Angle Recoding 2. Wu
1. Conventional CORDIC

20180920 Thr

⓪ Conventional CORDIC

elementary angle $a(i) = \tan^{-1}(2^{-i})$

the number of elementary angles N

the rotation sequence $\mu(i) = \{-1, +1\}$

$+1, -1, -1, +1, +1, \ldots\ldots$

the i-the rotation angle $a(i)$

the W-bit word length

the iteration number $N \leq W$

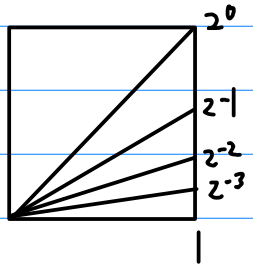
the angle quantization error

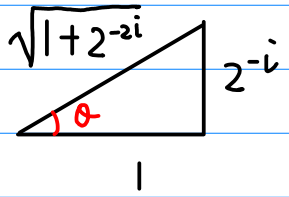
$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\, a(i)$$

rotation $\theta \rightarrow \cos\theta, \sin\theta$

$\uparrow$

$a(i) = \tan^{-1}(2^{-i})$





$$\begin{bmatrix} x(i+1) \\ y(i+1) \end{bmatrix} = (1+2^{-2i})^{0.5} \begin{bmatrix} x_p(i+1) \\ y_p(i+1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2^{-i} \\ 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}$$

ideal rotated vector

$$\begin{bmatrix} x_p(i+1) \\ y_p(i+1) \end{bmatrix} = \frac{1}{\sqrt{1+2^{-2i}}} \begin{bmatrix} 1 & -2^{-i} \\ 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x(i) \\ y(i) \end{bmatrix}$$

rotation of $a(i)$ — only two shift-and-add

AQ & conventional CORDIC

EAS (Elementary Angle Set)

comprises of all $a(i)$ for $0 \le i \le N-1$

$$S = \{a(i) : 0 \le i \le N-1\}$$

the CORDIC algorithm essentially performs AQ

tries to perform the rotation

by sequential applications of

micro-rotations of all elementary angles

given a target rotation angle θ

- the first rotation sequence $\mu(0)$
  - for the most significant elementary angle $a(0)$
- the second rotation sequence $\mu(1)$
  - for the next most significant elementary angle $a(1)$

repeated until the last elementary angle is applied.

the sub-angle θ_i in AQ

$\theta_i = \mu(i)\, a(i)$ in CORDIC

$\mu(i) = \{-1, +1\}$

$a(i) = \tan^{-1}(2^{-i})$

the number of sub-angles

N_A in AQ

N in CORDIC

CORDIC algorithm sequentially apply all θ_i's

for $i = 0, 1, \cdots, N-1$

to approximate the target angle θ

iteration number	elementary angle	value in radian
i=0	a(0)=atan(2^{-0})	
i=1	a(1)=atan(2^{-1})	
i=2	a(1)=atan(2^{-2})	
i=3	a(1)=atan(2^{-3})	
i=4	a(1)=atan(2^{-4})	
i=5	a(1)=atan(2^{-5})	
i=6	a(1)=atan(2^{-6})	
i=7	a(1)=atan(2^{-7})	

$$S = \{a(i) : 0 \leq i \leq N-1\}$$

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N-1} \mu(i)\,\alpha(i) \qquad \mu(i) = \{-1, 0, +1\}$$

$$= \theta - \sum_{j=0}^{N'} \tilde{\theta}(j)$$

$$N' \equiv \sum_{i=0}^{N-1} |\mu(i)| \qquad \{|+1|, |0|, |+1|\}$$

the effective iteration number N'

$S(j)$ the rotational sequence

determines the micro-rotation angle in the j-th iteration

$$i \leftarrow S(j)$$

index $0, 1, \cdots, N'$ (reduced number)

index $0, 1, \cdots, N$ (original number)

$$\mu(S(j)) \leftarrow \alpha(j)$$

$\{-1, +1\}$

$$\mu(i) = \begin{cases} \mu(S(j)) & i = S(j) \\ 0 & i \neq S(j) \end{cases}$$

--- reduced index

er

$i = 0, 1, 2, 3, \cdots, N-1$

$s(j) = 0, 1, 2, 3, \cdots, N-1$ rotational sequence

$\alpha(j) = -1, 0, 0, +1, \cdots, -1$ directional sequence

$j = 0, -, -, 1, \cdots, N'-1$ effective iteration number

$N' = N-2$

the j-th micro-rotation of $a(s(j))$

elementary angle

$$a(i) = \tan^{-1}(2^{-i})$$
$$a(s(j)) = \tan^{-1}(2^{-s(j)})$$

$\alpha(j)\, a(s(j)) = \alpha(j) \tan^{-1}(2^{-s(j)})$ $\quad \alpha(j) \in \{-1, +1\}$

$\Leftrightarrow \mu(i)\, a(i)$ $\quad \mu(i) \in \{-1, 0, +1\}$

$$\xi_{m,\,CORDIC} \equiv \theta - \sum_{i=0}^{N} \mu(i)\,a(i) \qquad \mu(i) \in \{-1, 0, +1\}$$

$$= \theta - \left[\sum_{j=0}^{N'} \tilde{\theta}(j)\right]$$

$$= \theta - \left[\sum_{j=0}^{N'} \tan^{-1}\left(\alpha(j)\cdot 2^{-s(j)}\right)\right] \qquad \alpha(j) \in \{-1, +1\}$$

$$\tilde{\theta}(j) = \alpha(j)\,\tan^{-1}\left(2^{-s(j)}\right)$$

$$= \tan^{-1}\left(\alpha(j)\cdot 2^{-s(j)}\right)$$

$$S_1 = \left\{ \tan^{-1}\left(\boxed{\alpha}\cdot 2^{-\boxed{s}}\right) \quad \boxed{\alpha} \in \{-1, 0, +1\},\ \boxed{s} \in \{0, 1, 2, \cdots N-1\} \right\}$$

.

```
>> mu = [1, -1, 1, 1, -1, 1, -1, -1, 1, -1, -1, 1, 1, -1, 1, 1]
>> length(mu)
ans =  16
>> s =[ 0: 15]
>> atan(1)
ans =  0.78540
>> pi/4
ans =  0.78540
>> sum(atan(2.^(-s)) .* mu)
ans =  0.63815
>> 13 * pi / 32
ans =  1.2763
```

```c
#include <stdio.h>
#include <math.h>

#define N 16


int main(void) {
  double a[N];
  // double theta = 0.63761;
  // double theta = 0.27623;
  double theta = 2*atan(pow(2,-2));

  int k, i;
  int u[N];
  double angle;


  for (i=0; i<N; ++i) {
    a[i] = atan(1./pow(2, i));
  }



  k = 0;
  for (k=0; k<N; ++k) {
    u[k] = (theta >= 0) ? +1 : -1;

    printf("k=%2d theta=%10.7f ", k, theta);
    printf("u[%2d]=%+d ", k, u[k]);

    theta = theta - u[k] * a[k];

    printf("a[%2d]=%10.7f, new theta=%10.7f \n", k, a[k], theta);

  }


  angle = 0.0;
  for (i=0; i<N; ++i) {
    angle += u[i] * a[i];

    printf("i=%2d u[%2d]=%+d a[%2d]=%f angle=%f \n", i, i, u[i], i, a[i], angle);
  }


}
```

```
k= 0 theta= 0.4899573 u[ 0]=+1 a[ 0]= 0.7853982, new theta=-0.2954408
k= 1 theta=-0.2954408 u[ 1]=-1 a[ 1]= 0.4636476, new theta= 0.1682068
k= 2 theta= 0.1682068 u[ 2]=+1 a[ 2]= 0.2449787, new theta=-0.0767719
k= 3 theta=-0.0767719 u[ 3]=-1 a[ 3]= 0.1243550, new theta= 0.0475831
k= 4 theta= 0.0475831 u[ 4]=+1 a[ 4]= 0.0624188, new theta=-0.0148357
k= 5 theta=-0.0148357 u[ 5]=-1 a[ 5]= 0.0312398, new theta= 0.0164041
k= 6 theta= 0.0164041 u[ 6]=+1 a[ 6]= 0.0156237, new theta= 0.0007804
k= 7 theta= 0.0007804 u[ 7]=+1 a[ 7]= 0.0078123, new theta=-0.0070319
k= 8 theta=-0.0070319 u[ 8]=-1 a[ 8]= 0.0039062, new theta=-0.0031257
k= 9 theta=-0.0031257 u[ 9]=-1 a[ 9]= 0.0019531, new theta=-0.0011726
k=10 theta=-0.0011726 u[10]=-1 a[10]= 0.0009766, new theta=-0.0001960
k=11 theta=-0.0001960 u[11]=-1 a[11]= 0.0004883, new theta= 0.0002923
k=12 theta= 0.0002923 u[12]=+1 a[12]= 0.0002441, new theta= 0.0000481
k=13 theta= 0.0000481 u[13]=+1 a[13]= 0.0001221, new theta=-0.0000740
k=14 theta=-0.0000740 u[14]=-1 a[14]= 0.0000610, new theta=-0.0000129
k=15 theta=-0.0000129 u[15]=-1 a[15]= 0.0000305, new theta= 0.0000176
i= 0 u[ 0]=+1 a[ 0]=0.785398 angle=0.785398
i= 1 u[ 1]=-1 a[ 1]=0.463648 angle=0.321751
i= 2 u[ 2]=+1 a[ 2]=0.244979 angle=0.566729
i= 3 u[ 3]=-1 a[ 3]=0.124355 angle=0.442374
i= 4 u[ 4]=+1 a[ 4]=0.062419 angle=0.504793
i= 5 u[ 5]=-1 a[ 5]=0.031240 angle=0.473553
i= 6 u[ 6]=+1 a[ 6]=0.015624 angle=0.489177
i= 7 u[ 7]=+1 a[ 7]=0.007812 angle=0.496989
i= 8 u[ 8]=-1 a[ 8]=0.003906 angle=0.493083
i= 9 u[ 9]=-1 a[ 9]=0.001953 angle=0.491130
i=10 u[10]=-1 a[10]=0.000977 angle=0.490153
i=11 u[11]=-1 a[11]=0.000488 angle=0.489665
i=12 u[12]=+1 a[12]=0.000244 angle=0.489909
i=13 u[13]=+1 a[13]=0.000122 angle=0.490031
i=14 u[14]=-1 a[14]=0.000061 angle=0.489970
i=15 u[15]=-1 a[15]=0.000031 angle=0.489940
```