

Temporal Characteristics of Random Processes

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July 17, 2020

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Based on
Probability, Random Variables and Random Signal Principles,
P.Z. Peebles,Jr. and B. Shi

Outline

1 Joint Distributions, Independence, and Moments

First Order Distribution Function

N Gaussian random variables

Definition

For one particular time t_1 , the distribution function associated with the random variable $X_1 = X(t_1)$

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\}$$

the density function

$$f_X(x_1; t_1) = \frac{dF_X(x_1; t_1)}{dx_1}$$

Second Order Distribution Function

N Gaussian random variables

Definition

For one particular time t_1, t_2 , the distribution function associated with the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

the density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

N -th Order Distribution Function

N Gaussian random variables

Definition

For one particular time t_1, t_2, \dots, t_N ,
the distribution function
associated with the random variables

$$X_1 = X(t_1), X_2 = X(t_2), \dots, X_N = X(t_N)$$

$$F_X(x_1, \dots, x_N; t_1, \dots, t_N) = P\{X(t_1) \leq x_1, \dots, X(t_N) \leq x_N\}$$

the density function

$$f_X(x_1, \dots, x_N; t_1, \dots, t_N) = \frac{\partial^N F_X(x_1, \dots, x_N; t_1, \dots, t_N)}{\partial x_1 \cdots \partial x_N}$$

Statistical Independence

N Gaussian random variables

Definition

Two processes $X(t)$, $Y(t)$ are **statistically independent** if the random variable group $X(t_1), X(t_2), \dots, X(t_N)$ is independent of the group $Y(t'_1), Y(t'_2), \dots, Y(t'_M)$ for any choice of time $t_1, t_2, \dots, t_N, t'_1, t'_2, \dots, t'_M$

Independence requires that the joint density be **factorable** by group

$$\begin{aligned} & f_{X,Y}(x_1, \dots, x_N, y_1, \dots, y_M; t_1, \dots, t_N, t'_1, \dots, t'_M) \\ &= f_X(x_1, \dots, x_N; t_1, \dots, t_N) f_Y(y_1, \dots, y_M; t'_1, \dots, t'_M) \end{aligned}$$

The 1st order moment

N Gaussian random variables

Definition

The mean of a random process

$$m_X(t) = E[X(t)]$$

$$m_X(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx$$

$$m_X[n] = E[X[n]]$$

The autocorrelation function

N Gaussian random variables

Definition

The **correlation** of a random process at two instants of time $X(t_1)$ and $X(t_2)$, in general varies with t_1 and t_2

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XX}[n, n + k] = E[X[n]X[n + k]]$$

The autocovariance function

N Gaussian random variables

Definition

$$\begin{aligned} C_{xx}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}] \\ &= R_{xx}(t, t + \tau) - m_X(t)m_X(t + \tau) \end{aligned}$$

$$\begin{aligned} C_{xx}[n, n + k] &= E[\{X[n] - m_X[n]\} \{X[n + k] - m_X[n + k]\}] \\ &= R_{xx}[n, n + k] - m_X[n]m_X[n + k] \end{aligned}$$

The variance of a random process

N Gaussian random variables

Definition

$$\begin{aligned}C_{xx}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}] \\&= R_{xx}(t, t + \tau) - m_X(t)m_X(t + \tau)\end{aligned}$$

$$C_{xx}(t, t) = R_{xx}(t, t) - m_X^2(t) = \sigma_X^2(t) \quad (\tau = 0)$$

The cross-correlation function

N Gaussian random variables

Definition

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$R_{XX}(t, t + \tau) = E[X(t)X(t + \tau)]$$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

$$R_{XY}(t, t + \tau) = E[X(t)Y(t + \tau)]$$

The cross-covariance function (1)

N Gaussian random variables

Definition

$$\begin{aligned}C_{XX}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{X(t + \tau) - m_X(t + \tau)\}] \\&= R_{XX}(t, t + \tau) - m_X(t)m_X(t + \tau)\end{aligned}$$

$$\begin{aligned}C_{XY}(t, t + \tau) &= E[\{X(t) - m_X(t)\} \{Y(t + \tau) - m_Y(t + \tau)\}] \\&= R_{XY}(t, t + \tau) - m_X(t)m_Y(t + \tau)\end{aligned}$$

The cross-covariance function (2)

N Gaussian random variables

Definition

$$\begin{aligned}C_{XX}[n, n+k] &= E[\{X[n] - m_X[n]\} \{X[n+k] - m_X[n+k]\}] \\ &= R_{XX}[n, n+k] - m_X[n]m_X[n+k]\end{aligned}$$

$$\begin{aligned}C_{XY}[n, n+k] &= E[\{X[n] - m_X[n]\} \{Y[n+k] - m_Y[n+k]\}] \\ &= R_{XY}[n, n+k] - m_X[n]m_Y[n+k]\end{aligned}$$

DT and CT relations

N Gaussian random variables

Definition

$$m_X[n] = m_Y(nT_s)$$

$$R_{XX}[n, n+k] = R_{YY}(nT_s, (n+k)T_s)$$

$$C_{XX}[n, n+k] = C_{YY}(nT_s, (n+k)T_s)$$

