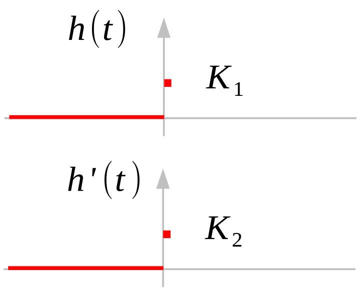
Impulse Matching (N>M) (1)

$$(D^2 + 5D + 6)\,y(t) = (D+1)\,x(t)$$

$$h''(t) + 5h'(t) + 6h(t) = \delta'(t) + \delta(t)$$

$$(m^2 + 5m + 6) = 0 \qquad m = -2,\ -3$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$$



$$h(0^+) = K_1 \quad \Rightarrow \quad h'(0) = K_1\delta(t)$$

$$h'(0^+) = K_2 \quad \Rightarrow \quad h''(0) = K_2\delta(t) + K_1\delta'(t)$$

$$h''(0) + 5h'(0) + 6h(0)$$

$$= [K_2\delta(t) + K_1\delta'(t)] + 5[K_1\delta(t)] + 6[const]$$

$$= K_1\delta'(t) + (5K_1 + K_2)\delta(t) + 6[const]$$

$$= \delta'(t) + \delta(t)$$

$$K_1 = 1 \quad \Rightarrow \quad h(0^+) = 1$$

$$K_2 = -4 \quad \Rightarrow \quad h'(0^+) = -4$$

Impulse Matching (N>M) (2)

$$(D^2 + 5D + 6)\,y(t) = (D+1)\,x(t)$$

$$h''(t) + 5h'(t) + 6h(t) = \delta'(t) + \delta(t)$$

$$(m^2 + 5m + 6) = 0 \qquad m = -2,\ -3$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t})\,u(t)$$

$$h'(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t})\,u(t) + (c_1 e^{-2t} + c_2 e^{-3t})\,\delta(t)$$

$$= (-2c_1 e^{-2t} - 3c_2 e^{-3t})\,u(t) + (c_1 + c_2)\,\delta(t)$$

$$h(0^+) = (c_1 + c_2)$$

$$h'(0^+) = (-2c_1 - 3c_2)$$

$$h(0^+) = 1$$

$$h'(0^+) = -4$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

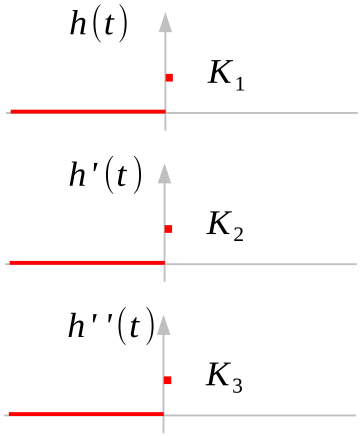
$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ -4 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{+1}{-1} = -1 \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = +2$$

$$h(t) = (-e^{-2t} + 2e^{-3t})\,u(t)$$

Impulse Matching (N>M) (3)

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$\left(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N\right) = 0 \quad \Rightarrow \quad h(t) = \left(\sum_{k=1}^{N} c_k e^{\lambda_k t}\right) u(t)$$



$$h(0^+) = K_1 \qquad h(0) = K_1 \delta(t)$$

$$\dot{h}(0^+) = K_2 \qquad \dot{h}(0) = K_2 \delta(t) + K_1 \delta'(t)$$

$$\ddot{h}(0^+) = K_3 \qquad \ddot{h}(0) = K_3 \delta''(t) + K_2 \delta'(t) + K_1 \delta''(t)$$

$$h(t) = \left(\sum_{k=1}^{N} c_k e^{\lambda_k t}\right) u(t)$$

Impulse matching

$$h(0^+) = K_1$$
$$\dot{h}(0^+) = K_2$$
$$\ddot{h}(0^+) = K_3$$

$\Rightarrow$

$$h(0)$$
$$\dot{h}(0)$$
$$\ddot{h}(0)$$

$\Rightarrow$

Find coeff's

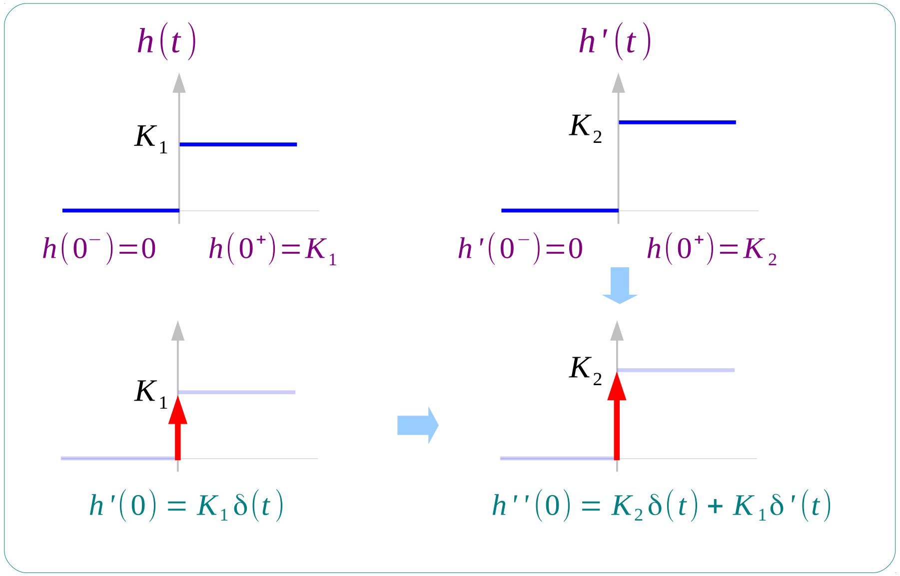
$$c_1$$
$$c_2$$
$$c_3$$

Impulse Matching (N=M) (1)

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2\, y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d\,x(t)}{dt} + b_2\, x(t)$$

$$\begin{cases} h''(t) + a_1 h'(t) + a_2 h(t) = b_0 \delta''(t) + b_1 \delta'(t) + b_2 \delta(t) \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

IVP (Initial Value Problem)



$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h(t) = b_0\delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})u(t)$$

$$h'(t) = b_0\delta'(t) + (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})\delta(t)$$

$$h''(t) = b_0\delta''(t) + (c_1 m_1^2 e^{m_1 x} + c_2 m_2^2 e^{m_2 x})u(t) + (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})(\delta(t) + \delta(t)) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})\delta'(t)$$

Impulse Matching (N=M) (2)

we can determine $K_1, \; K_2$

the initially at rest conditions $h(0^-) = h'(0^-) = 0$

and the property of step and delta $\frac{d}{dt}u(t) \; = \; \delta(t)$

$$h(t) \; = \; b_0\delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})u(t)$$

don't have to consider the term $b_0\delta(t)$

$$h''(0) + a_1 h'(0) + a_2 h(0) \; = \; b_0\delta''(t) + b_1\delta'(t) + b_2\delta(t)$$

thus h(0) has a finite jump

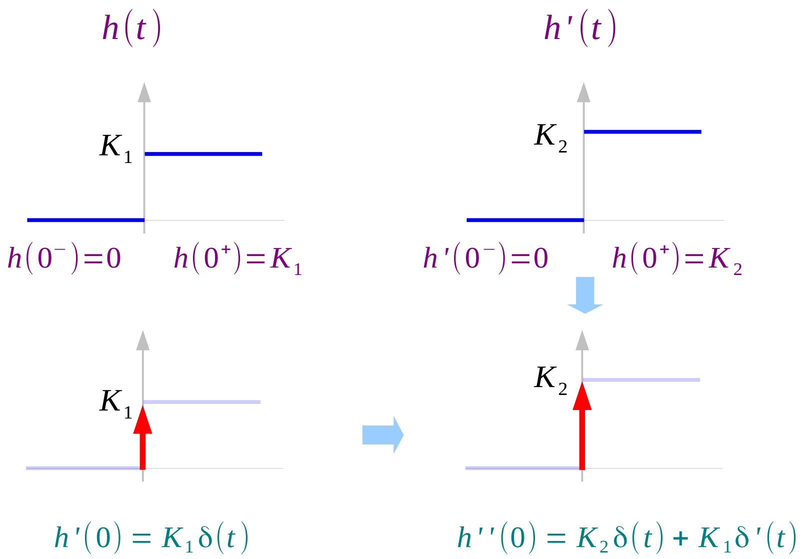
$$h''(0) + a_1 h'(0) \; = \; b_1\delta'(t) + b_2\delta(t)$$

$$K_2\delta(t) + K_1\delta'(t) + a_1 K_1\delta(t) \; = \; b_1\delta'(t) + b_2\delta(t)$$

$$K_1\delta'(t) + (a_1 K_1 + K_2)\delta(t) \; = \; b_1\delta'(t) + b_2\delta(t)$$

$$\begin{cases} K_1 \; = \; b_1 \\ a_1 K_1 + K_2 \; = \; b_2 \end{cases}$$

K_1
K_2



Impulse Matching (N=M) (3)

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d y(t)}{dt} + a_2 y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{d x(t)}{dt} + b_2 x(t)$$

$$\begin{cases} h''(t) + a_1 h'(t) + a_2 h(t) = b_0 \delta''(t) + b_1 \delta'(t) + b_2 \delta(t) \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

IVP (Initial Value Problem)

$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t)$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) \delta(t)$$

$$h(0^+) = (c_1 + c_2) \qquad h'(0^+) = (c_1 m_1 + c_2 m_2)$$

$$h(0^+) = K_1 \Rightarrow c_1 \qquad h'(0^+) = K_2 \Rightarrow c_2$$

$$h(t) = b_0 \delta(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x}) u(t)$$

Impulse Matching Example

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$h(t) = (c_1 e^{-2t} + c_2 e^{-3t}) \cdot u(t)$$

$$\dot{h}(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \cdot u(t) + (c_1 e^{-2t} + c_2 e^{-3t}) \cdot \delta(t)$$

$$\begin{cases} h(0^+) = (c_1 e^0 + c_2 e^0) \cdot u(t) \\ \dot{h}(0^+) = (-2c_1 e^0 - 3c_2 e^0) \cdot u(t) \end{cases}$$

$$\begin{cases} h(0^+) = c_1 + c_2 = +1 = K_1 \\ \dot{h}(0^+) = -2c_1 - 3c_2 = -4 = K_2 \end{cases}$$

$$c_1 = -1$$
$$c_2 = +2$$

$$h(t) = (-e^{-2t} + 2e^{-3t}) \cdot u(t)$$

$$\ddot{h}(t) + 5\dot{h}(t) + 6h(t) = \dot{\delta}(t) + 1\delta(t)$$

$$\begin{cases} h(0^-) = 0 \\ \dot{h}(0^-) = 0 \end{cases} \qquad \begin{cases} h(0^+) = K_1 \\ \dot{h}(0^+) = K_2 \end{cases}$$

$$\begin{cases} \dot{h}(0) = K_1\delta(t) \\ \ddot{h}(0) = K_1\dot{\delta}(t) + K_2\delta(t) \end{cases}$$

$$\ddot{h}(0) + 5\dot{h}(0) + 6h(0) = \dot{\delta}(t) + 1\delta(t)$$

$$(K_1\dot{\delta}(t) + K_2\delta(t)) + 5K_1\delta(t) + 6h(0)$$

$$K_1\dot{\delta}(t) + (5K_1 + K_2)\delta(t) + \cdots = \dot{\delta}(t) + 1\delta(t)$$

$$K_1 = 1 \qquad 5K_1 + K_2 = 1$$

$$\begin{cases} K_1 = 1 = h(0^+) \\ K_2 = -4 = \dot{h}(0^+) \end{cases}$$

* Simplified Impulse Matching

Simplified Impulse Matching (1)

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_2 y(t) = b_0 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + b_2 x(t)$$

$$\begin{cases} h''(t) + a_1 h'(t) + a_2 h(t) = b_0 \delta''(t) + b_1 \delta'(t) + b_2 \delta(t) \\ h(0^+) = K_1 \\ h'(0^+) = K_2 \end{cases}$$

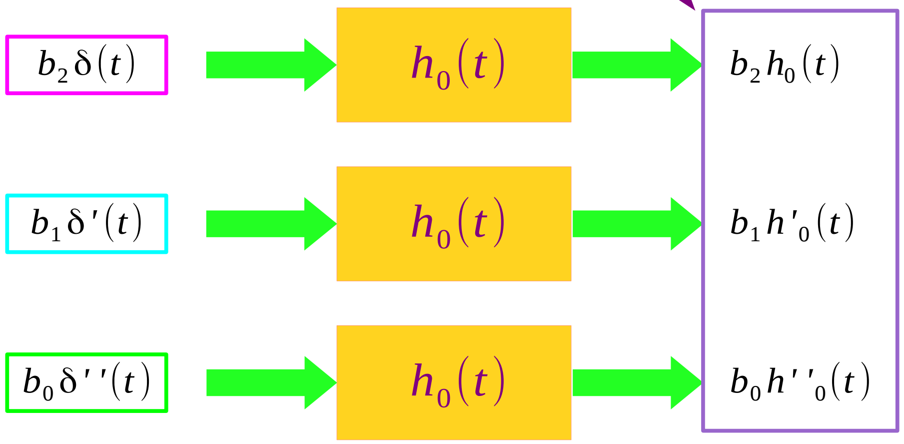
IVP (Initial Value Problem)

$$\begin{cases} h_0''(t) + a_1 h_0'(t) + a_2 h_0(t) = \delta(t) \\ h(0^+) = 0 \\ h'(0^+) = 1 \end{cases} \quad \Rightarrow h''(0^+) = \delta(t)$$

Simpler IVP

$$P(D) = (b_0 D^2 + b_1 D + b_2)$$

$$h(t) = b_0 \delta(t) + [P(D) h_0(t)]$$



Simplified Impulse Matching (2)

$$\begin{cases} h_0''(t) + a_1 h_0'(t) + a_2 h_0(t) = \delta(t) \\ h_0(0^+) = 0 \\ h_0'(0^+) = 1 \end{cases}$$

Simpler IVP

$y_n(t)$ Only a linear combination of characteristic modes Excluding u(t)

$$m^2 + a_1 m + a_2 = 0 \quad \Rightarrow \quad m = m_1,\ m_2$$

$$h_0(t) = y_n(t)\, u(t) \qquad \text{causality}$$

$$h_0(t) = (c_1 e^{m_1 x} + c_2 e^{m_2 x})\, u(t)$$

$$h_0'(t) = (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})\, u(t) + (c_1 e^{m_1 x} + c_2 e^{m_2 x})\, \delta(t)$$

$$y_n(t) = (c_1 e^{m_1 x} + c_2 e^{m_2 x})$$

$$y_n'(t) = (c_1 m_1 e^{m_1 x} + c_2 m_2 e^{m_2 x})$$

$$\begin{cases} h_0(0^+) = 0 = c_1 + c_2 \\ h_0'(0^+) = 1 = c_1 m_1 + c_2 m_2 \end{cases} \qquad c_1 \quad c_2$$

$$\begin{cases} y_n(0^+) = 0 = c_1 + c_2 \\ y_n'(0^+) = 1 = c_1 m_1 + c_2 m_2 \end{cases}$$

$$(b_0 D^2 + b_1 D + b_2)\,[(c_1 e^{m_1 x} + c_2 e^{m_2 x})\, u(t)]$$

$$[(b_0 D^2 + b_1 D + b_2)\,(c_1 e^{m_1 x} + c_2 e^{m_2 x})]\, u(t)$$

$$h(t) = b_0 \delta(t) + [P(D)\, h_0(t)]$$

$$\Leftrightarrow$$

$$h(t) = b_0 \delta(t) + [P(D)\, y_n(t)]\, u(t)$$

$$= b_0 \delta(t) + [P(D)\,[y_n(t)\, u(t)]]$$

Simplified Impulse Matching Example

$$\frac{d^2 y}{dt^2} + 5\frac{dy}{dt} + 6y(t) = \frac{dx}{dt} + 1x(t)$$

$$(D^2 + 5D + 6)y(t) = (D + 1)x(t)$$

$$(\lambda^2 + 5\lambda + 6) = (\lambda+2)(\lambda+3) = 0$$

$$\begin{cases} y_n(t) = (c_1 e^{-2t} + c_2 e^{-3t}) \\ \dot{y}_n(t) = (-2c_1 e^{-2t} - 3c_2 e^{-3t}) \end{cases} \quad \begin{cases} y_n(0) = 0 \\ \dot{y}_n(0) = 1 \end{cases}$$

$$\begin{cases} y_n(0) = (c_1 e^0 + c_2 e^0) = 0 \\ \dot{y}_n(0) = (-2c_1 e^0 - 3c_2 e^0) = 1 \end{cases}$$

$$\begin{cases} y_n(0) = c_1 + c_2 = 0 \\ \dot{y}_n(0) = -2c_1 - 3c_2 = 1 \end{cases}$$

$$c_1 = +1$$
$$c_2 = -1$$

$$y_n(t) = (+e^{-2t} - e^{-3t})$$

$$h(t) = b_0\delta(t) + [P(D)y_n(t)]u(t)$$

$$(D+1)y_n(t) = (-2e^{-2t} + 3e^{-3t}) + (e^{-2t} - e^{-3t})$$
$$= (-e^{-2t} + 2e^{-3t})$$

$$h(t) = (-e^{-2t} + 2e^{-3t})\cdot u(t)$$

- Integrating both sides successively

First Order ODE (n > m) – Impulse Response

$$y'(t) + a\,y(t) = x(t)$$

$$h'(t) + a\,h(t) = 0$$ Homogeneous equation

$$h'(t) + a\,h(t) = \delta(t)$$ Particular equation

$$\int_{0^-}^{0^+} h'(t)dt + a\int_{0^-}^{0^+} h(t)dt = \int_{0^-}^{0^+} \delta(t)dt$$

$$\underbrace{h(0^+) - h(0^-)}_{\to 0} + a\int_{0^-}^{0^+} h(t)dt = \underbrace{\int_{0^-}^{0^+} \delta(t)dt}_{\to 1}$$

$h(t) = 0$	$(t < 0)$	causal system
$h(t) = K e^{-at}$	$(t > 0)$	homogeneous solution
?	$(t = 0)$	particular solution

$$\int_{0^-}^{0^+} h(t)dt = 0$$ when h(t) does not have an impulse or its derivatives

$$\int_{0^-}^{0^+} h(t)dt \neq 0$$ Otherwise

if $\delta(t)$ is included in $h(t)$

$\delta'(t)$ must present in $h'(t)$

But RHS has no doublet or triplet

➡ no $\delta(t)$ included in $h(t)$

➡ $h(0^+) = 1$ $\quad h(0^+) = K e^{-a0^+} = K = 1$

$$h(t) = e^{-at} \quad (t > 0)$$

$$h(t) = e^{-at}u(t)$$

First Order ODE (n > m) - Verification

$$y'(t) + a\,y(t) \;=\; x(t)$$

$$h'(t) + a\,h(t) \;=\; \delta(t)$$

$$\begin{cases} h'(t) + a\,h(t) = 0 & (t < 0) \\ h'(t) + a\,h(t) = 0 & (t > 0) \\ h'(t) + a\,h(t) = \delta(t) & (t = 0) \end{cases}$$

$h(t) = e^{-at}\,u(t)$ satisfies

$h'(t) + a\,h(t) = \delta(t)$ for all t

$h(t)$ has exactly the right size discontinuity at time t=0

$$h(t) \;=\; 0 \qquad (t < 0)$$

$$h(0^+) \;=\; 1 \qquad (t > 0)$$

$$h(t) = e^{-at}\,u(t)$$

$$h'(t) = -a\,e^{-at}\,u(t) + a\,e^{-at}\,\delta(t)$$

$$h'(t) + a\,h(t)$$

$$= -a\,e^{-at}\,u(t) + e^{-at}\,\delta(t) + a\,e^{-at}\,u(t)$$

$$= e^{-at}\,\delta(t) = e^{0}\,\delta(t)$$

$$= \delta(t)$$

First Order ODE (n = m) – Impulse Response

$$h'(t) + a\,h(t) = x'(t)$$

$$u_1(t) = \frac{d}{dt}\delta(t)$$

$$h'(t) + a\,h(t) = 0$$ Homogeneous equation

$$h'(t) + a\,h(t) = \delta'(t) = u_1(t)$$ Particular equation

$h(t) = 0$	$(t < 0)$	causal system
$h(t) = K e^{-at}$	$(t > 0)$	homogeneous solution
?	$(t = 0)$	particular solution

The highest derivative is the same for the excitation and response (n = m)

➡ $\delta(t)$ included in $h(t)$

$$h(t) = Ke^{-at}u(t) + K_\delta \delta(t)$$

yh yp

Determining Coefficients

$$y'(t) + a\,y(t) = x'(t)$$

$$h'(t) + a\,h(t) = \delta'(t) = u_1(t)$$

$$\int_{0^-}^{0^+} h'(t)\,dt + a\int_{0^-}^{0^+} h(t)\,dt = \int_{0^-}^{0^+} \delta'(t)\,dt$$

$$h(0^+) - \underbrace{h(0^-)}_{\to 0} + a\int_{0^-}^{0^+} h(t)\,dt = \underbrace{\delta(0^+)}_{\to 0} - \underbrace{\delta(0^-)}_{\to 0}$$

$$K \qquad + a K_\delta \qquad = 0$$

$$\int_{-\infty}^{t} h'(t)\,dt + a\int_{-\infty}^{t} h(t)\,dt = \int_{-\infty}^{t} \delta'(t)\,dt$$

$$h(t) + a\int_{-\infty}^{t} h(t)\,dt = \delta(t)$$

$$\int_{0^-}^{0^+} h(t)\,dt + a\int_{0^-}^{0^+}\int_{-\infty}^{t} h(\lambda)\,d\lambda\,dt = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$\int_{0^-}^{0^+} h(t)\,dt + a\left\{\int_{0^-}^{0^+} dt\right\}\left\{\int_{-\infty}^{t} h(\lambda)\,d\lambda\right\} = \int_{0^-}^{0^+} \delta(t)\,dt$$

$$K_\delta = u(0^+) - u(0^-) = 1 - 0$$

$$K = -a K_\delta = -a$$

$$h(t) = \underbrace{K e^{-at} u(t)}_{y_h} + \underbrace{K_\delta \delta(t)}_{y_p}$$

$$h(t) = \delta(t) - a e^{-at} u(t)$$

First Order ODE (n = m) - Verification

$$h'(t) + a\,h(t) \;=\; x'(t)$$

$$h'(t) + a\,h(t) \;=\; \delta'(t) \;=\; u_1(t)$$

$$\begin{cases} h'(t) + a\,h(t) = 0 & (t < 0) \\ h'(t) + a\,h(t) = 0 & (t > 0) \\ h'(t) + a\,h(t) = \delta'(t) & (t = 0) \end{cases}$$

$h(t) \;=\; \delta(t) - a e^{-at} u(t)$ satisfies

$h'(t) + a\,h(t) = \delta'(t)$ for all t

$$h'(t) \;=\; \delta'(t) + a^2 e^{-at} u(t) - a e^{-at} \delta(t)$$

$$\;=\; \delta'(t) + a^2 e^{-at} u(t) - a\,\delta(t)$$

$$a\,h(t) \;=\; a\,\delta(t) - a^2 e^{-at} u(t)$$

$$h'(t) + a\,h(t)$$

$$= \delta'(t) + a^2 e^{-at} u(t) - a\,\delta(t)$$

$$+ \; a\,\delta(t) - a^2 e^{-at} u(t)$$

$$= \delta'(t)$$

* Superposition of Input Functions

References

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[4] M.J. Roberts, Fundamentals of Signals and Systems