Spectral Characteristics of System Response

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

Definition

$$S_{YY}(\omega) = S_{XX}(\omega)|H(\omega)|^2$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\omega)e^{-j\omega\tau}d\tau$$

Definition

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\xi_1) \int_{-\infty}^{\infty} h(\xi_2) \int_{-\infty}^{\infty} R_{XX}(\tau + \xi_1 - \xi_2) e^{-j\omega\tau} d\tau d\xi_2 d\xi_1$$

$$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\xi_1) e^{-j\omega\xi_1} d\xi_1 \int_{-\infty}^{\infty} h(\xi_2) e^{-j\omega\xi_2} d\xi_2 \int_{-\infty}^{\infty} R_{XX}(\xi) e^{-j\omega\xi} d\xi$$

$$S_{YY}(\omega) = H^*(\omega) H(\omega) S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Definition

$$S_{YY}(\omega) = H^*(\omega)H(\omega)S_{XX}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)|H(\omega)|^2 d\omega$$

Definition

$$S_{XY}(\omega) = H(\omega)S_{XX}(\omega)$$

$$S_{YX}(\omega) = H(-\omega)S_{XX}(\omega)$$

Definition

$$
\begin{aligned}
P_{YY}(\omega_f) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)|H(\omega)|^2 d\omega \\
&= \frac{1}{\pi}\int_{0}^{\infty} S_{XX}(\omega)|H(\omega)|^2 d\omega \\
&= \frac{1}{\pi} S_{XX}(\omega_f)\int_{0}^{\infty} |H(\omega)|^2 d\omega \\
&= \frac{1}{\pi} S_{XX}(\omega_f)|H(\omega_f)|^2 W_N
\end{aligned}
$$

Measurement of Power Density Spectrum (2)

N Gaussian random variables

Definition

$$P_{YY}(\omega_f) = \frac{1}{\pi} S_{XX}(\omega_f) |H(\omega)|^2 W_N$$

$$W_N = \frac{\int_0^\infty |H(\omega)|^2 d\omega}{|H(\omega_f)|^2}$$

$$S_{YY}(\omega_f) = \frac{\pi P_{YY}(\omega_f)}{W_N |H(\omega_f)|^2}$$

Definition

$$P_{YY}(\omega_{IF}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{A_0^2}{4}[S_{XX}A + S_{XX}B]|H(\omega)|^2 d\omega$$

$$S_{XX}A = S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f)$$

$$S_{XX}B = S_{XX}(\omega + \omega_0 + \omega_{IF} + \omega_f)$$

$$P_{YY}(\omega_f) = \frac{2}{2\pi}\int_{0}^{\infty} \frac{A_0^2}{4}[S_{XX}A]|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi}\int_{0}^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f)|H(\omega)|^2 d\omega$$

Definition

$$P_{YY}(\omega_f) = \frac{1}{2\pi}\int_{-0}^{\infty}\frac{A_0^2}{4}[S_{XX}A + S_{XX}B]|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f)|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} S_{XX}(-\omega_0 - \omega_f)\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega$$

Definition

$$P_{YY}(\omega_f) = \frac{1}{2\pi}\int_{-0}^{\infty}\frac{A_0^2}{4}[S_{XX}A + S_{XX}B]|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi}\int_{-\infty}^{\infty} S_{XX}(\omega - \omega_0 - \omega_{IF} - \omega_f)|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} S_{XX}(-\omega_0 - \omega_f)\int_0^{\infty}|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f)\int_0^{\infty}|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f) W_N |H(\omega_{IF})|^2$$

Definition

$$P_{YY}(\omega_f) = \frac{1}{2\pi}\int_{-0}^{\infty} \frac{A_0^2}{4}[S_{XX}A + S_{XX}B]|H(\omega)|^2 d\omega$$

$$= \frac{A_0^2}{4\pi} S_{XX}(\omega_0 + \omega_f) W_N |H(\omega_{IF})|^2$$

$$S_{XX}(\omega_0 + \omega_f) = \frac{4\pi P_{YY}(\omega_{IF})}{A_0^2 W_N |H(\omega_{IF})|^2}$$

Definition

$$S_{X_s X_s}(\omega) = S_{X_s X_s}(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} R_{XX}[n] e^{-jn\Omega}$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-jn\Omega}$$

Definition

$$R_{YY}[n] = E\left[Y[k]Y[k+n]\right]$$

$$= E\left[\sum_{m=-\infty}^{+\infty} X[k-m]h[m] \sum_{p=-\infty}^{+\infty} X[k+n-p]h[p]\right]$$

$$= \sum_{m=-\infty}^{+\infty} h[m]E\left[\sum_{p=-\infty}^{+\infty} X[k-m]X[k+n-p]h[p]\right]$$

$$= \sum_{m=-\infty}^{+\infty} h[m]E\left[\sum_{p=-\infty}^{+\infty} R_{XX}[n+m-p]h[p]\right]$$

Definition

$$S_{Y_sY_s}(\omega) = S_{Y_sY_s}(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} R_{YY}[n]e^{-jn\Omega}$$

$$= \sum_{m=-\infty}^{+\infty} h[m] \sum_{p=-\infty}^{+\infty} h[p] \sum_{n=-\infty}^{+\infty} R_{XX}[n+m-p]e^{-jn\Omega}$$

$$= \sum_{m=-\infty}^{+\infty} h[m]e^{+jm\Omega} \sum_{p=-\infty}^{+\infty} h[p]e^{-jp\Omega} \sum_{n=-\infty}^{+\infty} R_{XX}[r]e^{-jr\Omega}$$

$$S_{Y_sY_s}(\omega) = S_{Y_sY_s}(e^{j\Omega}) = S_{X_sX_s}(e^{j\Omega})|H(e^{j\Omega})|^2$$

Definition

$$E\left[Y^2[n]\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{Y_sY_s}(e^{j\Omega})d\Omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{X_sX_s}(e^{j\Omega})|H(e^{j\Omega})|^2 d\Omega$$

Definition

$$R_{XY}[m] = \sum_{k=-\infty}^{+\infty} h[k] R_{XX}[m-k]$$

$$R_{YX}[m] = \sum_{k=-\infty}^{+\infty} h[-k] R_{XX}[m-k]$$

$$S_{X_X Y_s} = S_{X_s X_s}(e^{j\Omega}) H(e^{j\Omega})$$

$$S_{Y_X X_s} = S_{X_s X_s}(e^{j\Omega}) H(e^{-j\Omega})$$