Background – LTI Systems (4A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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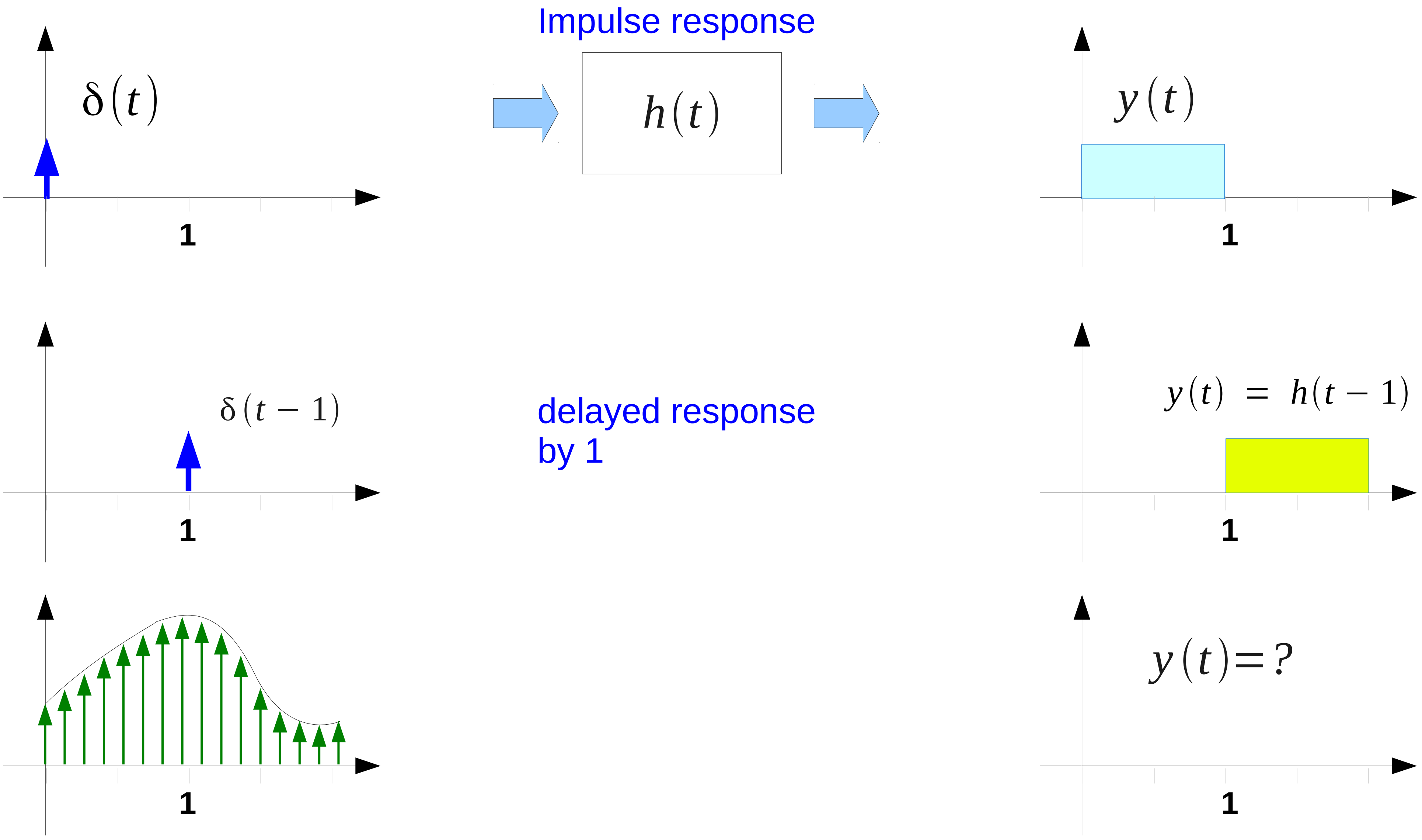
Convolution Integrals

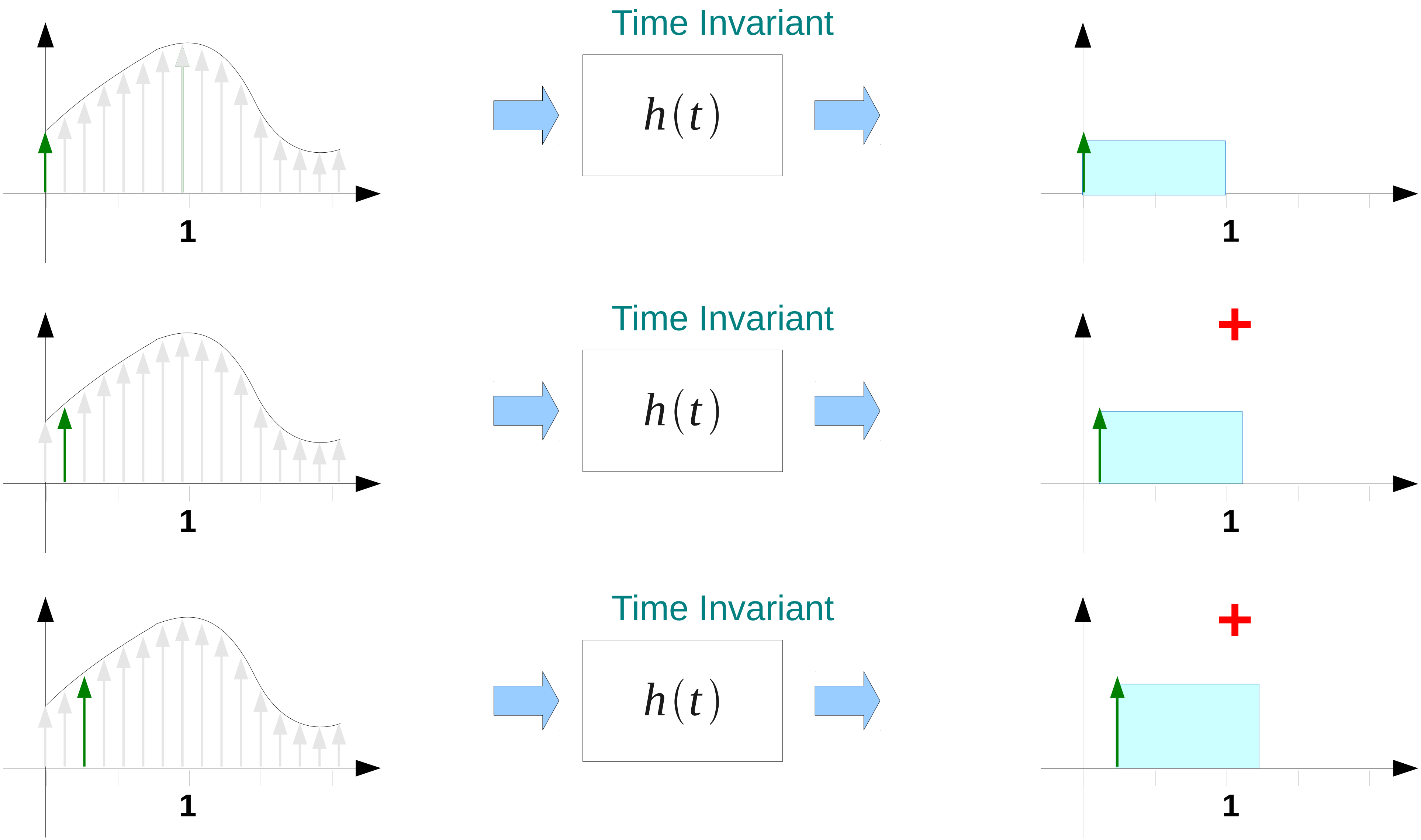
Convolution: delayed response of h(t) (1)



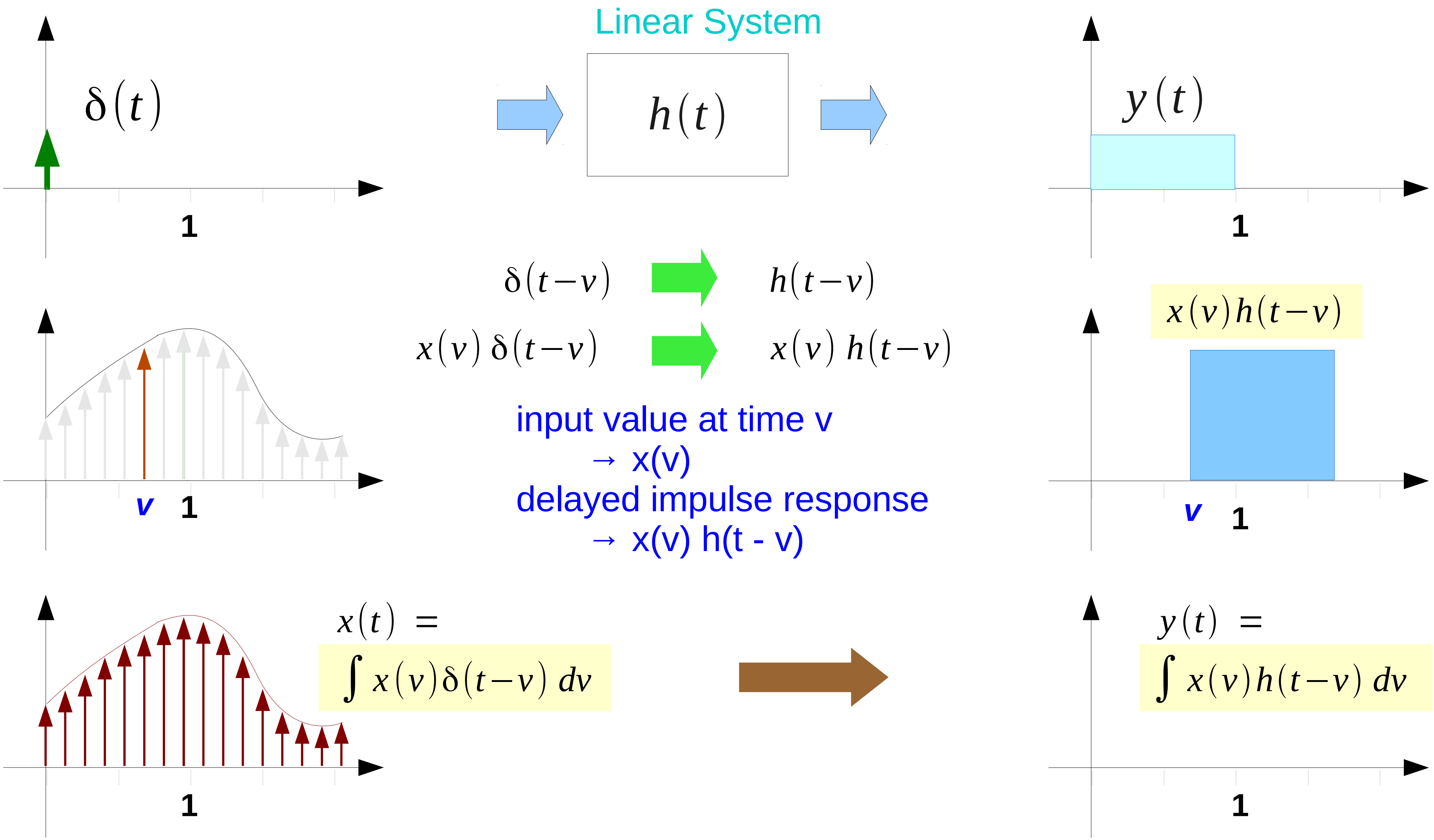
Impulse response

delayed response by 1

Convolution: delayed response of h(t) (2)



Convolution: delayed response of h(t) (3)



$$\delta(t-v) \rightarrow h(t-v)$$

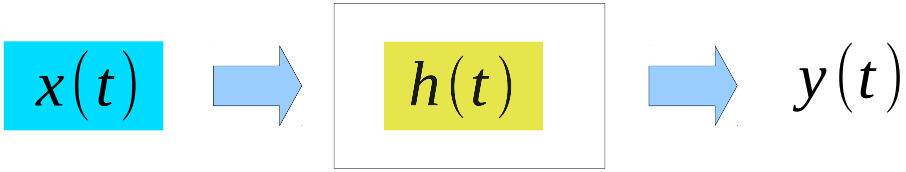
$$x(v)\,\delta(t-v) \rightarrow x(v)\,h(t-v)$$

input value at time v
→ x(v)
delayed impulse response
→ x(v) h(t - v)

$$x(t) = \int x(v)\delta(t-v)\,dv$$

$$y(t) = \int x(v)h(t-v)\,dv$$

Convolution: Commutative Law



$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

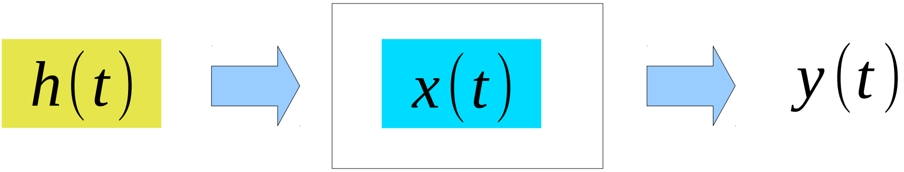
LTI

$x(v)$ $\qquad h(v) \xrightarrow{\text{Flip}} h(-v) \xrightarrow{\text{Shift}} h(t-v)$

$$\int x(v)h(t-v)\,dv = y(t)$$

$$\int h(v)x(t-v)\,dv = y(t)$$

$h(v)$ $\qquad x(v) \xrightarrow{\text{Flip}} x(-v) \xrightarrow{\text{Shift}} x(t-v)$



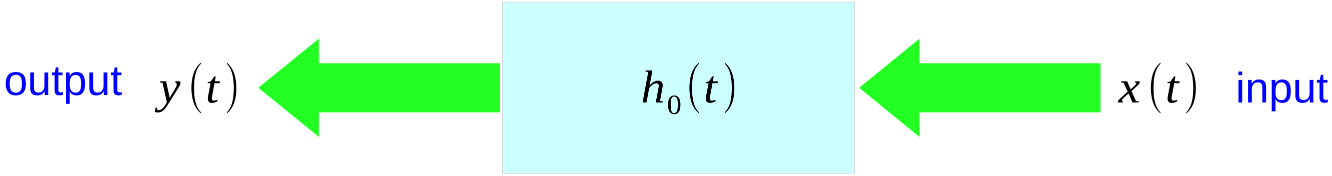
$h(t) \rightarrow \boxed{x(t)} \rightarrow y(t)$

LTI

Superposition of Derivatives of x(t)

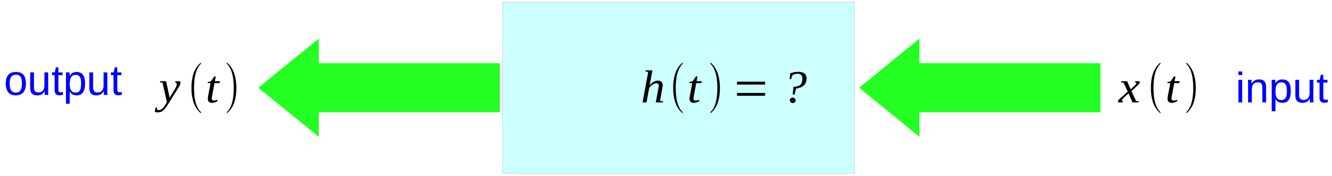
ODE's and Causal LTI Systems

$$a_N \frac{d^N y(t)}{d t^N} + a_{N-1} \frac{d^{N-1} y(t)}{d t^N} + \cdots + a_1 \frac{d\, y(t)}{d t} + a_0\, y(t) = x(t)$$



output $y(t)$ $h_0(t)$ $x(t)$ input

$$a_N \frac{d^N y(t)}{d t^N} + a_{N-1} \frac{d^{N-1} y(t)}{d t^N} + \cdots + a_1 \frac{d\, y(t)}{d t} + a_0\, y(t) = b_M \frac{d^M x(t)}{d t^M} + b_{M-1} \frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_1 \frac{d\, x(t)}{d t} + b_0\, x(t)$$

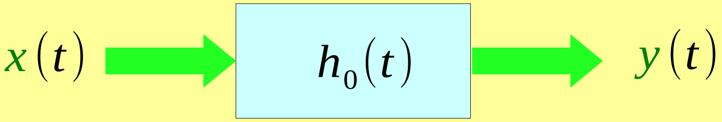


output $y(t)$ $h(t) = ?$ $x(t)$ input

Derivative of inputs

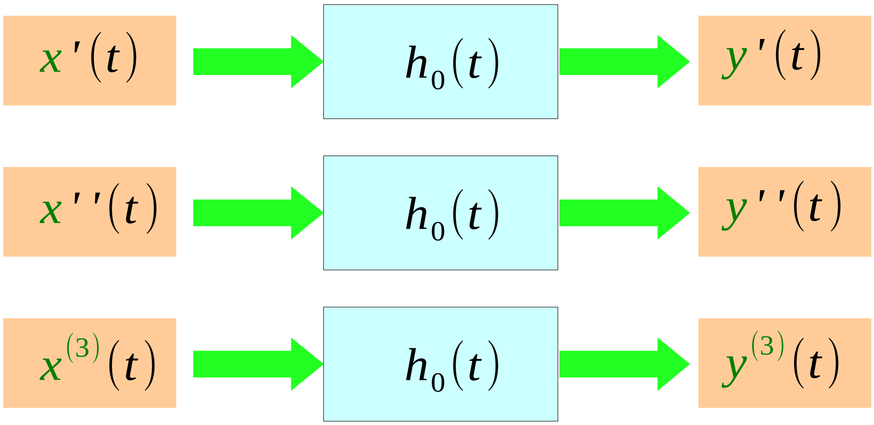
$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = x$$

base system



$$x(t) \rightarrow \boxed{h_0(t)} \rightarrow y(t)$$

$$\textit{notation}: \; y^{(N)} = \frac{d^N y}{dt^N} = \frac{d^N}{dt^N} y(t)$$



$$x'(t) \rightarrow \boxed{h_0(t)} \rightarrow y'(t)$$

$$x''(t) \rightarrow \boxed{h_0(t)} \rightarrow y''(t)$$

$$x^{(3)}(t) \rightarrow \boxed{h_0(t)} \rightarrow y^{(3)}(t)$$

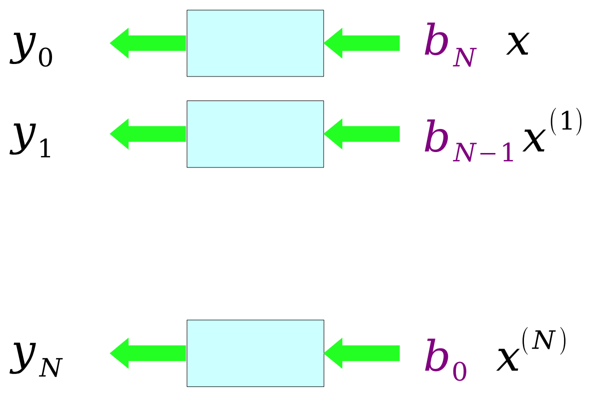
Superposition of derivatives of an input

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$$y_0^{(N)} + a_1 y_0^{(N-1)} + \cdots + a_{N-1} y_0^{(1)} + a_N y_0 = b_N x$$

$$y_1^{(N)} + a_1 y_1^{(N-1)} + \cdots + a_{N-1} y_1^{(1)} + a_N y_1 = b_{N-1} x^{(1)}$$

$$y_N^{(N)} + a_1 y_N^{(N-1)} + \cdots + a_{N-1} y_N^{(1)} + a_N y_N = b_0 x^{(N)}$$



$$y = y_0 + y_1 + \cdots + y_N$$

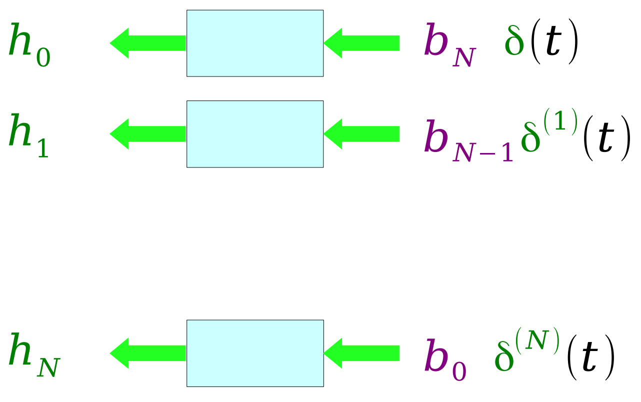
Superposition of derivatives of an delta function

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$

$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_{N-1} h_0^{(1)} + a_N h_0 = b_N \delta$$

$$h_1^{(N)} + a_1 h_1^{(N-1)} + \cdots + a_{N-1} h_1^{(1)} + a_N h_1 = b_{N-1} \delta^{(1)}$$

$$h_N^{(N)} + a_1 h_N^{(N-1)} + \cdots + a_{N-1} h_N^{(1)} + a_N h_N = b_0 \delta^{(N)}$$

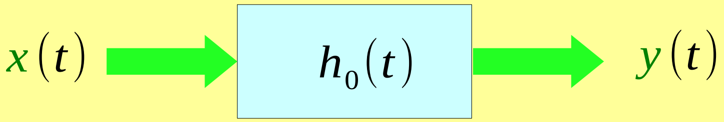


$$h = h_0 + h_1 + \cdots + h_N$$

Base System & Derived System

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = x$$

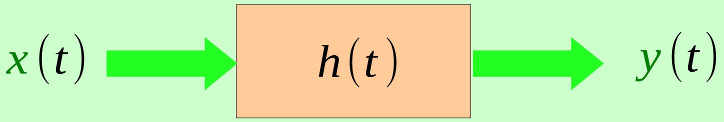
base system



notation: $y^{(N)} = \frac{d^N y}{dt^N} = \frac{d^N}{dt^N} y(t)$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

derived system

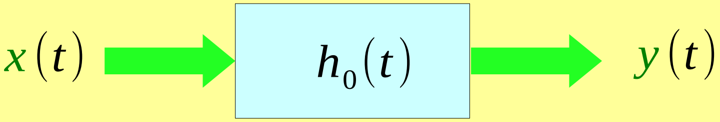


notation: $x^{(N)} = \frac{d^N x}{dt^N} = \frac{d^N}{dt^N} x(t)$

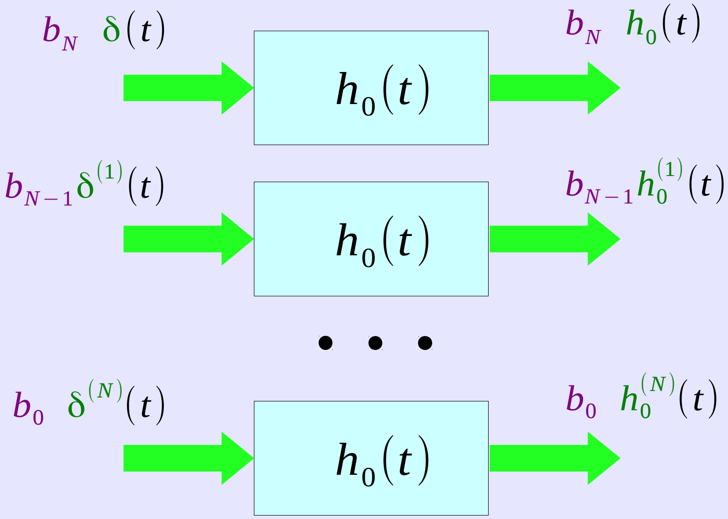
Superposition of derivatives of a delta function

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = x$$

base system



notation: $y^{(N)} = \frac{d^N y}{dt^N} = \frac{d^N}{dt^N} y(t)$



$$h_0^{(N)} + a_1 h_0^{(N-1)} + \cdots + a_N h_0 = b_N \ \delta$$

$$h_0^{(N+1)} + a_1 h_0^{(N-1)} + \cdots + a_N h_0^{(1)} = b_{N-1} \delta^{(1)}$$

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$$h_0^{(2N)} + a_1 h_0^{(2N-1)} + \cdots + a_N h_0^{(N)} = b_0 \ \delta^{(N)}$$

General System: Impulse Response

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$h = b_0 h_0^{(N)} + b_1 h_0^{(N-1)} + \cdots + b_{N-1} h_0^{(1)} + b_N h_0 \qquad = P(D) h_0(t)$$

$$Q(D) = (D^N + \boldsymbol{a_1} D^{N-1} + \cdots + \boldsymbol{a_{N-1}} D + \boldsymbol{a_N})$$

$$P(D) = (\boldsymbol{b_0} D^N + \boldsymbol{b_1} D^{N-1} + \cdots + \boldsymbol{b_{N-1}} D + \boldsymbol{b_N})$$

$$\delta(t) \longrightarrow \boxed{h_0(t)} \longrightarrow h_0(t)$$

$$Q(D) h_0(t) = \delta(t)$$

$$Q(D) \boxed{P(D) h_0(t)} = \boxed{P(D) \delta(t)}$$

$$P(D) h_0(t) \Rightarrow h(t)$$

General System: Output

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = b_0 w^{(N)} + b_1 w^{(N-1)} + \cdots + b_{N-1} w^{(1)} + b_N w \quad = P(D) w(t)$$

$$Q(D) = (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)$$

$$P(D) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)$$

$$x(t) \longrightarrow \boxed{h_0(t)} \longrightarrow w(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) \boxed{P(D) w(t)} = \boxed{P(D) x(t)}$$

$$P(D) w(t) \Rightarrow y(t)$$

ODE's and Causal LTI Systems

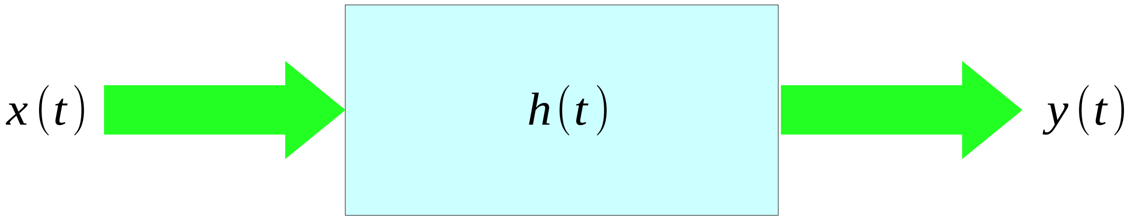
$$\overset{\downarrow}{a_N}\frac{d^N y(t)}{d t^N} + a_{N-1}\frac{d^{N-1} y(t)}{d t^N} + \cdots + a_1\frac{d\,y(t)}{d\,t} + a_0\,y(t) \;=\; \overset{\downarrow}{b_M}\frac{d^M x(t)}{d t^M} + b_{M-1}\frac{d^{M-1} x(t)}{d t^{M-1}} + \cdots + b_1\frac{d\,x(t)}{d\,t} + b_0\,x(t)$$

N: the highest order of derivatives of the output y(t) (LHS)

M: the highest order of derivatives of the input x(t) (RHS)

N < M : (M-N) differentiator – magnify high frequency components of noise (seldom used)

N > M : (N-M) Integrator



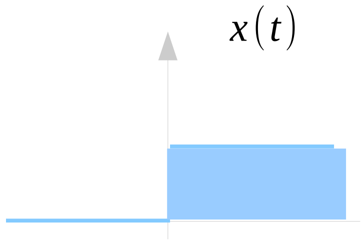
- System Response

  (a) Zero Input Response
  (b) Zero State Response
  (c) Natural Response
  (d) Forced Response

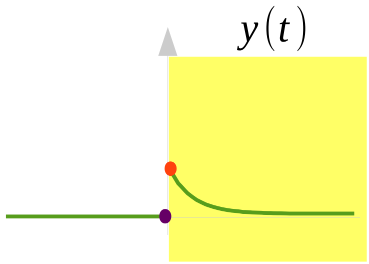
Interval of Interest

Total = ZSR + ZIR

together with Laplace transform,
the time interval [0, ∞) is assumed



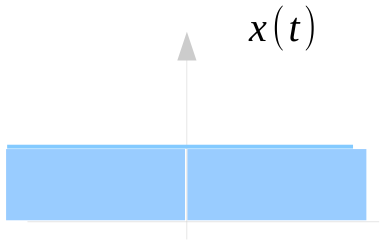
causal x(t)



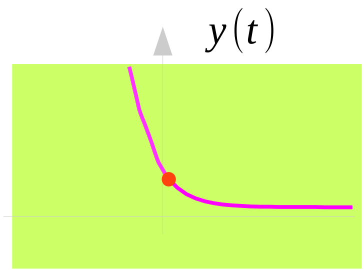
$$\boxed{0^{-} < t < \infty}$$

x(t) * h(t)

No restriction in the time interval
But if the same time interval [0, ∞)
can be assumed, then the two total
responses will be the same

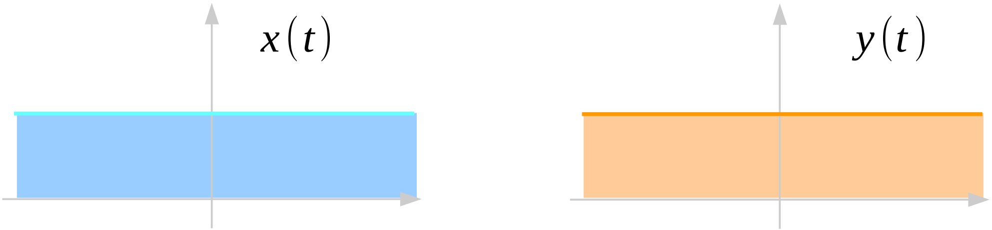


non-causal x(t)



$$\boxed{-\infty < t < \infty}$$

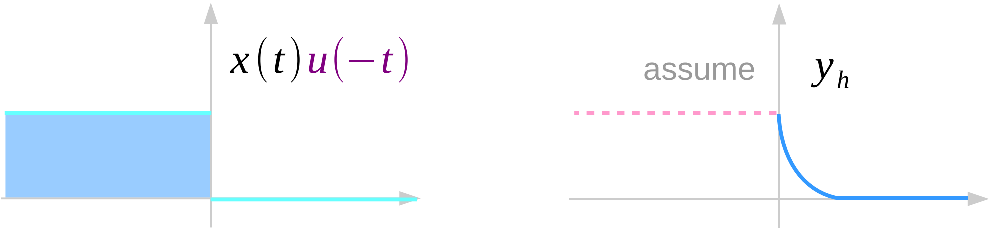
Responses after *t* = 0



$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

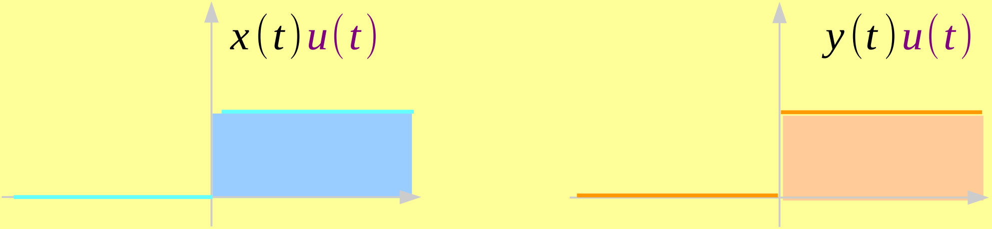
ZSR

$$y_{x+}(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$



ZIR

$$y^{+}(t) = \left(\int_{-\infty}^{0} x(\tau)h(t-\tau)d\tau\right) \cdot u(t)$$



ZSR + ZIR

Natural + Forced

ODE Solutions and System Responses

$y_h(t) = \sum_i c_i e^{\lambda}$	Characteristic Mode Terms	$-\infty < t < +\infty$
$y_p(t) \Leftarrow x(t)$	Similar form as the input x(t)	$-\infty < t < +\infty$

$y_n(t)$	Natural response with all coefficients determined
$y_p(t)$	Forced response
$y_{zi}(t)$	Zero input response
$y_{zs}(t)$	Zero state response

$$y_{zi}(t) \quad + \quad y_{zs}(t)$$

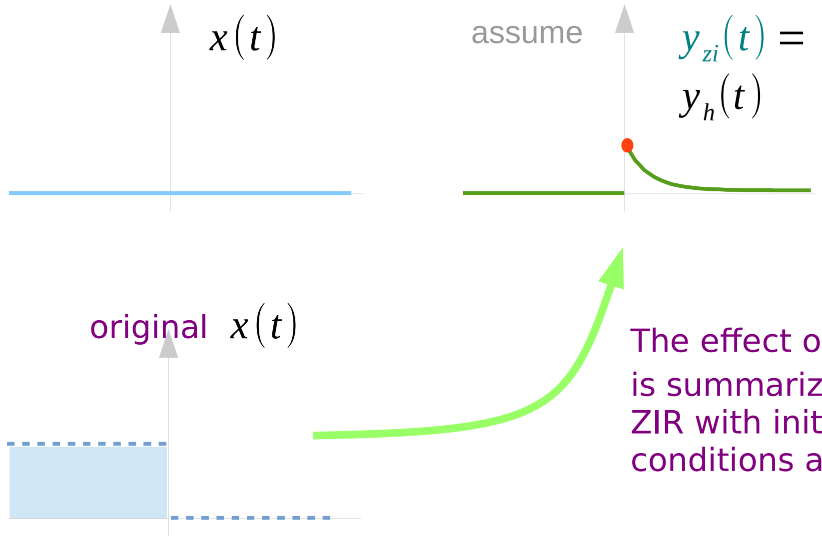
$$y_n(t) \quad + \quad y_p(t)$$

Responses after $t = 0$ $(t > 0)$
to the applied input x(t) $(t \geq 0)$

Causal Input
Causal System
Causal Output

ZIR & ZSR in terms of y_h & y_p

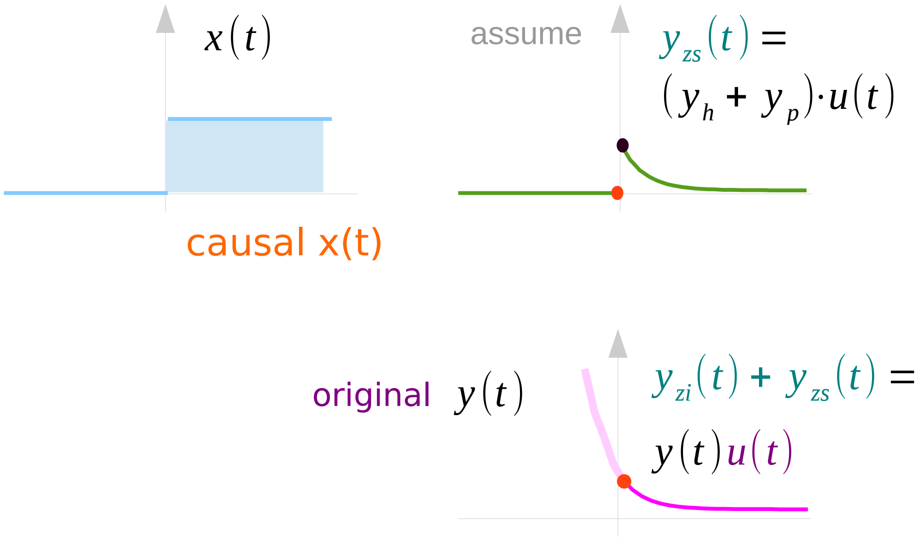
ZIR



The effect of $x(t)$ $(t < 0)$ is summarized as a ZIR with initial conditions at t=0-

$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$$

ZSR



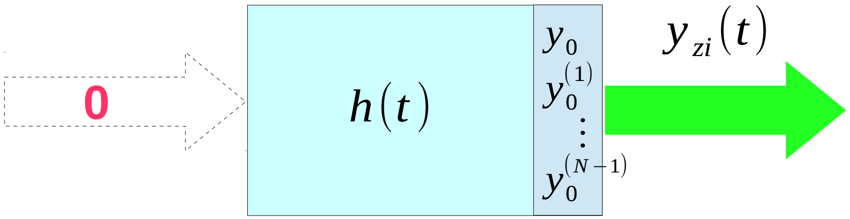
causal x(t)

$$h(t) * x(t) = \left(b_0\delta(t) + \sum_i d_i e^{\lambda_i t}\right) * x(t)$$

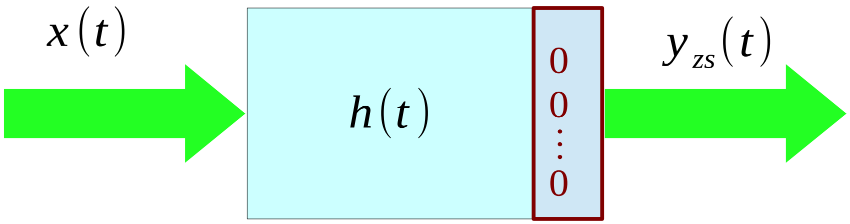
$$u(t)\cdot(y_h + y_p) = u(t)\cdot\left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

Types of System Responses

- **Zero Input Response** State only



- **Zero State Response** Input only



- **Natural Response** Homogeneous

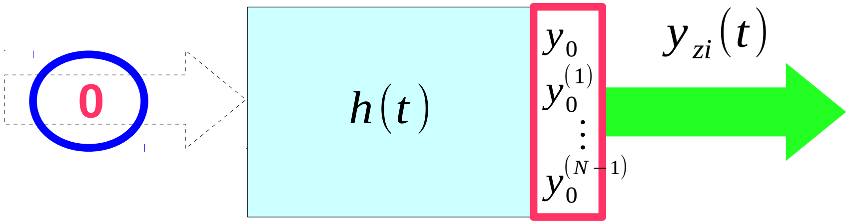
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = 0$$

- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)$$

Comparison of System Responses (1)

- **Zero Input Response** State only



Response of a system when the input x(t) is zero (no input)

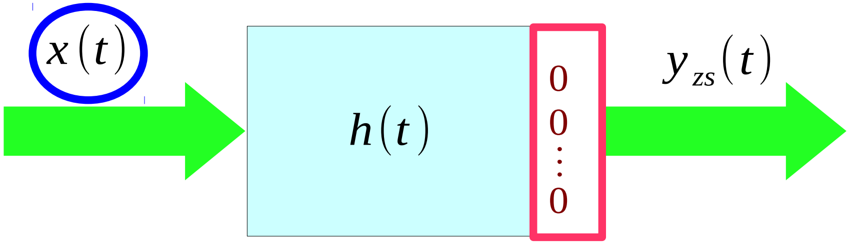
≠

- **Natural Response** Homogeneous

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = 0$$

Solution due to characteristic modes only

- **Zero State Response** Input only



Response of a system when system is at rest initially

≠

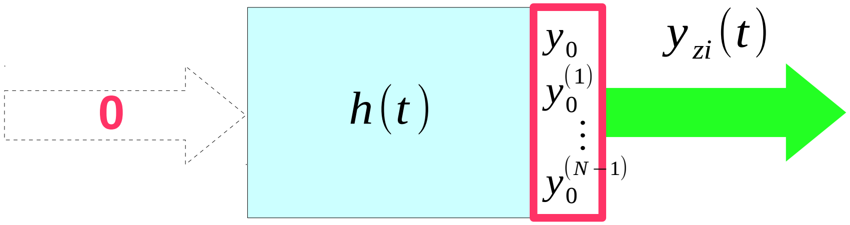
- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = \boxed{b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)}$$

Solution <u>excluding</u> the effect of characteristic modes

Comparison of System Responses (2)

- **Zero Input Response** State only



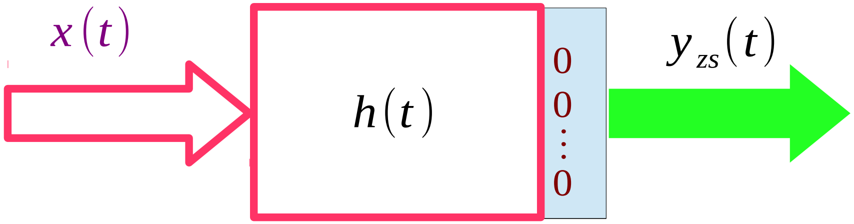
response to the initial conditions only

- **Natural Response** Homogeneous

$$\boxed{\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t)} = 0$$

all characteristic modes response

- **Zero State Response** Input only



response to the input only

- **Forced Response** Particular

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y(t) = \boxed{b_0 \frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt} + \cdots + b_n x(t)}$$

non-characteristic mode response

Comparison of System Responses (3)

- **Zero Input Response** State only

response to the ***initial conditions only***

$$\{y^{(N-1)}(0^-),\ \cdots,\ y^{(1)}(0^-),\ y(0^-)\}$$

- **Natural Response** Homogeneous

all characteristic ***modes*** response

$$y_n(t) = \sum_i K_i e^{\lambda_i t}$$

- **Zero State Response** Input only

response to the ***input* only**

$$h(t) * x(t) = \left(b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}\right) * x(t)$$

causal x(t)

- **Forced Response** Particular

non-characteristic mode response

$$y_p(t)$$

Forms of System Responses

- **Zero Input Response** State only

$$y_{zi}(t) = \sum_i c_i e^{\lambda_i t} \qquad \{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$$

determines the coefficients

- **Zero State Response** Input only

convolution form

Impulse matching

$$y_{zs}(t) = x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t)\right)$$

step function form

balancing singularities

$$y_{zs}(t) = u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

$y_p(t)$ is similar to the input $x(t)$

- **Natural Response** Homogeneous

$$y_n(t) = \sum_i K_i e^{\lambda_i t}$$

the coefficients K_i's are determined by the initial conditions.

$$y_n(t) + y_p(t) \qquad \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$$

determines the coefficients

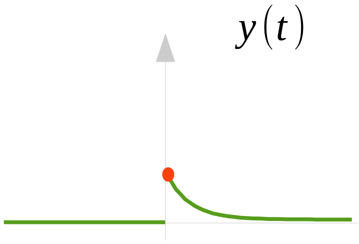
- **Forced Response** Particular

$$y_p(t) = \begin{cases} \beta e^{\xi t} \quad \text{or} \\ (t^r + \beta_{r-1} t^{r-1} + \cdots + \beta_1 t + \beta_0) \end{cases}$$

$y_p(t)$ similar to the input, with the coefficients determined by equating the similar terms

Valid Intervals of System Responses

ZIR + ZSR



$$\boxed{0^- < t < \infty}$$

General Assumption

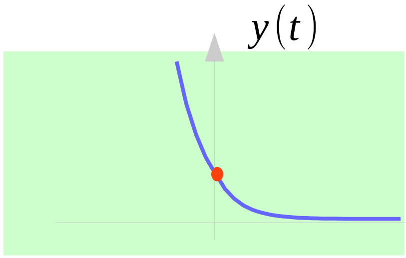
$x(t) = 0 \quad (t < 0)$ Causal Input

$h(t) = 0 \quad (t < 0)$ Causal System

$y(t) = 0 \quad (t < 0)$ Causal Output

x(t) * h(t)

non-causal x(t)



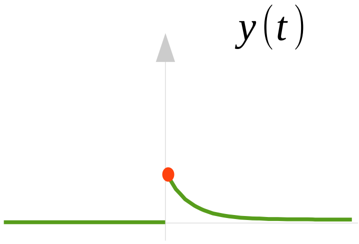
$$\boxed{-\infty < t < \infty}$$

General Assumption

$h(t) = 0 \quad (t < 0)$ Causal System

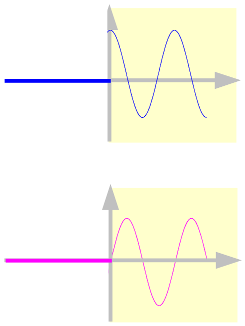
Causal and Everlasting Exponential Inputs

ZIR + ZSR



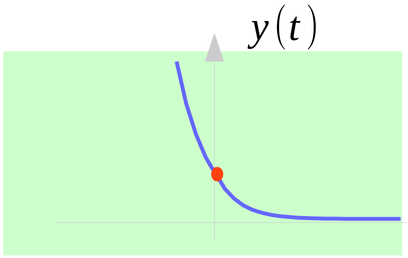
$$\boxed{0^- < t < \infty}$$

Suitable for causal exponential inputs



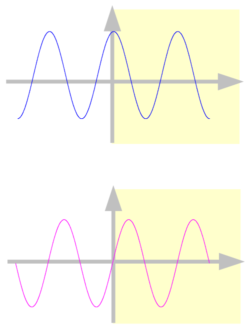
x(t) * h(t)

non-causal x(t)



$$\boxed{-\infty < t < \infty}$$

Suitable for everlasting exponential inputs



The effect of an input for t < 0

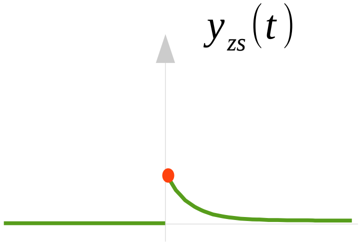
ZIR + ZSR

causal x(t)

$x(t) = 0 \quad (t < 0)$

$h(t) = 0 \quad (t < 0)$

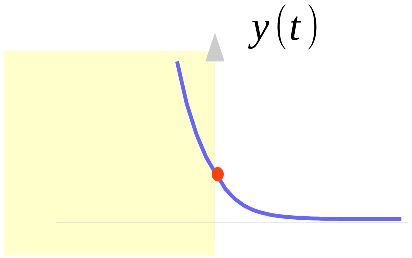
$y(t) = 0 \quad (t < 0)$

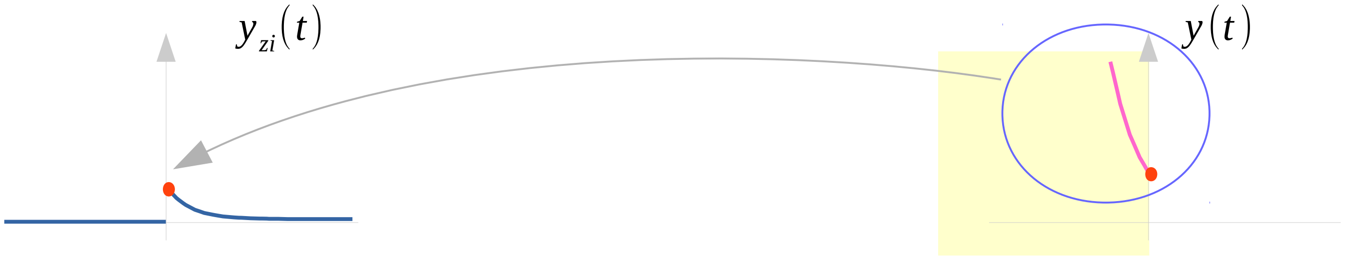


x(t) * h(t)

non-causal x(t)

$h(t) = 0 \quad (t < 0)$

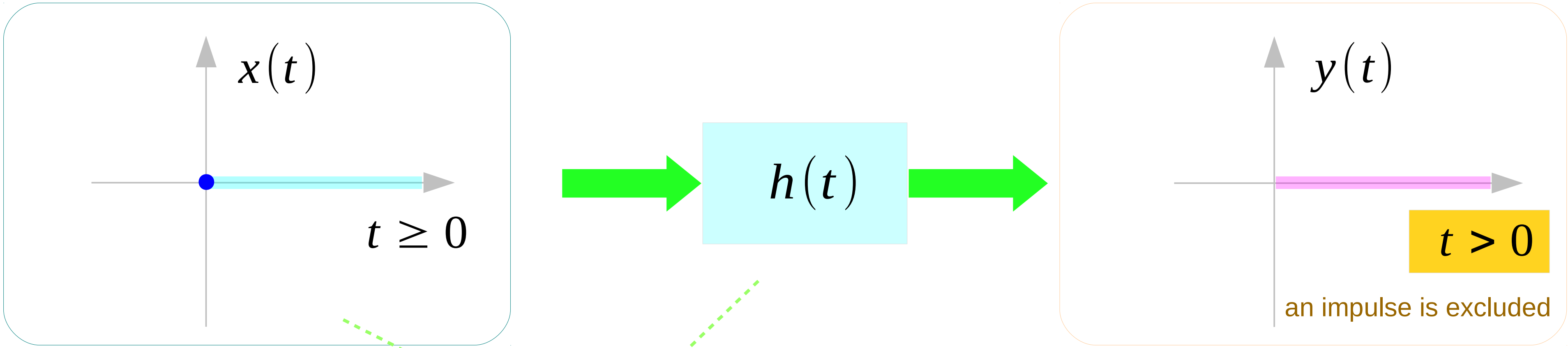




The effect of $x(t) \quad (t < 0)$

is summarized as a ZIR with initial conditions at t=0-

Interval of Validity



an impulse is excluded

an impulse can be present in *x(t)* and *h(t)* at *t = 0*

$\delta(t)$

discontinuity at *t = 0*
creates initial conditions

$$\begin{matrix} x^{(N-1)}(0^-) \\ x^{(N-2)}(0^-) \\ \vdots \\ x^{(1)}(0^-) \\ x(0^-) \end{matrix}$$

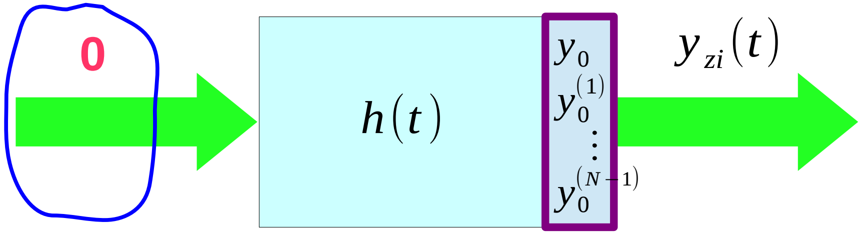
The initial conditions in the term “**Zero State**” refers this initial conditions

$y^{(N-1)}(0^-)$	$y^{(N-1)}(0)$	$y^{(N-1)}(0^+)$
$y^{(N-2)}(0^-)$	$y^{(N-2)}(0)$	$y^{(N-2)}(0^+)$
$\vdots$	$\vdots$	$\vdots$
$y^{(1)}(0^-)$	$y^{(1)}(0)$	$y^{(1)}(0^+)$
$y(0^-)$	$y(0)$	$y(0^+)$
Usually known IC		Needed IC

- Zero Input Response

Zero Input Response : $y_{zi}(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) \;=\; (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$



$x(t) = 0$

$$\frac{d^2 y(t)}{dt^2} + a_1 \frac{d\,y(t)}{dt} + a_2 y(t) = \boxed{0}$$

$x(t) = 0$

$Q(D)\, y_{zi}(t) \;=\; 0$ ⇒ $(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_{zi}(t) \;=\; 0$

⇒ linear combination of {y_{zi}(t) and its derivatives} = 0

⇒ $ce^{\lambda t}$ only this form can be the solution of $y_{zi}(t)$

$Q(\lambda) \;=\; 0$ ⇔ $\underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N)}_{=\;0}\; \underbrace{ce^{\lambda t}}_{\neq\;0} \;=\; 0$

Characteristic Modes

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

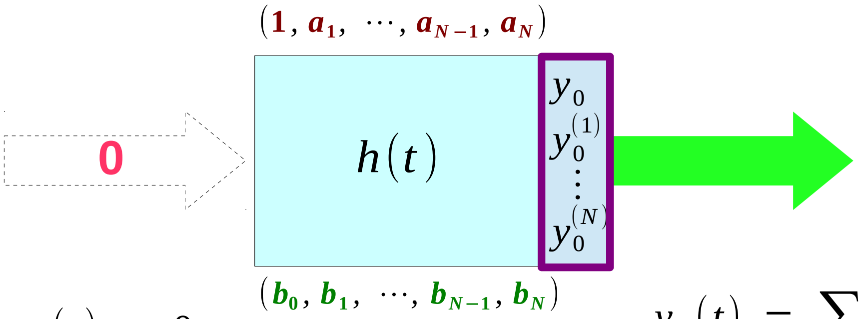
$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$$y_{zi}(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = \sum_i c_i e^{\lambda_i t}$$

λ_i	characteristic roots
$e^{\lambda_i t}$	characteristic modes

ZIR a linear combination of the characteristic modes of the system



$x(t) = 0$

$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$

the initial condition **before** t=0 is used

$$\{y^{(N-1)}(0^-), \cdots, y^{(1)}(0^-), y(0^-)\}$$

$$= \{y^{(N-1)}(0^+), \cdots, y^{(1)}(0^+), y(0^+)\}$$

any input is applied at time t=0, but in the ZIR: the initial condition does not change before and after time t=0 since no input is applied

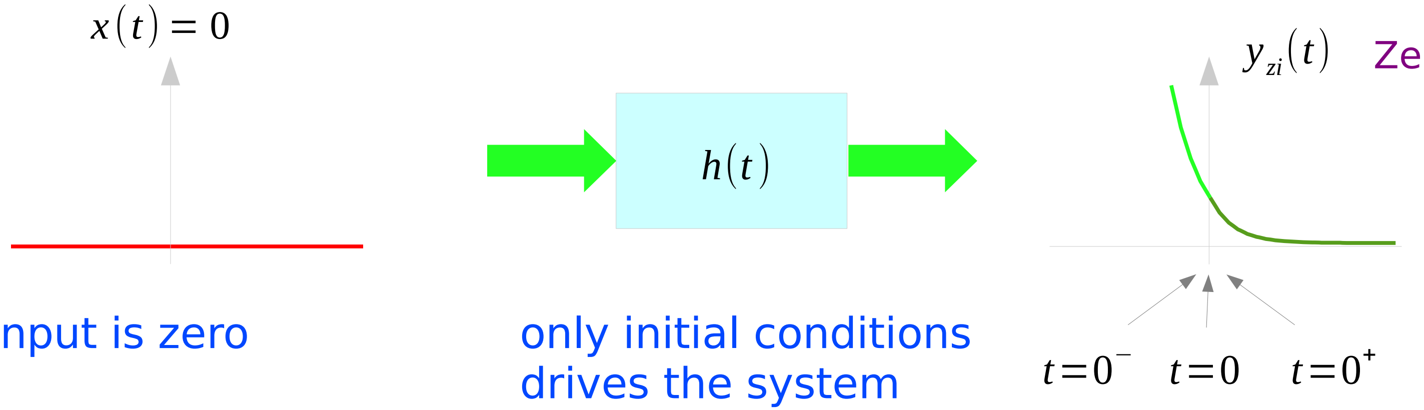
Zero Input Response IVP

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = 0$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = 0$$

$$\begin{array}{ccccccc}
y^{(N-1)}(0^-) & = & y^{(N-1)}(0) & = & y^{(N-1)}(0^+) & = & k_{N-1} \\
y^{(N-2)}(0^-) & = & y^{(N-2)}(0) & = & y^{(N-2)}(0^+) & = & k_{N-2} \\
\vdots & & \vdots & & \vdots & & \vdots \\
y^{(1)}(0^-) & = & y^{(1)}(0) & = & y^{(1)}(0^+) & = & k_1 \\
y(0^-) & = & y(0) & = & y(0^+) & = & k_0
\end{array}$$

ZIR Initial Value Problem (IVP)



Zero Input Response

input is zero

only initial conditions drives the system

$$\begin{array}{ccccc}
y_{zi}(0^-) & = & y_{zi}(0) & = & y_{zi}(0^+) \\
\dot{y}_{zi}(0^-) & = & \dot{y}_{zi}(0) & = & \dot{y}_{zi}(0^+) \\
\ddot{y}_{zi}(0^-) & = & \ddot{y}_{zi}(0) & = & \ddot{y}_{zi}(0^+)
\end{array}$$

- Zero State Response

Zero State Response y(t)

Zero State Response $$y(t) = x(t)*h(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$$ Convolution

Impulse response $h(t)$

causal system *h(t)*: *response cannot begin before the input*

$$\boxed{h(t-\tau) = 0} \quad t-\tau < 0$$

causal input *x(t)*: *the input starts at t=0*

$$\boxed{x(\tau) = 0} \quad \tau < 0$$

Causality $$y(t) = \int_{0^-}^{t} x(\tau)h(t-\tau)\,d\tau \quad , \quad t \geq 0$$

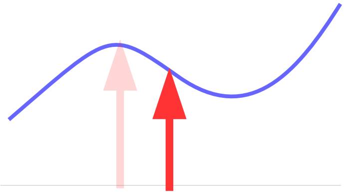
Delayed Impulse Response

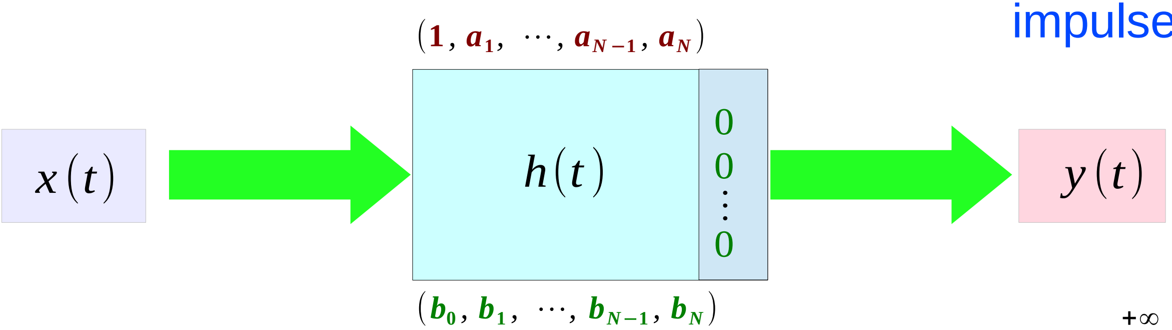
$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^N + b_1 D^{N-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

All initial conditions are zero

$$y^{(N-1)}(0^-) = \cdots = y^{(1)}(0^-) = y^{(0)}(0^-) = 0$$

superposition of inputs
– delayed impulse





the sum of delayed
impulse responses

$$y(t) = h(t) * x(t)$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

scaling

delayed
impulse
response

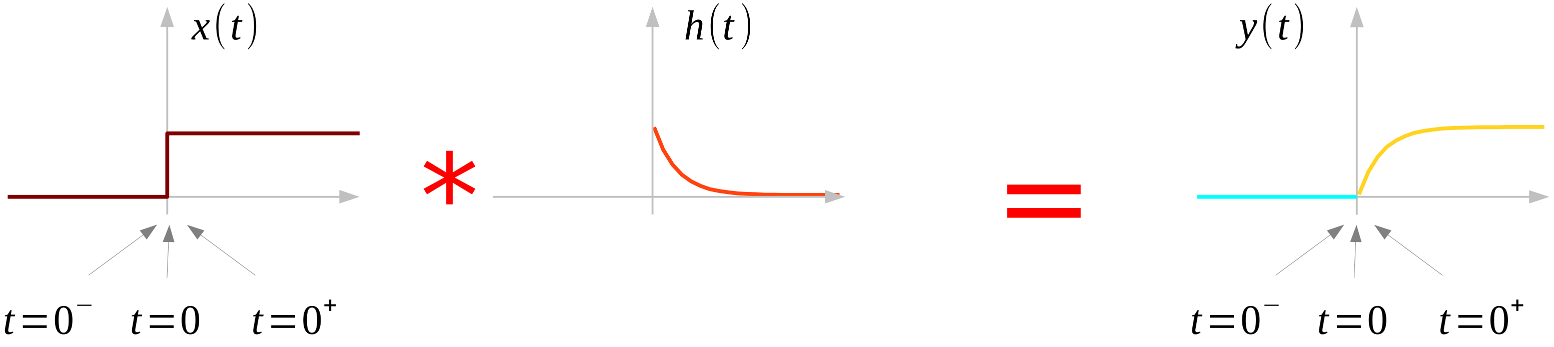
Zero State Response IVP

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^{M-1} x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$\boxed{(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)} \cdot y(t) = \boxed{(b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)}$$

all initial conditions are zero $y(0^-) = y^{(1)}(0^-) = y^{(2)}(0^-) \cdots = y^{(N-2)}(0^-) = y^{(N-1)}(0^-) = 0$

ZSR Initial Value Problem (IVP)



* an impulse in x(t) & h(t) at t = 0 creates non-zero initial conditions $y(0^+) = k_0, \quad y^{(1)}(0^+) = k_1, \quad \cdots \quad y^{(N-2)}(0^+) = k_{N-2}, \quad y^{(N-1)}(0^+) = k_{N-1}$

- Total Response

Total Response

$$y(t) = \underline{\sum_{k=1}^{N} c_k e^{\lambda_k t}} + \underline{x(t) * h(t)}$$

Zero Input Response   Zero State Response

$$y(t) = \underline{y_n(t)} + \underline{y_p(t)}$$

Natural Response   Forced Response

Classical Approach

Total Response = ZIR + ZSR (1)

$$y(t) = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{ZIR} \quad y_{zi}(t)} + \underbrace{x(t) * h(t)}_{\text{ZSR} \quad y_{zs}(t)}$$

$$y(t) = y_{zi}(t) \Leftarrow \boxed{t \le 0^-}$$

because the input has not started yet

$$y_{zi}(0^-) = y_{zi}(0^+)$$

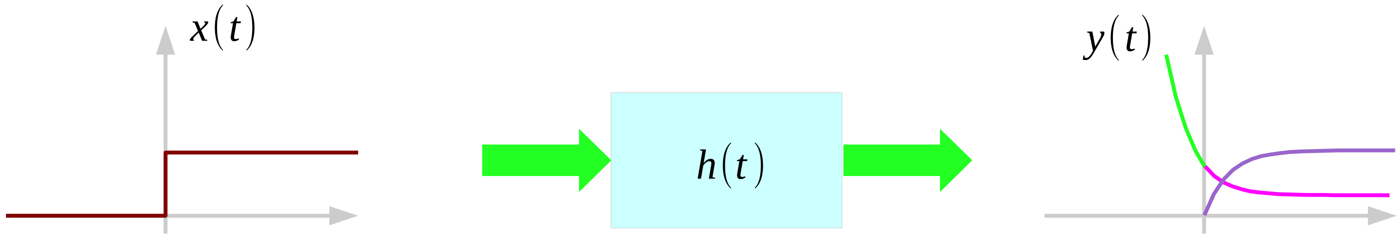
$$\dot{y}_{zi}(0^-) = \dot{y}_{zi}(0^+)$$

possible <u>discontinuity</u> at t = 0

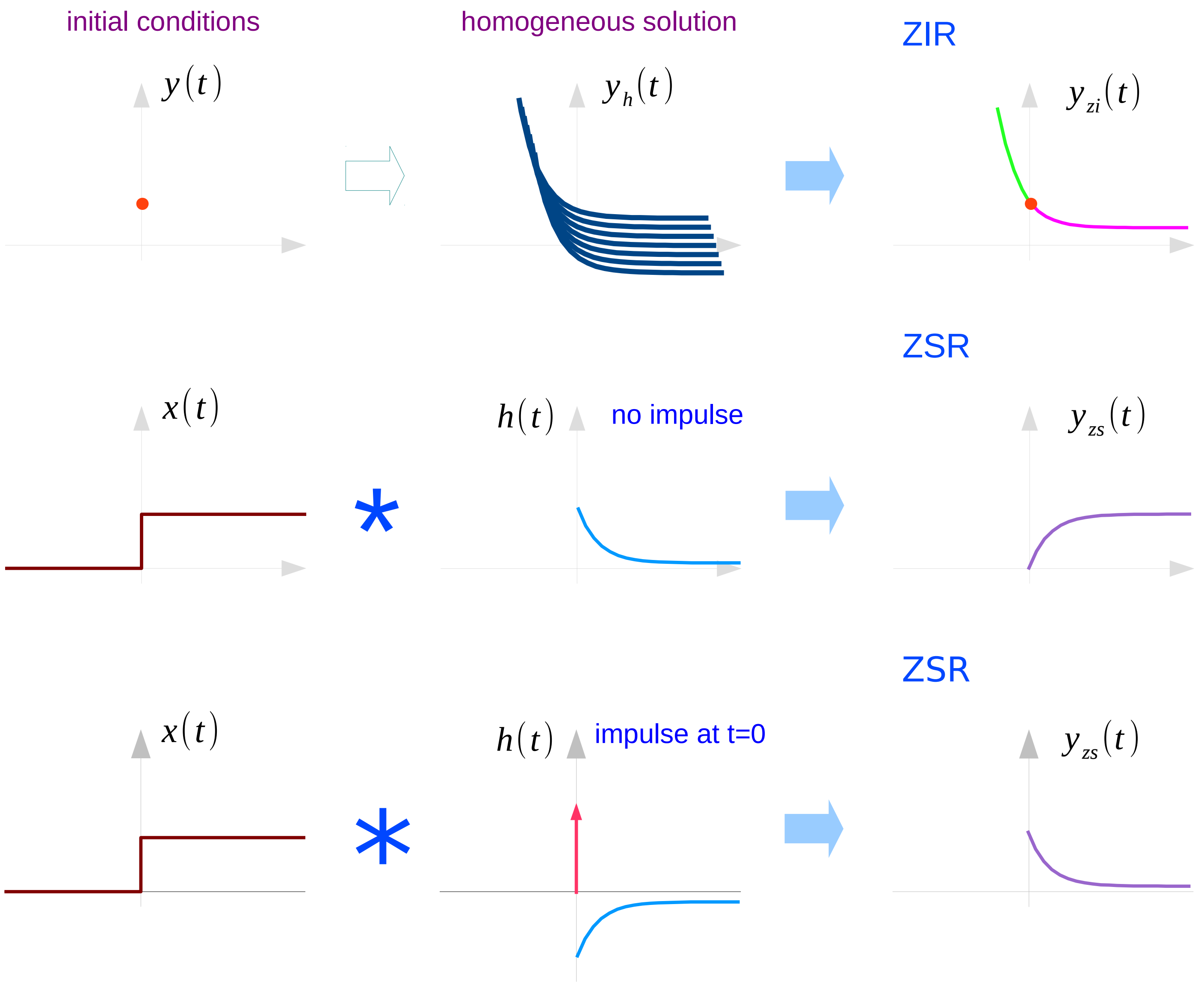
in general, the total response

$$\cancel{y_{zs}(0^-) = y_{zs}(0^+)}$$

$$\cancel{\dot{y}_{zs}(0^-) = \dot{y}_{zs}(0^+)}$$



Total Response = ZIR + ZSR (2)



Total Response = $y_h + y_p$ (1)

$$y(t) = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{Zero Input Response}} + \underbrace{x(t) * h(t)}_{\text{Zero State Response}}$$

$$x(t) * \left(\sum_i d_i e^{\lambda_i t} + b_0 \delta(t)\right)$$

convolution form

$$u(t) \cdot \left(\sum_i k_i e^{\lambda_i t} + y_p(t)\right)$$

step function form

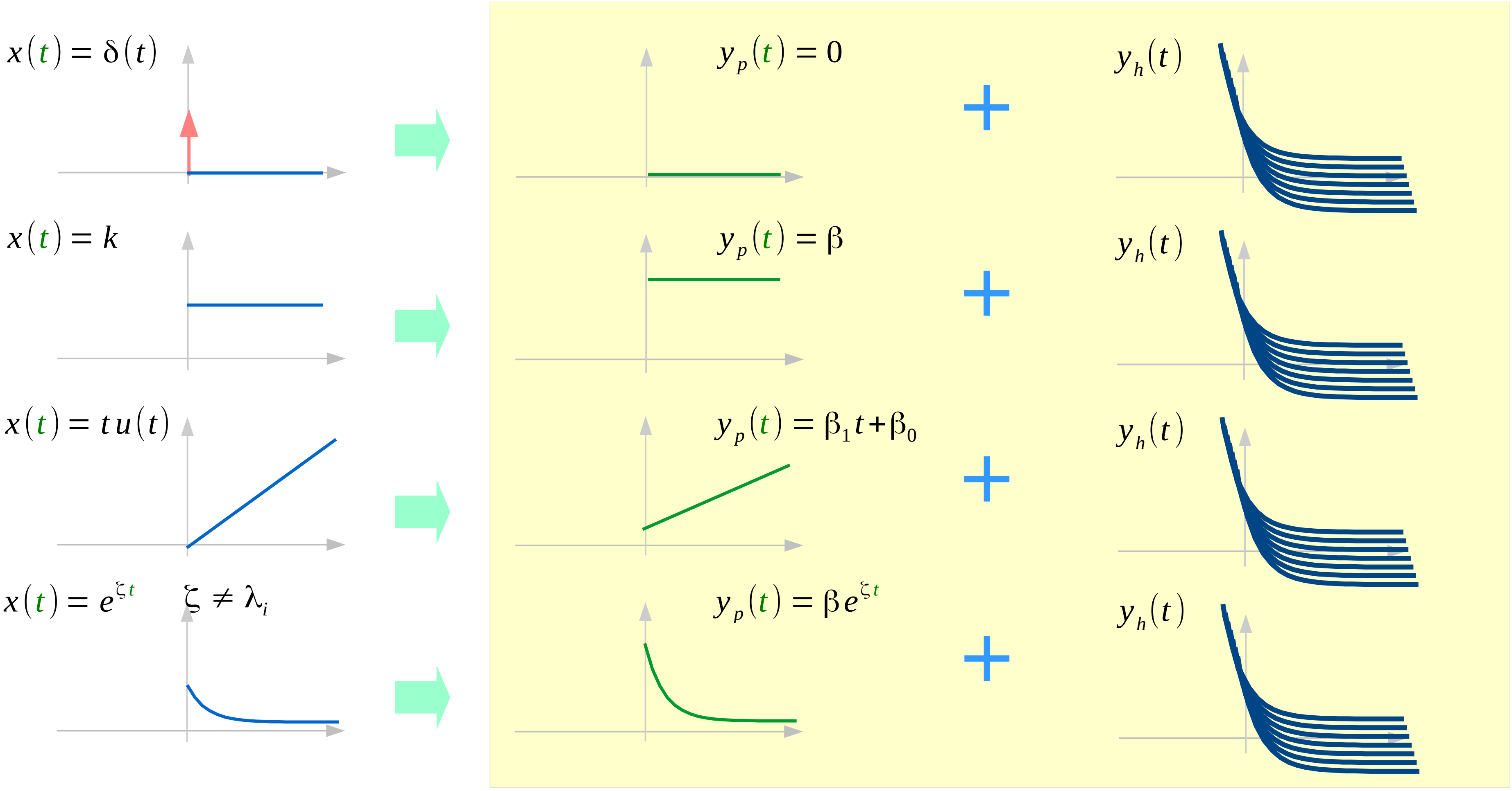
$$y(t) = \underbrace{y_n(t)}_{\text{Natural Response}} + \underbrace{y_p(t)}_{\text{Forced Response}}$$

Classical Approach

$y_p(t)$ is similar to the input $x(t)$

Total Response = y_h + y_p (2)

$$y(t) = \underbrace{y_p(t)}_{\text{Forced Response}} + \underbrace{y_n(t)}_{\text{Natural Response}}$$



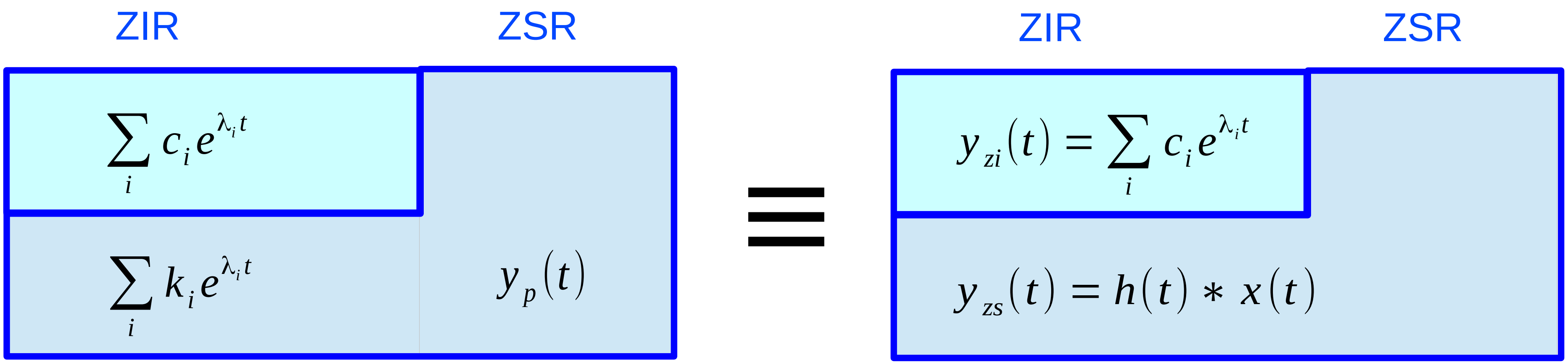
$x(t) = \delta(t)$ → $y_p(t) = 0$ + $y_h(t)$

$x(t) = k$ → $y_p(t) = \beta$ + $y_h(t)$

$x(t) = t\,u(t)$ → $y_p(t) = \beta_1 t + \beta_0$ + $y_h(t)$

$x(t) = e^{\zeta t}$ $\quad \zeta \neq \lambda_i$ → $y_p(t) = \beta e^{\zeta t}$ + $y_h(t)$

Total Response



ZIR ZSR

$\sum_i c_i e^{\lambda_i t}$

$\sum_i k_i e^{\lambda_i t}$ $y_p(t)$

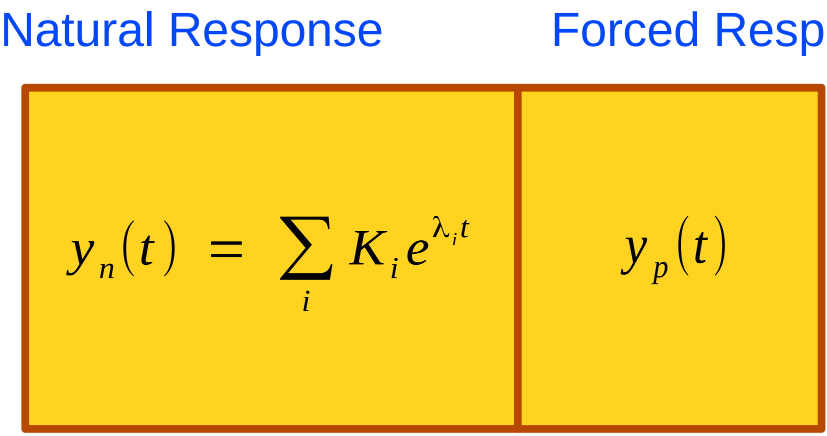
$\equiv$

ZIR ZSR

$y_{zi}(t) = \sum_i c_i e^{\lambda_i t}$

$y_{zs}(t) = h(t) * x(t)$

$$h(t) = b_0 \delta(t) + \sum_i d_i e^{\lambda_i t}$$



Natural Response Forced Response

$y_n(t) \ = \sum_i K_i e^{\lambda_i t}$ $y_p(t)$

$y_p(t) = 0 \Leftarrow x(t) = \delta(t)$

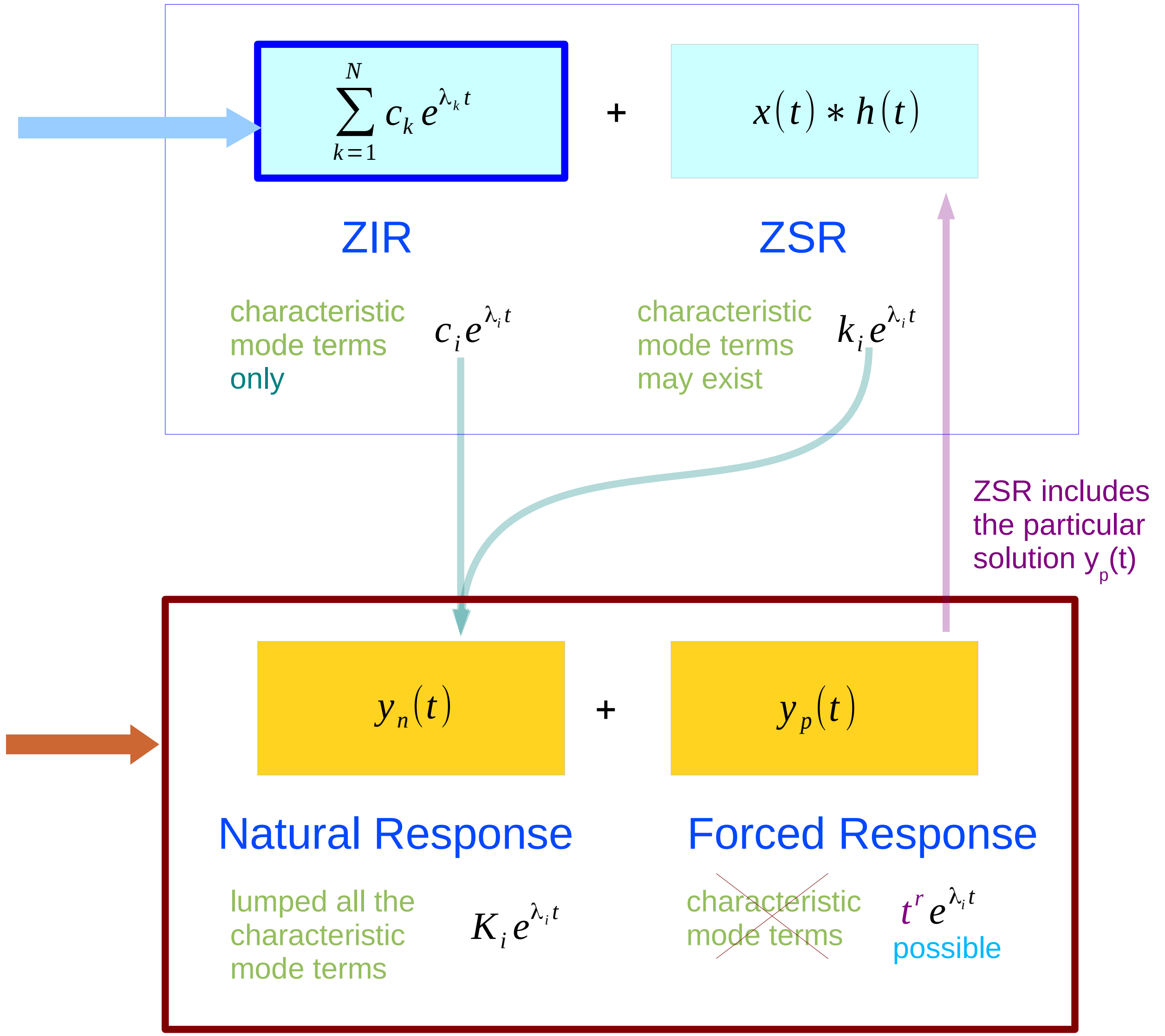
$y_p(t) = \beta \Leftarrow x(t) = k$

$y_p(t) = \beta_1 t + \beta_0 \Leftarrow x(t) = t u(t)$

$y_p(t) = \beta e^{\zeta t} \Leftarrow x(t) = e^{\zeta t} \quad \zeta \neq \lambda_i$

Characteristic Mode Coefficients

the initial condition **after** t=0 is used
the same initial condition **before** t=0
since no input is applied



$$\sum_{k=1}^{N} c_k e^{\lambda_k t}$$ + $$x(t) * h(t)$$

ZIR

ZSR

characteristic mode terms only $c_i e^{\lambda_i t}$

characteristic mode terms may exist $k_i e^{\lambda_i t}$

ZSR includes the particular solution y_p(t)

$$y_n(t)$$ + $$y_p(t)$$

Natural Response

Forced Response

lumped all the characteristic mode terms $K_i e^{\lambda_i t}$

~~characteristic mode terms~~ $t^r e^{\lambda_i t}$ possible

the initial condition **after** t=0 is used
So the effects of the char. modes of ZSR are included.

Total Response and Initial Conditions

zero input response

+

zero state response

$$[-\infty, \; 0^-]$$

$$y(t) = y_{zi}(t) \Leftarrow t \le 0^-$$

because the input has not started yet

continuous at *t = 0*

$$\begin{array}{l} y(0^-) = \boxed{y_{zi}(0^-)} = y_{zi}(0^+) \\ \dot{y}(0^-) = \boxed{\dot{y}_{zi}(0^-)} = \dot{y}_{zi}(0^+) \end{array}$$

$$[0^+, \; +\infty]$$

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$y(0^+) \neq y(0^-)$$

possible discontinuity at *t = 0*

$$\begin{array}{l} y(0^+) = \boxed{y_{zi}(0^+) + y_{zs}(0^+)} \\ \dot{y}(0^+) = \boxed{\dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)} \end{array}$$

natural response

+

forced response

$$\left\{ \begin{array}{l} y_h(0^-) \neq \boxed{y_{zi}(0^-)} \\ \dot{y}_h(0^-) \neq \boxed{\dot{y}_{zi}(0^-)} \end{array} \right.$$

$$\left\{ \begin{array}{l} y_p(0^-) \neq \boxed{y_{zi}(0^-)} \\ \dot{y}_p(0^-) \neq \boxed{\dot{y}_{zi}(0^-)} \end{array} \right.$$

$$[0^+, \; +\infty]$$

$$y(t) = y_h(t) + y_p(t)$$

$$\left\{ \begin{array}{l} y(0^+) = \boxed{y_{zi}(0^+) + y_{zs}(0^+)} \\ \dot{y}(0^+) = \boxed{\dot{y}_{zi}(0^+) + \dot{y}_{zs}(0^+)} \end{array} \right.$$

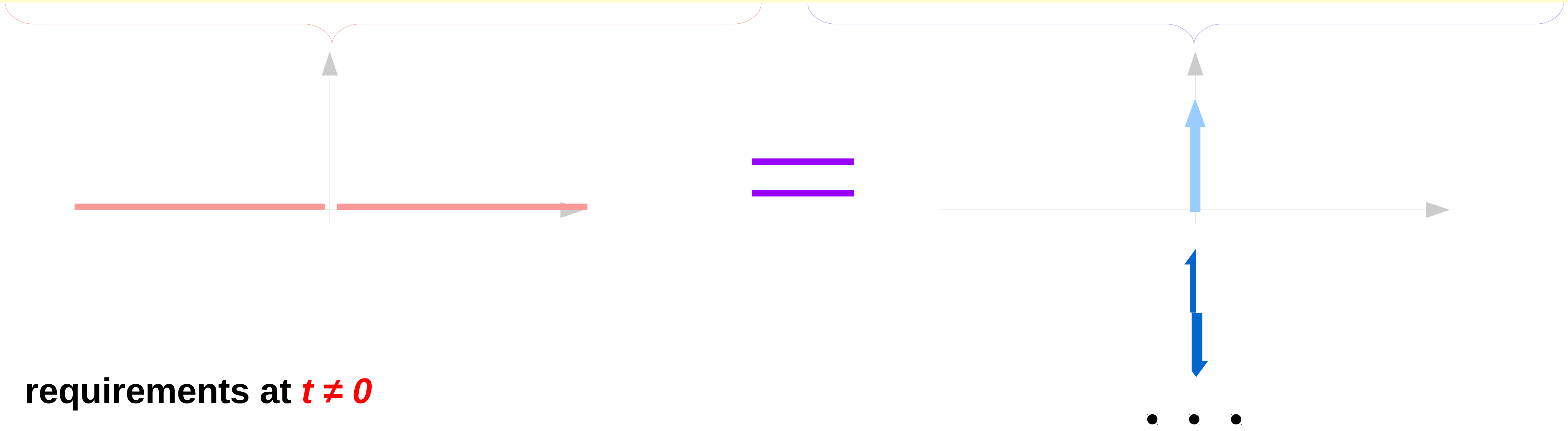
$$\left\{ \begin{array}{l} y(0^+) = \boxed{y_h(0^+) + y_p(0^+)} \\ \dot{y}(0^+) = \boxed{\dot{y}_h(0^+) + \dot{y}_p(0^+)} \end{array} \right.$$

Interval of validity t > 0

- Requirements of an Impulse Response

Requirements of h(t) at t≠0 (1)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



requirements at *t ≠ 0*

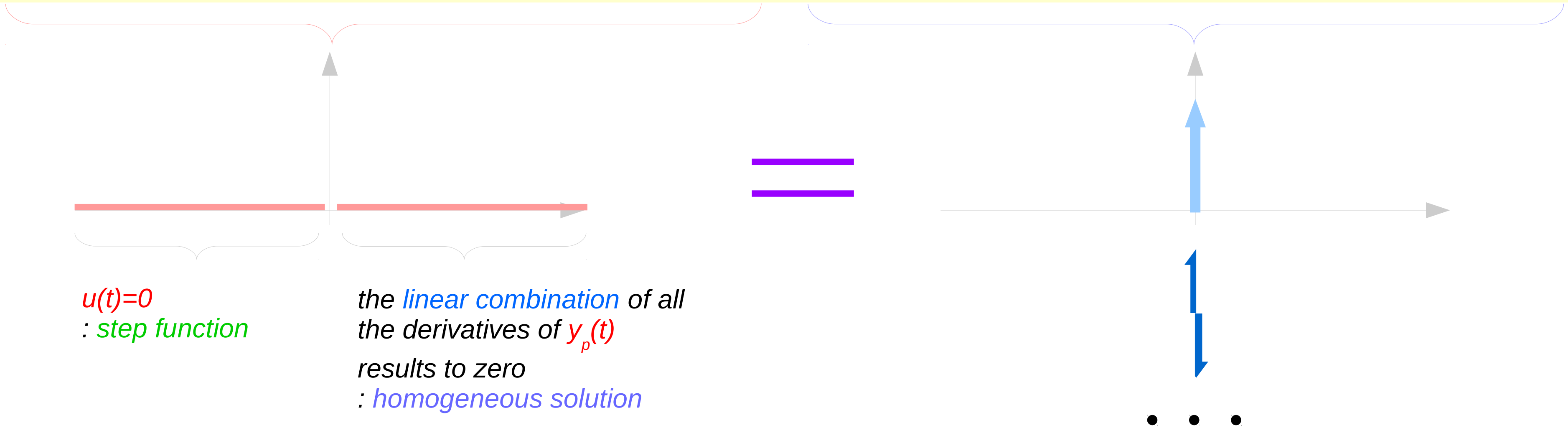
$$h^{(N)}(t) + a_1 h^{(N-1)}(t) \cdots + a_N h(t) = 0 \qquad (t \neq 0)$$

The linear combination
of all the derivatives of *h(t)*
must add to zero for any time t≠0

all the derivatives of δ(t)
exists only t=0.
It is zero for any time t≠0

Requirements of h(t) at t≠0 (2)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\, \delta(t)}{dt} + b_M \delta(t)$$



u(t)=0
: step function

the linear combination of all the derivatives of $y_p(t)$ results to zero
: homogeneous solution

$u(t) = 0$

$y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$

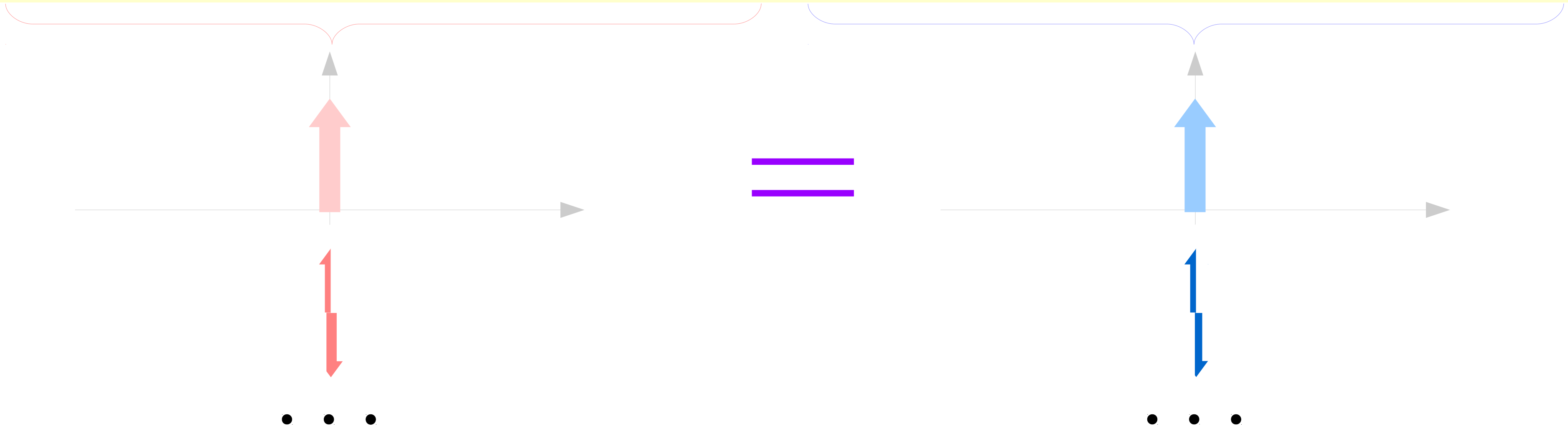
for $t < 0$, $u(t) = 0$
for $t > 0$, $y_h^{(N)} + a_1 y_h^{(N-1)} + \cdots + a_N y_h = 0$
derivatives of $\{y_h \cdot u\}$ produce
derivatives of δ when $t = 0$

$y_h(t)u(t)$ when $t \neq 0$

➡ A possible candidate of $h(t)$

Requirements of h(t) at t=0 (1)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



requirements at *t = 0*

All the <u>derivatives</u> of *h(t)* up to *N* must **match** the corresponding <u>derivatives</u> of an impulse *δ(t)* up to *M* at time t=0

Need to add a δ(t) and its derivatives in case that (N ≤ M)

Need to integrate $y_h(t) \cdot u(t)$ several times in case that (N > M)

Derivatives of $y_h(t)\cdot u(t)$

$$h(t) = y_h(t)u(t)$$

$$u^{(i)}(t) = \delta^{(i-1)}(t)$$
$$f(t)\delta(t) = f(0)\delta(t)$$

$$h = y_h u$$
$$h^{(1)} = y_h^{(1)}u + y_h u^{(1)} \longrightarrow y_h(0)\delta(t)$$
$$h^{(2)} = y_h^{(2)}u + 2\,y_h^{(1)}u^{(1)} + y_h u^{(2)} \longrightarrow 2\,y_h^{(1)}(0)\delta(t) + y_h(0)\delta^{(1)}(t)$$
$$h^{(3)} = y_h^{(3)}u + 3\,y_h^{(2)}u^{(1)} + 3\,y_h^{(1)}u^{(2)} + y_h u^{(3)} \longrightarrow 3\,y_h^{(2)}(0)\delta(t) + 3\,y_h^{(1)}(0)\delta^{(1)}(t) + y_h(0)\delta^{(2)}(t)$$
$$\cdots \quad \cdots \quad \cdots \qquad\qquad \cdots \quad \cdots \quad \cdots$$

$$h(t) = y_h(t)u(t)$$

⇒

All the derivatives of *h(t)* up to *N* incurs
the derivatives of an impulse *δ(t)* up to *N-1*

$$h^{(N)}(t) = \frac{d^N}{dt^N}\{y_h(t)u(t)\} \longrightarrow K_1\delta^{(N-1)}(t) + K_2\delta^{(N-2)}(t) + \cdots + K_{N-1}\delta^{(1)}(t) + K_N\delta(t)$$

h(t) can have at most a δ(t) for most systems

$y(t) \leftarrow \boxed{h(t)} \leftarrow x(t)$

$$y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y^{(1)} + a_N y = b_0 x^{(N)} + b_1 x^{(N-1)} + \cdots + b_{N-1} x^{(1)} + b_N x$$

$h(t) \leftarrow \boxed{h(t)} \leftarrow \delta(t)$

$$h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-1} h^{(1)} + a_N h = b_0 \delta^{(N)} + b_1 \delta^{(N-1)} + \cdots + b_{N-1} \delta^{(1)} + b_N \delta$$

$if\ h\ contain\ \delta$ $\quad C\delta^{(N)} \cdots \cdots = b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

cannot be matched

$if\ h\ contain\ \delta^{(1)}$ $\quad C\delta^{(N+1)} \cdots \cdots \neq b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

cannot be matched

$if\ h\ contain\ \delta^{(2)}$ $\quad C\delta^{(N+2)} \cdots \cdots \neq b_0 \delta^{(N)}$ — the highest order derivatives of δ(t)

$t=0$

h(t) can have at most an impulse $b_0\delta(t)$
no derivatives of $\delta(t)$ possible at all

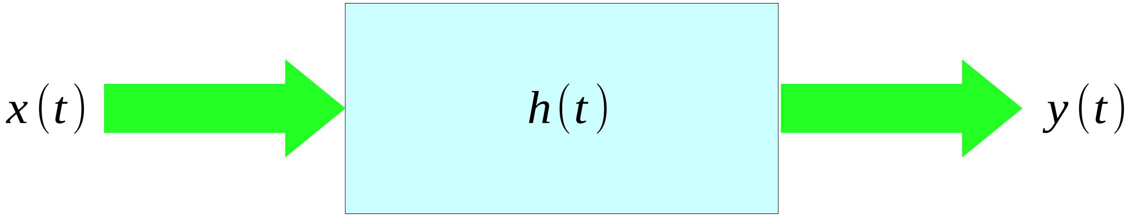
in most systems N ≥ M

ODE's and Causal LTI Systems

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

N > M : (N-M) integrator

N < M : (M-N) differentiator – magnify high frequency components of noise (seldom used)



N ≥ M *in most systems*

$$h(t) = y_h(t)u(t) \qquad (N > M)$$

$$h(t) = y_h(t)u(t) + m_0\delta(t) \qquad (N = M)$$

- An Impulse Response and System Responses

h(t) as a ZIR (t > 0)

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$

$$h(t) \;=\; y_h(t) u(t)$$

The solution of the IVP with the following I.C.

Interval of Validity : (t > 0)

Zero Input (no input for t > 0)

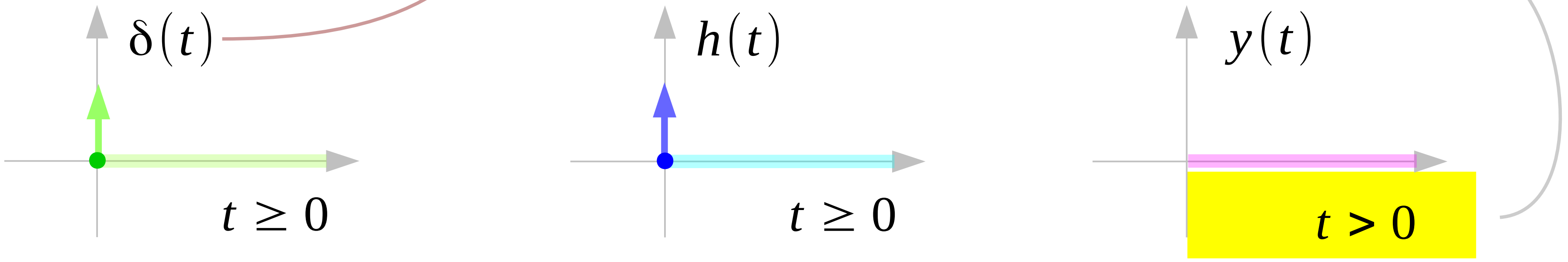
$\delta(t) = 0 \quad for \quad t > 0$

creates nonzero i. c

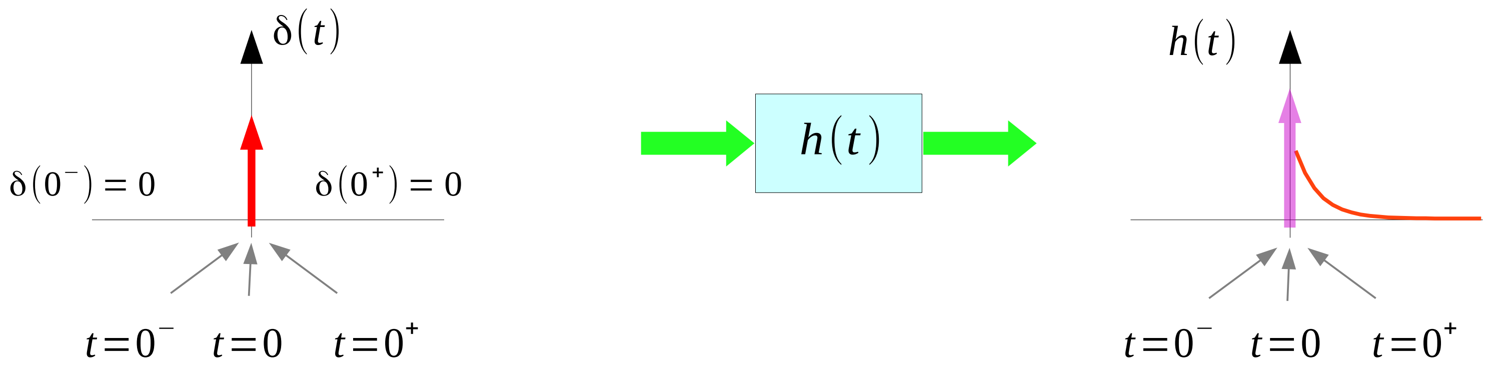
$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

an impulse may be present

No impulse is assumed



Impulse Response h(t)



* general **y(t)** cannot be a **ZIR** for a general input **x(t)** (generally x(t) ≠ 0 for t > 0)

* impulse input vanishes (x(t) = δ(t) = 0 for t > 0)

$(N \geq M)$

h(t) : ZIR with the newly created I.C. (t > 0)

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

non-zero i. c.

$\exists\, i, \quad k_i \neq 0$

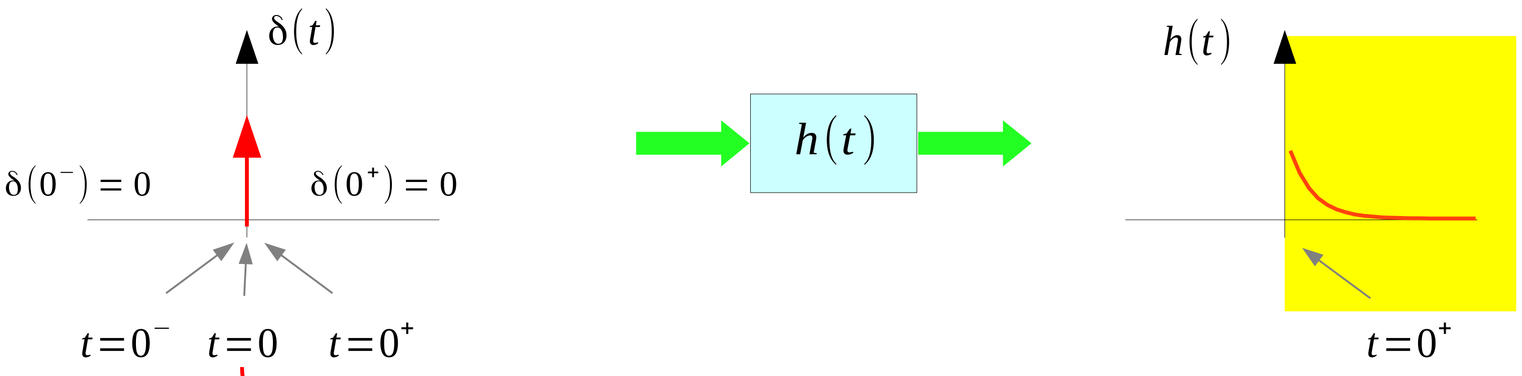
$t \geq 0^+$ $(t \neq 0)$ $h(t) \;=\; char\ mode\ terms$

$t = 0$ $h(t) \;=\; b_0\delta(t)$ at most an impulse

$t \geq 0$ $h(t) \;=\; b_0\delta(t) + char\ mode\ terms$

h(t) : **ZSR** to an impulse input

Assumed Finite Jumps



- generates energy storage
- creates nonzero i. c. at $t = 0^+$

$$\begin{aligned} h^{(N-1)}(0^-) &= 0 \\ h^{(N-2)}(0^-) &= 0 \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^-) &= 0 \\ h(0^-) &= 0 \end{aligned}$$

initially at rest

$$\begin{aligned} h^{(N)}(0) &= f_{N-1}(k_i, \delta^{(i)}) \\ h^{(N-1)}(0) &= f_{N-2}(k_i, \delta^{(i)}) \\ \vdots \quad & \quad \vdots \\ h^{(2)}(0) &= f_1(k_i, \delta^{(i)}) \\ h^{(1)}(0) &= f_0(k_i, \delta^{(i)}) \end{aligned}$$

Impulse matching

$$\begin{aligned} h^{(N-1)}(0^+) &= k_{N-1} \\ h^{(N-2)}(0^+) &= k_{N-2} \\ \vdots \quad & \quad \vdots \\ h^{(1)}(0^+) &= k_1 \\ h(0^+) &= k_0 \end{aligned}$$

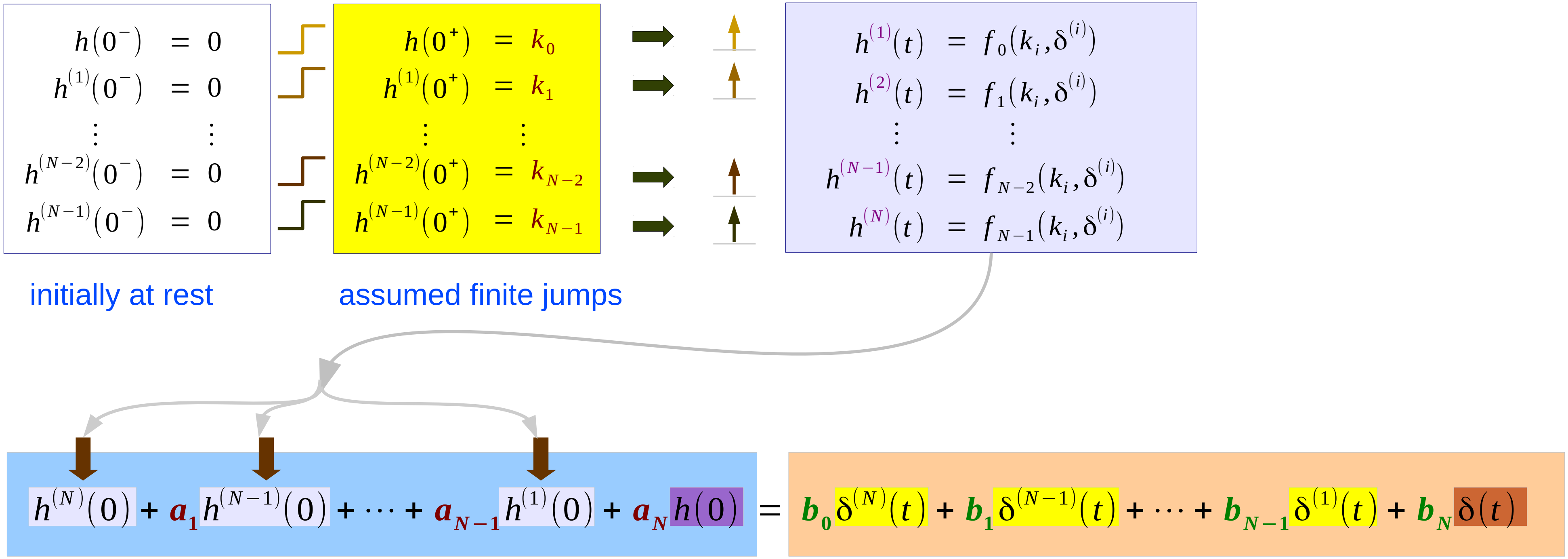
non-zero initial conditions

$\exists\, i, \quad k_i \neq 0$

Assumed finite jumps

Impulse Matching Method Summary

$$\frac{d^N h(t)}{dt^N} + a_1 \frac{d^{N-1} h(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,h(t)}{dt} + a_N h(t) = b_0 \frac{d^M \delta(t)}{dt^M} + b_1 \frac{d^{M-1} \delta(t)}{dt^{M-1}} + \cdots + b_{M-1} \frac{d\,\delta(t)}{dt} + b_M \delta(t)$$



$$\begin{aligned} h(0^-) &= 0 \\ h^{(1)}(0^-) &= 0 \\ &\vdots \\ h^{(N-2)}(0^-) &= 0 \\ h^{(N-1)}(0^-) &= 0 \end{aligned}$$

$$\begin{aligned} h(0^+) &= k_0 \\ h^{(1)}(0^+) &= k_1 \\ &\vdots \\ h^{(N-2)}(0^+) &= k_{N-2} \\ h^{(N-1)}(0^+) &= k_{N-1} \end{aligned}$$

$$\begin{aligned} h^{(1)}(t) &= f_0(k_i, \delta^{(i)}) \\ h^{(2)}(t) &= f_1(k_i, \delta^{(i)}) \\ &\vdots \\ h^{(N-1)}(t) &= f_{N-2}(k_i, \delta^{(i)}) \\ h^{(N)}(t) &= f_{N-1}(k_i, \delta^{(i)}) \end{aligned}$$

initially at rest

assumed finite jumps

$$h^{(N)}(0) + a_1 h^{(N-1)}(0) + \cdots + a_{N-1} h^{(1)}(0) + a_N h(0) = b_0 \delta^{(N)}(t) + b_1 \delta^{(N-1)}(t) + \cdots + b_{N-1} \delta^{(1)}(t) + b_N \delta(t)$$

Impulse matching

$$h(t) = \left(c_0 e^{\lambda_0 t} + c_1 e^{\lambda_1 t} + \cdots + c_{N-2} e^{\lambda_{N-2} t} + c_{N-1} e^{\lambda_{N_1} t}\right) \cdot u(t)$$

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