Ergodic Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

Sample Average
N Gaussian random variables

Definition

The sample average

$$\overline{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$$

N independent sample realizations of the process
$X_i(t)$ for $i = 1, \ldots, N$
each realization is a time function
it is difficult to get many sample realizations

Time Average
N Gaussian random variables

Definition

The time average

$$A_T[\bullet] = \frac{1}{2T} \int_{-T}^{T} [\bullet] dt$$

average over time of a single realization

Time-Autocorrelation Function

N Gaussian random variables

Definition

The time average

$$\overline{x}_T = A_T\left[x(t)\right] = \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

The time autocorrelation function

$$R_T(\tau) = A_T\left[x(t)x(t+\tau)\right] = \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

Stationary Processes
N Gaussian random variables

Definition

first order stationary processes

$$m_X(t) = E[X(t)] = \overline{X} = constant$$

second order stationary processes

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

Stationary Processe Example

N Gaussian random variables

Definition

As a example of a stationary process
for which any single realisation has
an apparently noise-free structure,
let Y have a uniform distribution on $(0, 2\pi]$
and define the time series $X(t)$ by

$$X(t) = \cos(t + Y)$$

.
Then $X(t)$ is strictly stationary.

Expectation of Time-Autocorrelation Function

N Gaussian random variables

Definition

assuming the expectation can be brought inside the integrals

$$E[\overline{x}_T] = A_T[E[x(t)]] = \overline{X}$$
$$E[R_T(\tau)] = A_T[E[x(t)x(t+\tau)]] = R_{XX}(\tau)$$

- $\overline{x}_T$: a unbiased estimate of $\overline{X}$
- $\overline{X}$ - the process mean
- $R_T(\tau)$: a unbiased estimate of $R_{XX}(\tau)$
- $R_{XX}(\tau)$ - the autocorrelation of time lag τ

Convergence in square
N Gaussian random variables

Definition

the conditions under which random sequences of time average $A[\bullet]$ converge as $T \to \infty$

convergence in square

A random sequence X_n is said to **converge** to a random variable X in mean squre if

$$\lim_{n\to\infty} E\left[(X_n - X)^2\right] = 0$$

$$A[\bullet] = \lim_{n\to\infty} A_T[\bullet]$$

Conditions
N Gaussian random variables

1. $X(t)$ has a finite constant mean $\overline{X}$ for all t
2. $X(t)$ is bounded $x(t) < \infty$ for all t and all $x(t)$
3. Bounded time average of E[|X(t)|]

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E[|X(t)|]dt < \infty$$

4. $X(t)$ is a regular process

$$E\left[|X(t)|^2\right] = R_{XX}(t,t) < \infty$$

Regular process

N Gaussian random variables

$X(t)$ is a regular process

$$E\left[|X(t)|^2\right] = R_{XX}(t,t) < \infty$$

for a real WSS process $X(t)$

$$E\left[|X(t)|^2\right] = R_{XX}(0) < \infty$$

since $\overline{X}$ is finit by assumption

$$C_{XX}(0) = R_{XX}(0) - \overline{X}^2 < \infty$$

Mean Erogodic
N Gaussian random variables

Definition

A wide snese stationary (WSS) process $X(t)$
with a constant **mean value** $\overline{X}$
is called **mean-ergodic**
if the **time average** $\overline{x}_T = A_T[x(t)]$ converges to $\overline{X}$ as $T \to \infty$

$$\lim_{T\to\infty} E\left[(\overline{x}_T - \overline{X})^2\right] = 0$$

$$\lim_{T\to\infty} \sigma^2_{\overline{x}_T} = 0$$

Covariance Functions
N Gaussian random variables

Definition

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t) m_X(t+\tau)$$

for a WSS process $X(t)$

$$C_{XX}(\tau) = E\left[\left\{X(t) - \overline{X}\right\}\left\{X(t+\tau) - \overline{X}\right\}\right]$$
$$= R_{XX}(\tau) - \overline{X}^2$$

Variance of $\overline{x}_T$ (1)

N Gaussian random variables

$$\sigma^2_{\overline{x}_T} = E\left[\left\{\frac{1}{2T}\int_{-T}^{T}\left(X(t)-\overline{X}\right)dt\right\}^2\right]$$

$$= E\left[\left(\frac{1}{2T}\right)^2\left\{\int_{-T}^{T}\left(X(t)-\overline{X}\right)dt\right\}\left\{\int_{-T}^{T}\left(X(t_1)-\overline{X}\right)dt_1\right\}\right]$$

$$= E\left[\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\left(X(t)-\overline{X}\right)\left(X(t_1)-\overline{X}\right)dt\,dt_1\right]$$

Variance of $\overline{x}_T$ (2)

N Gaussian random variables

$$\sigma^2_{\overline{x}_T} = E\left[(\overline{x}_T - \overline{X})^2\right]$$

$$= \left(\frac{1}{2T}\right)^2 \int_{-T}^{T} E\left[\left(X(t) - \overline{X}\right)\left(X(t_1) - \overline{X}\right)\right] dt dt_1$$

$$= \left(\frac{1}{2T}\right)^2 \int_{-T}^{T} C_{XX}(t, t_1) dt dt_1$$

for the WSS $X(t)$, let $C_{XX}(t, t_1) = C_{XX}(\tau)$, $\tau = t_1 - t$, and $d\tau = dt_1$

$$= \left(\frac{1}{2T}\right)^2 \int_{t=-T}^{T} \int_{\tau=-T-t}^{T-t} C_{XX}(\tau) d\tau dt$$

Variance of $\overline{x}_T$ (3)

N Gaussian random variables

$$\sigma^2_{\overline{x}_T} = \left(\frac{1}{2T}\right)^2 \int_{t=-T}^{T} \int_{\tau=-T-t}^{T-t} C_{XX}(\tau) d\tau dt$$

the Riemann strips in $\tau - t$ plane

1. using horizontal Rieman strips
2. using vertical Rieman strips
   using the symmetry $C_{XX}(-\tau) = C_{XX}(-\tau)$

$$\sigma^2_{\overline{x}_T} = \frac{1}{2T} \int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{XX}(\tau) d\tau$$

Variance of $\overline{x}_T$ (4)

N Gaussian random variables

the necessary and sufficient condition
for a WSS process X(t) to be mean ergodic

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau\right\}=0$$

$$\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau<\frac{1}{2T}\int_{-2T}^{2T}|C_{XX}(\tau)|\,d\tau$$

1. $C_{XX}(0)<\infty$ and $C_{XX}(\tau)\to 0$ as $|\tau|\to\infty$
2. $\int_{-\infty}^{\infty}|C_{XX}(\tau)|\,d\tau<\infty$

Mean Ergodic Condition
N Gaussian random variables

Definition

a necessary and sufficient condition
for a WSS process $X(t)$ to be **mean ergodic**

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau\right\}=0$$

Mean Ergodic Process - continuous time

N Gaussian random variables

Definition

$X(t)$ is a **mean ergodic** if

1. $C_{XX}(0) < \infty$ and $C_{XX}(\tau) \to 0$ as $|\tau| \to \infty$
2. $\int_{-\infty}^{\infty} |C_{XX}(\tau)| \, d\tau < \infty$

Mean Ergodic Process - discrete time

N Gaussian random variables

Definition

$X[n]$ is a mean ergodic if

$$\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-N}^{+N} X[n]\right\} = \overline{X}$$

$$\lim_{T\to\infty}\left\{\frac{1}{2N+1}\sum_{n=-2N}^{+2N}\left(1-\frac{|n|}{2N+1}\right)C_{XX}[n]\right\} = 0$$

Auto-correlation Ergodic Process - Continuous Time

N Gaussian random variables

Definition

A stationary continuous process $X(t)$
with autocorrelation function $R_{XX}(\tau)$
is called **autocorrelation ergodic**
if for all τ, $R_T(\tau) = A_T[x(t)x(t+\tau)]$
converges to $R_{XX}(\tau)$ as $T \to \infty$

Auto-correlation Ergodic Process - Discrete Time

N Gaussian random variables

Definition

A stationary sequence $X[n]$
with autocorrelation function $R_N[k]$
is called **autocorrelation ergodic**
if for all k, $R_N[k] = \frac{1}{2N+1} \sum_{n=-N}^{+N} x[n]x[n+k]$
converges to $R_{XX}[k]$ as $N \to \infty$

A necessary and sufficient condition
N Gaussian random variables

Definition

$$W(t) = X(t)X(t+\tau)$$
$$E[W(t)] = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

$$\begin{aligned} R_{WW}(\lambda) &= E[W(t)W(t+\lambda)] \\ &= E[X(t)X(t+\tau)X(t+\lambda)X(t+\tau+\lambda)] \end{aligned}$$

$$\begin{aligned} C_{WW}(\lambda) &= R_{WW}(\lambda) - \{E[W(t)]\}^2 \\ &= R_{WW}(\lambda) - R_{XX}^2(\tau) \end{aligned}$$

Auto-correlation Ergodic Condition
N Gaussian random variables

Definition

a necessary and sufficient condition
for a WSS process $X(t)$ to be **auto-correlation ergodic**

$$\lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{WW}(\tau)d\tau\right\}=0$$

Auto-correlation Ergodic

N Gaussian random variables

- auto-correlation ergodicity requires that the 4-th order moments of X(t)
  $R_{WW}(\lambda) = E[X(t)X(t+\tau)X(t+\lambda)X(t+\tau+\lambda)]$
- for Gaussian processes, 4-th order moments are known via 2nd and 1st order moments