Temporal Characteristics of Random Processes

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Based on
Probability, Random Variables and Random Signal Principles,
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Outline

First Order Distribution Function

N Gaussian random variables

Definition

For one particular time t_1, the distribution function associated with the random variable $X_1 = X(t_1)$

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$$

the density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

Second Order Distribution Function

N Gaussian random variables

Definition

For one particular time t_1, t_2, the distribution function associated with the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

the density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

N-th Order Distribution Function
N Gaussian random variables

Definition

For one particular time t_1, t_2, ... , t_N,
the distribution function
associated with the random variables
$X_1 = X(t_1)$, $X_2 = X(t_2)$, ..., $X_N = X(t_N)$

$$F_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = P\{X(t_1) \leq x_1, \ldots, X(t_N) \leq x_N\}$$

the density function

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = \frac{\partial^N F_X(x_1, \ldots, x_N; t_1, \ldots, t_N)}{\partial x_1 \cdots \partial x_N}$$

Statistical Independence
N Gaussian random variables

Definition

Two processes $X(t)$, $Y(t)$ are **statistically independent**
if the random variable group $X(t_1), X(t_2), \cdots X(t_N)$
is independent of the group $Y(t_1^{'}), Y(t_2^{'}), \cdots Y(t_M^{'})$
for any choice of time $t_1, t_2, \cdots, t_N, t_1^{'}, t_2^{'}, \cdots, t_M^{'}$
Independence requires that
the joint density be **factorable** by group

$$f_{X,Y}(x_1, \ldots, x_N, y_1, \ldots, y_M; t_1, \ldots, t_N, t_1^{'}, \ldots, t_M^{'})$$

$$= f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) f_Y(y_1, \ldots, y_M; t, \ldots, t_M^{'})$$

The 1st order moment
N Gaussian random variables

Definition

The mean of a random process

$$m_X(t) = E[X(t)]$$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x;t)\,dx$$

$$m_X[n] = E[X[n]]$$

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The autocorrelation function
N Gaussian random variables

Definition

The **correlation** of a random process
at two instants of time $X(t_1)$ and $X(t_2)$,
in general varies with t_1 and t_2

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$R_{XX}[n, n+k] = E[X[n]X[n+k]]$$

—

The autocovariance function
N Gaussian random variables

Definition

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XX}[n, n+k] = E\left[\{X[n] - m_X[n]\}\{X[n+k] - m_X[n+k]\}\right]$$
$$= R_{XX}[n, n+k] - m_X[n]m_X[n+k]$$

The variance of a random process
N Gaussian random variables

Definition

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XX}(t, t) = R_{XX}(t, t) - m_X^2(t) = \sigma_X^2(t) \qquad (\tau = 0)$$

The cross-correlation function
N Gaussian random variables

Definition

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$
$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)]$$

The cross-covariance function (1)

N Gaussian random variables

Definition

$$C_{XX}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{X(t+\tau) - m_X(t+\tau)\}\right]$$
$$= R_{XX}(t, t+\tau) - m_X(t)m_X(t+\tau)$$

$$C_{XY}(t, t+\tau) = E\left[\{X(t) - m_X(t)\}\{Y(t+\tau) - m_Y(t+\tau)\}\right]$$
$$= R_{XY}(t, t+\tau) - m_X(t)m_Y(t+\tau)$$

The cross-covariance function (2)

N Gaussian random variables

Definition

$$C_{XX}[n, n+k] = E\left[\{X[n] - m_X[n]\}\{X[n+k] - m_X[n+k]\}\right]$$
$$= R_{XX}[n, n+k] - m_X[n]m_X[n+k]$$

$$C_{XY}[n, n+k] = E\left[\{X[n] - m_X[n]\}\{Y[n+k] - m_Y[n+k]\}\right]$$
$$= R_{XY}[n, n+k] - m_X[n]m_Y[n+k]$$

DT and CT relations

N Gaussian random variables

Definition

$$m_X[n] = m_Y(nT_s)$$

$$R_{XX}[n, n+k] = R_{YY}(nT_s, (n+k)T_s)$$

$$C_{XX}[n, n+k] = C_{YY}(nT_s, (n+k)T_s)$$