

Spectral Characteristics of System Response

Young W Lim

January 4, 2020

Copyright (c) 2018 Young W. Lim. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

This work is licensed under a Creative Commons "Attribution-NonCommercial-ShareAlike 3.0 Unported" license.



Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles, Jr. and B. Shi

Power Spectral Density

N Gaussian random variables

Definition

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\omega)e^{-j\omega\tau}d\tau$$

Power Spectral Density

N Gaussian random variables

Definition

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\xi_1) \int_{-\infty}^{\infty} h(\xi_2) \int_{-\infty}^{\infty} R_{XX}(\tau + \xi_1 - \xi_2) e^{-j\omega\tau} d\tau d\xi_2 d\xi_1$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\xi_1) e^{-j\omega\xi_1} d\xi_1 \int_{-\infty}^{\infty} h(\xi_2) e^{-j\omega\xi_2} d\xi_2 \int_{-\infty}^{\infty} R_{XX}(\xi) e^{-j\omega\xi} d\xi$$

$$S_{YY}(\omega) = H^*(\omega)H(\omega)S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Average Power

N Gaussian random variables

Definition

$$S_{YY}(\omega) = H^*(\omega)H(\omega)S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$P_{YY} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) |H(\omega)|^2 d\omega$$

Cross-Correlation Function

N Gaussian random variables

Definition

$$S_{XY}(\omega) = H(\omega)S_{XX}(\omega)$$

$$S_{YX}(\omega) = H(-\omega)S_{XX}(\omega)$$

Measurement of Power Density Spectrum (1)

N Gaussian random variables

Definition

$$\begin{aligned}P_{YY}(\omega_f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega) |H(\omega)|^2 d\omega \\&= \frac{1}{\pi} \int_0^{\infty} S_{XX}(\omega) |H(\omega)|^2 d\omega \\&= \frac{1}{\pi} S_{XX}(\omega_f) \int_0^{\infty} |H(\omega)|^2 d\omega \\&= \frac{1}{\pi} S_{XX}(\omega_f) |H(\omega_f)|^2 W_N\end{aligned}$$

Measurement of Power Density Spectrum (2)

N Gaussian random variables

Definition

$$P_{YY}(\omega_f) = \frac{1}{\pi} S_{XX}(\omega_f) |H(\omega)|^2 W_N$$

$$W_N = \frac{\int_0^\infty |H(\omega)|^2 d\omega}{|H(\omega_f)|^2}$$

$$S_{YY}(\omega_f) = \frac{\pi P_{YY}(\omega_f)}{W_N |H(\omega_f)|^2}$$

Measurement of Power Density Spectrum (4)

N Gaussian random variables

Definition

$$P_{YY}(\omega_{IF}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2}{4} [S_{XX}A + S_{XX}B] |H(\omega)|^2 d\omega$$

$$S_{XX}A = S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f)$$

$$S_{XX}B = S_{XX}(\omega + \omega_0 + \omega_{IF} + \omega_f)$$

$$P_{YY}(\omega_f) = \frac{2}{2\pi} \int_0^{\infty} \frac{A_0^2}{4} [S_{XX}A] |H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} \int_0^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f) |H(\omega)|^2 d\omega$$

Measurement of Power Density Spectrum (5)

N Gaussian random variables

Definition

$$\begin{aligned}P_{YY}(\omega_f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2}{4} [S_{XX}A + S_{XX}B] |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} \int_{-\infty}^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f) |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} S_{XX}(-\omega_0 - \omega_f) \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega\end{aligned}$$

Measurement of Power Density Spectrum (5)

N Gaussian random variables

Definition

$$\begin{aligned} P_{YY}(\omega_f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2}{4} [S_{XX}A + S_{XX}B] |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} \int_{-\infty}^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f) |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} S_{XX}(-\omega_0 - \omega_f) \int_0^{\infty} |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f) \int_0^{\infty} |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f) W_N |H(\omega_{IF})|^2 \end{aligned}$$

Measurement of Power Density Spectrum (6)

N Gaussian random variables

Definition

$$\begin{aligned}P_{YY}(\omega_f) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A_0^2}{4} [S_{XX}A + S_{XX}B] |H(\omega)|^2 d\omega \\ &= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f) W_N |H(\omega_{IF})|^2 \\ S_{XX}(\omega_0 + \omega_f) &= \frac{4\pi P_{YY}(\omega_{IF})}{A_0^2 W_N |H(\omega_{IF})|^2}\end{aligned}$$

Power Density Spectrum for Discrete-Time System Response

N Gaussian random variables

Definition

$$S_{X_s, X_s}(\omega) = S_{X_s, X_s}(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} R_{XX}[n]e^{-jn\Omega}$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-jn\Omega}$$

Auto-Correlation Function for Discrete-Time System Response

N Gaussian random variables

Definition

$$\begin{aligned}R_{YY}[n] &= E[Y[k]Y[k+n]] \\&= E\left[\sum_{m=-\infty}^{+\infty} X[k-m]h[m] \sum_{p=-\infty}^{+\infty} X[k+n-p]h[p]\right] \\&= \sum_{m=-\infty}^{+\infty} h[m]E\left[\sum_{p=-\infty}^{+\infty} X[k-m]X[k+n-p]h[p]\right] \\&= \sum_{m=-\infty}^{+\infty} h[m]E\left[\sum_{p=-\infty}^{+\infty} R_{XX}[n+m-p]h[p]\right]\end{aligned}$$

DTFT of Auto-Correlation Function for Discrete-Time System Response (1)

N Gaussian random variables

Definition

$$\begin{aligned} S_{Y_s Y_s}(\omega) &= S_{Y_s Y_s}(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} R_{YY}[n] e^{-jn\Omega} \\ &= \sum_{m=-\infty}^{+\infty} h[m] \sum_{p=-\infty}^{+\infty} h[p] \sum_{n=-\infty}^{+\infty} R_{XX}[n+m-p] e^{-jn\Omega} \\ &= \sum_{m=-\infty}^{+\infty} h[m] e^{+jm\Omega} \sum_{p=-\infty}^{+\infty} h[p] e^{-jp\Omega} \sum_{n=-\infty}^{+\infty} R_{XX}[r] e^{-jr\Omega} \\ S_{Y_s Y_s}(\omega) &= S_{Y_s Y_s}(e^{j\Omega}) = S_{X_s X_s}(e^{j\Omega}) |H(e^{j\Omega})|^2 \end{aligned}$$

Average Power

N Gaussian random variables

Definition

$$\begin{aligned} E[Y^2[n]] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{Y_s Y_s}(e^{j\Omega}) d\Omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{X_s X_s}(e^{j\Omega}) |H(e^{j\Omega})|^2 d\Omega \end{aligned}$$

Cross-Power Density Spectrums of Input and Output

N Gaussian random variables

Definition

$$R_{XY}[m] = \sum_{k=-\infty}^{+\infty} h[k]R_{XX}[m-k]$$

$$R_{YX}[m] = \sum_{k=-\infty}^{+\infty} h[-k]R_{XX}[m-k]$$

$$S_{X_s Y_s} = S_{X_s X_s}(e^{j\Omega})H(e^{j\Omega})$$

$$S_{Y_s X_s} = S_{X_s X_s}(e^{j\Omega})H(e^{-j\Omega})$$

